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# Positive Integer Solutions of Some Pell Equations

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**Abstract**. Let $k$ be a natural number and $d = k^2 \pm 4$ or $k^2 \pm 1$. In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and we get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ in terms of generalized Fibonacci and Lucas sequences. Moreover, we find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences.

## 1 Introduction

Let $d$ be a positive integer that is not a perfect square. It is well known that the Pell equation $x^2 - dy^2 = 1$ have always positive integer solutions. When $N \neq 1$, the Pell equation $x^2 - dy^2 = N$ may not has any positive integer solution. It can be seen that the equations $x^2 - 3y^2 = -1$ and $x^2 - 7y^2 = -4$ have no positive integer solutions. Whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{d}$ (See section 2 for more detailed information). When $k$ is a positive integer and $d \in \left\{k^2 \pm 4,\ k^2 \pm 1\right\}$, positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ and $x^2 - dy^2 = \pm 1$ have been investigated by Jones in [6] and the method used in the proofs of the theorems is the method of descent of Fermat. The same or similar equations are investigated by some other authors in [18], [9], [10], [17], [8], and [16]. Especially, when a solution exists, all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ and $x^2 - dy^2 = \pm 1$ are given in terms of the generalized Fibonacci and Lucas sequences. In this paper, if a solution exists, we will use continued fraction expansion of $\sqrt{d}$ in order to get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ when $d \in \left\{k^2 \pm 4,\ k^2 \pm 1\right\}$. Moreover, we will find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ when $d \in \left\{k^2 \pm 4,\ k^2 \pm 1\right\}$.

Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$. Let $k$ and $s$ be two nonzero integers with $k^2 + 4s > 0$. Generalized Fibonacci sequence is defined by

$$U_0(k,s) = 0, U_1(k,s) = 1 \text{ and } U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$$

for $n \geqslant 1$ and generalized Lucas sequence is defined by

$$V_0(k,s) = 2, V_1(k,s) = k \text{ and } V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$$

for $n \geqslant 1$, respectively. For $k = s = 1$, the sequences $(U_n)$ and $(V_n)$ are called Fibonacci and Lucas sequences and they are denoted as $(F_n)$ and $(L_n)$, respectively. For $k = 2$ and $s = 1$, the sequences $(U_n)$ and $(V_n)$ are called Pell and Pell-Lucas sequences and they are denoted as $(P_n)$ and $(Q_n)$, respectively. It is well known that

$$U_n(k,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k,s) = \alpha^n + \beta^n$$

where $\alpha = \left(k + \sqrt{k^2 + 4s}\right)/2$ and $\beta = \left(k - \sqrt{k^2 + 4s}\right)/2$. The above identities are known as Binet's formulae. Clearly, $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 + 4s}$, and $\alpha\beta = -s$. Especially, if

$\alpha = \left(k + \sqrt{k^2+4}\right)/2$ and $\beta = \left(k - \sqrt{k^2+4}\right)/2$ , then we get

$$U_n(k,1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k,1) = \alpha^n + \beta^n. \tag{1.1}$$

If $\alpha = \left(k + \sqrt{k^2-4}\right)/2$ and $\beta = \left(k - \sqrt{k^2-4}\right)/2$ , then we get

$$U_n(k,-1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k,-1) = \alpha^n + \beta^n. \tag{1.2}$$

Also, if $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$ , then we get

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n. \tag{1.3}$$

Moreover, if $k$ is even, then it can be easily seen that

$$U_n(k,\pm 1) \text{ is odd} \Leftrightarrow n \text{ is odd},$$

$$U_n(k,\pm 1) \text{ is even} \Leftrightarrow n \text{ is even},$$

$$V_n(k,\pm 1) \text{ is even for all } n \in \mathbb{N}. \tag{1.4}$$

If $k$ is odd, then

$$2 \mid V_n(k,\pm 1) \Leftrightarrow 2 \mid U_n(k,\pm 1) \Leftrightarrow 3 \mid n.$$

For more information about generalized Fibonacci and Lucas sequences, one can consult [14], [7], [13], [9], and [10].

## 2 Preliminaries

Let $d$ be a positive integer which is not a perfect square and $N$ be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = \pm 1$, the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are known as classical Pell equation. If $a^2 - db^2 = N$, we say that $(a,b)$ is a solution to the Pell equation $x^2 - dy^2 = N$. We use the notations $(a,b)$ and $a + b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also, if $a$ and $b$ are both positive, we say that $a + b\sqrt{d}$ is positive solution to the equation $x^2 - dy^2 = N$. Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Let $d$ be a positive integer that is not a perfect square. Then there is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = \left[a_0, \overline{a_1, a_2, ..., a_{l-1}, 2a_0}\right]$, where $l$ is the period length and the $a_j$'s are given by the recursion formula;

$$\alpha_0 = \sqrt{d},\ a_k = \lfloor \alpha_k \rfloor \text{ and } \alpha_{k+1} = \frac{1}{\alpha_k - a_k},\ k = 0,1,2,3,...$$

Recall that $a_l = 2a_0$ and $a_{l+k} = a_k$ for $k \geq 1$. The $n^{th}$ convergent of $\sqrt{d}$ for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}.$$

Let $x_1 + y_1\sqrt{d}$ be a positive solution to the equation $x^2 - dy^2 = N$. We say that $x_1 + y_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = N$, if $x_2 + y_2\sqrt{d}$ is a different solution to the equation $x^2 - dy^2 = N$, then $x_1 + y_1\sqrt{d} < x_2 + y_2\sqrt{d}$. Recall that if $a + b\sqrt{d}$ and $r + s\sqrt{d}$ are two solutions to the equation $x^2 - dy^2 = N$, then $a = r$ if and only if $b = s$, and $a + b\sqrt{d} < r + s\sqrt{d}$ if and only if $a < r$ and $b < s$. The following lemmas and theorems can be found many elementary textbooks.

**Lemma 2.1.** *If* $x_1 + y_1\sqrt{d}$ *is the fundamental solution to the equation* $x^2 - dy^2 = -1$*, then* $(x_1 + y_1\sqrt{d})^2$ *is the fundamental solution to the equation* $x^2 - dy^2 = 1$.

If we know fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$, then we can give all positive integer solutions to these equations. For more information about Pell equation, one can consult [12], [15], and [4]. Now we give the fundamental solution of the equations $x^2 - dy^2 = \pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{d}$.

**Lemma 2.2.** *Let* $l$ *be the period length of continued fraction expansion of* $\sqrt{d}$*. If* $l$ *is even, then the fundamental solution to the equation* $x^2 - dy^2 = 1$ *is given by*

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}$$

*and the equation* $x^2 - dy^2 = -1$ *has no integer solutions. If* $l$ *is odd, then the fundamental solution of the equation* $x^2 - dy^2 = 1$ *is given by*

$$x_1 + y_1\sqrt{d} = p_{2l-1} + q_{2l-1}\sqrt{d}.$$

*and the fundamental solution to the equation* $x^2 - dy^2 = -1$ *is given by*

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}.$$

**Theorem 2.3.** *Let* $x_1 + y_1\sqrt{d}$ *be the fundamental solution to the equation* $x^2 - dy^2 = 1$*. Then all positive integer solutions to the equation* $x^2 - dy^2 = 1$ *are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

*with* $n \geq 1$.

**Theorem 2.4.** *Let* $x_1 + y_1\sqrt{d}$ *be the fundamental solution to the equation* $x^2 - dy^2 = -1$*. Then all positive integer solutions to the equation* $x^2 - dy^2 = -1$ *are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}$$

*with* $n \geq 1$.

Now we give the following two theorems from [15]. See also [4].

**Theorem 2.5.** *Let* $x_1 + y_1\sqrt{d}$ *be the fundamental solution to the equation* $x^2 - dy^2 = 4$*. Then all positive integer solutions to the equation* $x^2 - dy^2 = 4$ *are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}}$$

*with* $n \geq 1$.

**Theorem 2.6.** *Let* $x_1 + y_1\sqrt{d}$ *be the fundamental solution to the equation* $x^2 - dy^2 = -4$*. Then all positive integer solutions to the equation* $x^2 - dy^2 = -4$ *are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^{2n-1}}{4^{n-1}}$$

*with* $n \geq 1$.

From now on, we will assume that $k$ is a natural number. We give continued fraction expansion of $\sqrt{d}$ for $d = k^2 \pm 4$. The proofs of the following two theorems are easy and they can be found many text books on number theory as an exercise (see, for example [2]).

**Theorem 2.7.** *Let $k > 1$. Then*

$$\sqrt{k^2+4} = \begin{cases} \left[k, \overline{\frac{k}{2}, 2k}\right], & \text{if } k \text{ is even,} \\ \left[k, \overline{\frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2k}\right], & \text{if } k \text{ is odd.} \end{cases}$$

**Theorem 2.8.** *Let $k > 3$. Then*

$$\sqrt{k^2-4} = \begin{cases} \left[k-1, \overline{1, \frac{k-3}{2}, 2, \frac{k-3}{2}, 1, 2(k-1)}\right], & \text{if } k \text{ is odd,} \\ \left[k-1, \overline{1, \frac{k-4}{2}, 1, 2(k-1)}\right], & \text{if } k \text{ is even and } k \neq 4 \\ [3, \overline{2, 6}], & \text{if } k = 4 \end{cases}$$

**Corollary 2.9.** *Let $k > 1$ and $d = k^2 + 4$. If $k$ is odd, then the fundamental solution to the equation $x^2 - dy^2 = -1$ is*

$$x_1 + y_1\sqrt{d} = \frac{k^3+3k}{2} + \frac{k^2+1}{2}\sqrt{d}.$$

*If $k$ is even, the equation $x^2 - dy^2 = -1$ has no positive integer solutions.*

*Proof.* Assume that $k$ is odd. Then the period length of the continued fraction expansion of $\sqrt{k^2+4}$ is 5 by Theorem 2.7. Therefore the fundamental solution of the equation $x^2 - dy^2 = -1$ is $p_4 + q_4\sqrt{d}$ by Lemma 2.2. Since

$$\frac{p_4}{q_4} = k + \cfrac{1}{(k-1)/2 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{(k-1)/2}}}} = \frac{\frac{k^3+3k}{2}}{\frac{k^2+1}{2}},$$

the proof follows. If $k$ is even, then the period length is even by Theorem 2.7 and therefore $x^2 - dy^2 = -1$ has no positive integer solutions by Lemma 2.2. □

**Corollary 2.10.** *Let $k > 1$ and $d = k^2 + 4$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = \begin{cases} \frac{k^2+2}{2} + \frac{k}{2}\sqrt{d}, & \text{if } k \text{ is even,} \\ \left(\frac{k^3+3k}{2} + \frac{k^2+1}{2}\sqrt{d}\right)^2, & \text{if } k \text{ is odd.} \end{cases}$$

*Proof.* If $k$ is even, then the proof follows from Lemma 2.2 and Theorem 2.7. If $k$ is odd, then the proof follows from Corollary 2.9 and Lemma 2.1. □

From Lemma 2.2 and Theorem 2.8, we can give the following corollary.

**Corollary 2.11.** *Let $k > 3$ and $d = k^2 - 4$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by*

$$x_1 + y_1\sqrt{d} = \begin{cases} \frac{k^2-2}{2} + \frac{k}{2}\sqrt{d}, & \text{if } k \text{ is even,} \\ \frac{k^3-3k}{2} + \frac{k^2-1}{2}\sqrt{d}, & \text{if } k \text{ is odd.} \end{cases}$$

**Corollary 2.12.** *Let $k > 3$. Then the equation $x^2 - (k^2-4)y^2 = -1$ has no integer solutions.*

*Proof.* The period length of continued fraction expansion of $\sqrt{k^2-4}$ is always even by Theorem 2.8. Thus by Lemma 2.2, it follows that there is no positive integer solutions of the equation $x^2 - (k^2-4)y^2 = -1$. □

## 3 Main Theorems

**Theorem 3.1.** *Let $k>1$ and $d=k^2+4$. Then all positive integer solutions of the equation $x^2-dy^2=1$ are given by*

$$(x,y)=\begin{cases}\left(\frac{V_{2n}(k,1)}{2},\frac{U_{2n}(k,1)}{2}\right), & \text{if } k \text{ is even,}\\ \left(\frac{V_{6n}(k,1)}{2},\frac{U_{6n}(k,1)}{2}\right), & \text{if } k \text{ is odd,}\end{cases}$$

*with $n\geq 1$.*

*Proof.* Assume that $k$ is even. Thenby Corollary 2.10 and Theorem 2.3, all positive integer solutions of the equation $x^2-dy^2=1$ are given by

$$x_n+y_n\sqrt{d}=\left(\frac{k^2+2}{2}+\frac{k}{2}\sqrt{d}\right)^n$$

with $n\geq 1$. Let $\alpha_1=\frac{k^2+2}{2}+\frac{k}{2}\sqrt{d}$ and $\beta_1=\frac{k^2+2}{2}-\frac{k}{2}\sqrt{d}$. Then

$$x_n+y_n\sqrt{d}=\alpha_1^n \text{ and } x_n-y_n\sqrt{d}=\beta_1^n.$$

Thus it follows that $x_n=\frac{\alpha_1^n+\beta_1^n}{2}$ and $y_n=\frac{\alpha_1^n-\beta_1^n}{2\sqrt{d}}$. Let

$$\alpha=\frac{k+\sqrt{k^2+4}}{2} \text{ and } \beta=\frac{k-\sqrt{k^2+4}}{2}.$$

Then it is seen that $\alpha^2=\alpha_1$ and $\beta^2=\beta_1$. Thus it follows that

$$x_n=\frac{\alpha^{2n}+\beta^{2n}}{2}=\frac{V_{2n}(k,1)}{2}$$

and

$$y_n=\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{d}}=\frac{\alpha^{2n}-\beta^{2n}}{2(\alpha-\beta)}=\frac{U_{2n}(k,1)}{2}$$

by (1.1). Now assume that $k$ is odd. Then by Corollary 2.10 and Theorem 2.3, we get

$$x_n+y_n\sqrt{d}=\left(\left(\frac{k^3+3k}{2}+\frac{k^2+1}{2}\sqrt{d}\right)^2\right)^n$$

with $n\geq 1$. Let

$$\alpha_1=\left(\frac{k^3+3k}{2}+\frac{k^2+1}{2}\sqrt{d}\right)^2$$

and

$$\beta_1=\left(\frac{k^3+3k}{2}-\frac{k^2+1}{2}\sqrt{d}\right)^2.$$

Then

$$x_n+y_n\sqrt{d}=\alpha_1^n \text{ and } x_n-y_n\sqrt{d}=\beta_1^n.$$

Thus it is seen that $x_n=\frac{\alpha_1^n+\beta_1^n}{2}$ and $y_n=\frac{\alpha_1^n-\beta_1^n}{2\sqrt{d}}$. Let

$$\alpha=\frac{k+\sqrt{k^2+4}}{2} \text{ and } \beta=\frac{k-\sqrt{k^2+4}}{2}.$$

Since $\alpha_1=\left(\frac{k^3+3k}{2}+\frac{k^2+1}{2}\sqrt{d}\right)^2=(\alpha^3)^2=\alpha^6$ and thus $\beta_1=\beta^6$, we get

$$x_n=\frac{\alpha^{6n}+\beta^{6n}}{2}=\frac{V_{6n}(k,1)}{2}$$

and

$$y_n=\frac{\alpha^{6n}-\beta^{6n}}{2\sqrt{d}}=\frac{\alpha^{6n}-\beta^{6n}}{2(\alpha-\beta)}=\frac{U_{6n}(k,1)}{2}$$

by (1.1). □

**Theorem 3.2.** *Let $k>1$ be an odd integer and $d=k^2+4$. Then all positive integer solutions of the equation $x^2-dy^2=-1$ are given by*

$$(x,y)=\left(\frac{V_{6n-3}(k,1)}{2},\frac{U_{6n-3}(k,1)}{2}\right)$$

with $n\geq 1$.

*Proof.* Assume that $k>1$ be an odd integer. Then by Corollary 2.9 and Theorem 2.4, all positive integer solutions of the equation $x^2-dy^2=-1$ are given by

$$x_n+y_n\sqrt{d}=\left(\frac{k^3+3k}{2}+\frac{k^2+1}{2}\sqrt{d}\right)^{2n-1}$$

with $n\geq 1$. Let $\alpha_1=\frac{k^3+3k}{2}+\frac{k^2+1}{2}\sqrt{d}$ and $\beta_1=\frac{k^3+3k}{2}-\frac{k^2+1}{2}\sqrt{d}$. Then it follows that

$$x_n+y_n\sqrt{d}=\alpha_1^{2n-1} \text{ and } x_n-y_n\sqrt{d}=\beta_1^{2n-1}$$

and therefore $x_n=\frac{\alpha_1^{2n-1}+\beta_1^{2n-1}}{2}$ and $y_n=\frac{\alpha_1^{2n-1}-\beta_1^{2n-1}}{2\sqrt{d}}$. Let

$$\alpha=\frac{k+\sqrt{k^2+4}}{2} \text{ and } \beta=\frac{k-\sqrt{k^2+4}}{2}.$$

Then it is seen that

$$\alpha^3=\left(\frac{k+\sqrt{k^2+4}}{2}\right)^3=\frac{k^3+3k}{2}+\frac{k^2+1}{2}\sqrt{d}=\alpha_1$$

and

$$\beta^3=\left(\frac{k-\sqrt{k^2+4}}{2}\right)^3=\frac{k^3+3k}{2}-\frac{k^2+1}{2}\sqrt{d}=\beta_1.$$

Thus it follows that

$$x_n=\frac{(\alpha^3)^{2n-1}+(\beta^3)^{2n-1}}{2}=\frac{\alpha^{6n-3}+\beta^{6n-3}}{2}=\frac{V_{6n-3}(k,1)}{2}$$

and

$$y_n=\frac{(\alpha^3)^{2n-1}-(\beta^3)^{2n-1}}{2\sqrt{d}}=\frac{\alpha^{6n-3}-\beta^{6n-3}}{2(\alpha-\beta)}=\frac{U_{6n-3}(k,1)}{2}$$

by (1.1). □

**Theorem 3.3.** *Let $k>3$ and $d=k^2-4$. Then all positive integer solutions of the equation $x^2-dy^2=1$ are given by*

$$(x,y)=\begin{cases}\left(\frac{V_{2n}(k,-1)}{2},\frac{U_{2n}(k,-1)}{2}\right), & \text{if } k \text{ is even,}\\ \left(\frac{V_{3n}(k,-1)}{2},\frac{U_{3n}(k,-1)}{2}\right), & \text{if } k \text{ is odd,}\end{cases}$$

*with $n\geq 1$.*

*Proof.* Assume that $k$ is even. By Corollary 2.11 and Theorem 2.3, all positive integer solutions of the equation $x^2-dy^2=1$ are given by

$$x_n+y_n\sqrt{d}=\left(\frac{k^2-2}{2}+\frac{k}{2}\sqrt{d}\right)^n.$$

Let $\alpha_1=\frac{k^2-2}{2}+\frac{k}{2}\sqrt{d}$ and $\beta_1=\frac{k^2-2}{2}-\frac{k}{2}\sqrt{d}$. Then it follows that

$$x_n+y_n\sqrt{d}=\alpha_1^n \text{ and } x_n-y_n\sqrt{d}=\beta_1^n$$

and therefore $x_n = \frac{\alpha_1^n + \beta_1^n}{2}$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{2\sqrt{d}}$. Let

$$\alpha = \frac{k + \sqrt{k^2 - 4}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 - 4}}{2}.$$

Then it is seen that $\alpha^2 = \alpha_1$ and $\beta^2 = \beta_1$. Thus it follows that

$$x_n = \frac{\alpha^{2n} + \beta^{2n}}{2} = \frac{V_{2n}(k, -1)}{2}$$

and

$$y_n = \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{d}} = \frac{\alpha^{2n} - \beta^{2n}}{2(\alpha - \beta)} = \frac{U_{2n}(k, -1)}{2}$$

by (1.2). Now assume that $k$ is odd. Then by Corollary 2.11 and Theorem 2.3, we get

$$x_n + y_n\sqrt{d} = \left(\frac{k^3 - 3k}{2} + \frac{k^2 - 1}{2}\sqrt{d}\right)^n.$$

Let $\alpha_1 = \frac{k^3-3k}{2} + \frac{k^2-1}{2}\sqrt{d}$ and $\beta_1 = \frac{k^3-3k}{2} - \frac{k^2-1}{2}\sqrt{d}$. Then $x_n + y_n\sqrt{d} = \alpha_1^n$ and $x_n - y_n\sqrt{d} = \beta_1^n$. Thus it follows that $x_n = \frac{\alpha_1^n + \beta_1^n}{2}$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{2\sqrt{d}}$. Let $\alpha = \frac{k+\sqrt{k^2-4}}{2}$ and $\beta = \frac{k-\sqrt{k^2-4}}{2}$. Since

$$\alpha^3 = \left(\frac{k + \sqrt{k^2 - 4}}{2}\right)^3 = \frac{k^3 - 3k}{2} + \frac{k^2 - 1}{2}\sqrt{d} = \alpha_1$$

and

$$\beta^3 = \left(\frac{k - \sqrt{k^2 - 4}}{2}\right)^3 = \frac{k^3 - 3k}{2} - \frac{k^2 - 1}{2}\sqrt{d} = \beta_1,$$

we get

$$x_n = \frac{\alpha^{3n} + \beta^{3n}}{2} = \frac{V_{3n}(k, -1)}{2}$$

and

$$y_n = \frac{\alpha^{3n} - \beta^{3n}}{2\sqrt{d}} = \frac{\alpha^{3n} - \beta^{3n}}{2(\alpha - \beta)} = \frac{U_{3n}(k, -1)}{2}$$

by (1.2). □

Now we give all positive integer solutions of the equations $x^2 - (k^2 + 4)y^2 = \pm 4$ and $x^2 - (k^2 - 4)y^2 = \pm 4$. Before giving all positive integer solutions of the equations $x^2 - (k^2 + 4)y^2 = \pm 4$, we give the following lemma which will be useful for finding the solutions.

**Lemma 3.4.** *Let $a + b\sqrt{d}$ be a positive integer solution to the equation $x^2 - dy^2 = 4$. If $a > b^2 - 2$ , then $a + b\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = 4$.*

*Proof.* If $b = 1$, then the proof is trivial. Assume that $b > 1$. Suppose that $x_1 + y_1\sqrt{d}$ is a positive solution to the equation $x^2 - dy^2 = 4$ such that $1 \le y_1 < b$. Then it follows that $a^2 - db^2 = 4 = x_1^2 - dy_1^2$ and thus $d = (x_1^2 - 4)/y_1^2 = (a^2 - 4)/b^2$. This shows that $x_1^2 b^2 - y_1^2 a^2 = 4b^2 - 4y_1^2 = 4(b^2 - y_1^2) > 0$. Thus

$$[(x_1 b + y_1 a)/2][(x_1 b - y_1 a)/2] = b^2 - y_1^2 > 1.$$

It can be seen that $x_1 b + y_1 a$ and $x_1 b - y_1 a$ are even integers. Let $k_1 = (x_1 b + y_1 a)/2$ and $k_2 = (x_1 b - y_1 a)/2$. Then $k_1 k_2 = b^2 - y_1^2$ and $a = (k_1 - k_2)/y_1$. Thus

$$a = \frac{k_1 - k_2}{y_1} \le \frac{k_1 k_2 - 1}{y_1} = \frac{b^2 - y_1^2 - 1}{y_1} \le b^2 - y_1^2 - 1 \le b^2 - 2,$$

which is a contradiction since $a > b^2 - 2$. □

**Theorem 3.5.** *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2+4)y^2 = 4$ are given by*

$$(x, y) = (V_{2n}(k, 1), U_{2n}(k, 1))$$

*with $n \geq 1$.*

*Proof.* Let $a = k^2+2$ and $b = k$. Then $a + b\sqrt{k^2+4}$ is a positive integer solution of the equation $x^2 - (k^2+4)y^2 = 4$. Since $a = k^2+2 > k^2 - 2 = b^2 - 2$, it follows that $k^2+2+k\sqrt{k^2+4}$ is the fundamental solution of the equation $x^2 - (k^2+4)y^2 = 4$, by Lemma 3.4. Thus by Theorem 2.5, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(k^2+2+k\sqrt{k^2+4})^n}{2^{n-1}} = 2\left(\frac{k^2+2+k\sqrt{k^2+4}}{2}\right)^n.$$

Let $\alpha_1 = \frac{k^2+2+k\sqrt{k^2+4}}{2}$ and $\beta_1 = \frac{k^2+2-k\sqrt{k^2+4}}{2}$. Then it is seen that

$$x_n + y_n\sqrt{d} = 2\alpha_1^n \text{ and } x_n - y_n\sqrt{d} = 2\beta_1^n.$$

Thus it follows that $x_n = \alpha_1^n + \beta_1^n$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{\sqrt{d}}$. Let

$$\alpha = \frac{k+\sqrt{k^2+4}}{2} \text{ and } \beta = \frac{k-\sqrt{k^2+4}}{2}.$$

Then

$$\alpha^2 = \left(\frac{k+\sqrt{k^2+4}}{2}\right)^2 = \frac{k^2+2+k\sqrt{k^2+4}}{2} = \alpha_1$$

and

$$\beta^2 = \left(\frac{k-\sqrt{k^2+4}}{2}\right)^2 = \beta_1.$$

Therefore we get

$$x_n = \alpha^{2n} + \beta^{2n} = V_{2n}(k, 1) \text{ and } y_n = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{d}} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = U_{2n}(k, 1)$$

by (1.1). □

**Theorem 3.6.** *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2+4)y^2 = -4$ are given by*

$$(x, y) = (V_{2n-1}(k, 1), U_{2n-1}(k, 1))$$

*with $n \geq 1$.*

*Proof.* Since $k^2 - (k^2+4) = -4$, it follows that $k + \sqrt{k^2+4}$ is the fundamental solution of the equation $x^2 - (k^2+4)y^2 = -4$. Thus by Theorem 2.6, all positive integer solutions of the equation $x^2 - dy^2 = -4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(k+\sqrt{k^2+4})^{2n-1}}{4^{n-1}} = 2\left(\frac{k+\sqrt{k^2+4}}{2}\right)^{2n-1}.$$

Let $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$. Then it follows that

$$x_n + y_n\sqrt{d} = 2\alpha^{2n-1} \text{ and } x_n - y_n\sqrt{d} = 2\beta^{2n-1}.$$

Therefore

$$x_n = \alpha^{2n-1} + \beta^{2n-1} = V_{2n-1}(k, 1)$$

and

$$y_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{\sqrt{d}} = \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta} = U_{2n-1}(k, 1)$$

by (1.1). □

**Theorem 3.7.** *Let* $k > 3$. *Then all positive integer solutions of the equation* $x^2 - (k^2-4)y^2 = 4$ *are given by*

$$(x, y) = (V_n(k, -1), U_n(k, -1))$$

*with* $n \geq 1$.

*Proof.* Since $k^2 - (k^2 - 4) = 4$, it is seen that $k + \sqrt{k^2-4}$ is the fundamental solution of the equation $x^2 - (k^2-4)y^2 = 4$. Let $d = k^2 - 4$. Then by Theorem 2.5, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(k+\sqrt{k^2-4})^n}{2^{n-1}} = 2\left(\frac{k+\sqrt{k^2-4}}{2}\right)^n.$$

Let $\alpha = \frac{k+\sqrt{k^2-4}}{2}$ and $\beta = \frac{k-\sqrt{k^2-4}}{2}$. Then it follows that $x_n + y_n\sqrt{d} = 2\alpha^n$ and $x_n - y_n\sqrt{d} = 2\beta^n$. Thus we get

$$x_n = \alpha^n + \beta^n = V_n(k, -1) \text{ and } y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = U_n(k, -1)$$

by (1.2). □

The following theorem is given in [5].

**Theorem 3.8.** *Let* $d$ *be an odd positive integer. If the equation* $x^2 - dy^2 = -4$ *has a positive integer solution, then the equation* $x^2 - dy^2 = -1$ *has positive integer solutions.*

Now we give the continued fraction expansions of $\sqrt{k^2+1}$ and $\sqrt{k^2-1}$. Since the continued fraction expansions of them are given in [3], we omit their proofs.

**Theorem 3.9.** *If* $k \geq 1$, *then* $\sqrt{k^2+1} = [k, \overline{2k}]$. *If* $k > 1$, *then* $\sqrt{k^2-1} = \left[k-1, \overline{1, 2(k-1)}\right]$.

The proofs of the following corollaries follow from Lemma 2.2 and Theorem 3.9 and therefore we omit their proofs.

**Corollary 3.10.** *Let* $k \geq 1$ *and* $d = k^2 + 1$. *Then the fundamental solution of the equation* $x^2 - dy^2 = 1$ *is*

$$x_1 + y_1\sqrt{d} = 2k^2 + 1 + 2k\sqrt{d}.$$

**Corollary 3.11.** *Let* $k \geq 1$ *and* $d = k^2 + 1$. *Then the fundamental solution of the equation* $x^2 - dy^2 = -1$ *is*

$$x_1 + y_1\sqrt{d} = k + \sqrt{d}.$$

**Corollary 3.12.** *Let* $k > 1$ *and* $d = k^2 - 1$. *Then the fundamental solution of the equation* $x^2 - dy^2 = 1$ *is*

$$x_1 + y_1\sqrt{d} = k + \sqrt{d}.$$

**Theorem 3.13.** *Let* $k \geq 1$. *Then all positive integer solutions of the equation* $x^2 - (k^2+1)y^2 = 1$ *are given by*

$$(x, y) = \left(\frac{V_{2n}(2k, 1)}{2}, U_{2n}(2k, 1)\right)$$

*with* $n \geq 1$.

*Proof.* By Corollary 3.10 and Lemma 2.2, it follows that all positive integer solutions of the equation $x^2 - (k^2+1)y^2 = 1$ are given by

$$x_n + y_n\sqrt{k^2+1} = \left(2k^2 + 1 + 2k\sqrt{k^2+1}\right)^n = \left(2k^2 + 1 + k\sqrt{(2k)^2+4}\right)^n.$$

Let $\alpha = \frac{2k+\sqrt{(2k)^2+4}}{2}$ and $\beta = \frac{2k-\sqrt{(2k)^2+4}}{2}$. Then

$$\alpha^2 = \left(\frac{2k+\sqrt{(2k)^2+4}}{2}\right)^2 = 2k^2 + 1 + k\sqrt{(2k)^2+4}$$

and

$$\beta^2 = \left(\frac{2k - \sqrt{(2k)^2+4}}{2}\right)^2 = 2k^2 + 1 - k\sqrt{(2k)^2+4}.$$

Thus it follows that

$$x_n + y_n\sqrt{k^2+1} = x_n + \frac{y_n}{2}\sqrt{(2k)^2+4} = \alpha^{2n}$$

and

$$x_n - y_n\sqrt{k^2+1} = x_n - \frac{y_n}{2}\sqrt{(2k)^2+4} = \beta^{2n}.$$

Then it is seen that

$$x_n = \frac{\alpha^{2n}+\beta^{2n}}{2} = \frac{V_{2n}(2k,1)}{2}$$

and

$$y_n = \frac{\alpha^{2n}-\beta^{2n}}{\sqrt{(2k)^2+4}} = \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} = U_{2n}(2k,1)$$

by (1.1). □

Since the proof of the following theorems are similar to that of the above theorems, we omit them.

**Theorem 3.14.** *Let $k \geq 1$. Then all positive integer solutions of the equation $x^2 - (k^2+1)y^2 = -1$ are given by*

$$(x,y) = \left(\frac{V_{2n-1}(2k,1)}{2}, U_{2n-1}(2k,1)\right)$$

*with $n \geq 1$.*

**Theorem 3.15.** *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2-1)y^2 = 1$ are given by*

$$(x,y) = \left(\frac{V_n(2k,-1)}{2}, U_n(2k,-1)\right)$$

*with $n \geq 1$.*

**Corollary 3.16.** *Let $k > 1$. Then the equation $x^2 - (k^2-1)y^2 = -1$ has no positive integer solutions.*

*Proof.* The period length of continued fraction expansion of $\sqrt{k^2-1}$ is always even by Theorem 3.9. Thus by Lemma 2.2, it follows that there is no positive integer solutions of the equation $x^2 - (k^2-1)y^2 = -1$. □

**Theorem 3.17.** *Let $k > 3$. Then the equation $x^2 - (k^2-4)y^2 = -4$ has no positive integer solutions.*

*Proof.* Assume that $k$ is odd. Then $k^2-4$ is odd and thus the proof follows from Theorem 3.8 and Corollary 2.12. Now assume that $k$ is even. If $(a,b)$ is a solution to the equation $x^2-(k^2-4)y^2 = -4$, then $a$ is even. Thus we get

$$(a/2)^2 - ((k/2)^2-1)b^2 = -1,$$

which is impossible by Corollary 3.16. □

Now we give all positive integer solutions of the equations $x^2 - (k^2+1)y^2 = \pm 4$ and $x^2 - (k^2-1)y^2 = \pm 4$.

**Theorem 3.18.** *Let $k \geq 1$ and $k \neq 2$. Then all positive integer solutions of the equation $x^2 - (k^2+1)y^2 = -4$ are given by*

$$(x, y) = (V_{2n-1}(2k, 1), 2U_{2n-1}(2k, 1))$$

*with $n \geq 1$.*

*Proof.* Since $k \geq 1$ and $k \neq 2$, it can be shown that $2k + 2\sqrt{k^2+1}$ is the fundamental solution to the equation $x^2 - (k^2+1)y^2 = -4$. Then by Theorem 2.6, all positive integer solutions of the equation $x^2 - (k^2+1)y^2 = -4$ are given by

$$x_n + y_n\sqrt{k^2+1} = 2\left(\frac{2k + 2\sqrt{k^2+1}}{2}\right)^{2n-1} = 2\left(\frac{2k + \sqrt{(2k)^2+4}}{2}\right)^{2n-1}.$$

Let $\alpha = \frac{2k+\sqrt{(2k)^2+4}}{2}$ and $\beta = \frac{2k-\sqrt{(2k)^2+4}}{2}$. Then we get

$$x_n + y_n\sqrt{k^2+1} = x_n + \frac{y_n}{2}\sqrt{(2k)^2+4} = 2\alpha^{2n-1}$$

and

$$x_n - y_n\sqrt{k^2+1} = x_n - \frac{y_n}{2}\sqrt{(2k)^2+4} = 2\beta^{2n-1}.$$

Thus it follows that

$$x_n = \alpha^{2n-1} + \beta^{2n-1} = V_{2n-1}(2k, 1)$$

and

$$y_n = 2\frac{\alpha^{2n-1} - \beta^{2n-1}}{\sqrt{(2k)^2+4}} = 2\frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta} = 2U_{2n-1}(2k, 1)$$

by (1.1). □

Now we can give the following corollary from Theorem 3.18 and identity (1.4).

**Corollary 3.19.** *If $(a, b)$ is a positive integer solution of the equation $x^2 - (k^2+1)y^2 = -4$, then $a$ and $b$ are even.*

Since the proof of the following theorem is similar to that of Theorem 3.18 , we omit it.

**Theorem 3.20.** *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2-1)y^2 = 4$ are given by*

$$(x, y) = (V_n(2k, -1), 2U_n(2k, -1))$$

*with $n \geq 1$.*

**Theorem 3.21.** *Let $k \geq 1$ and $k \neq 2$. Then all positive integer solutions of the equation $x^2 - (k^2+1)y^2 = 4$ are given by*

$$(x, y) = (V_{2n}(2k, 1), 2U_{2n}(2k, 1))$$

*with $n \geq 1$.*

*Proof.* Firstly, we show that if $(a, b)$ is a solution to the equation $x^2 - (k^2+1)y^2 = 4$, then $a$ and $b$ are even. Assume that $k$ is odd. Then $k^2 + 1 = 2t$ for some odd integer $t$. Since $a^2 - 2tb^2 = 4$, it follows that $a$ is even and therefore $b$ is even. Now assume that $k$ is even. Let $d = k^2 + 1$. Then $d$ is odd. Assume that $a$ and $b$ are odd integers. Let $x_1 = |db - ka|$, $y_1 = |a - kb|$. Then $x_1$ and $y_1$ are odd integers. Moreover,

$$\begin{aligned} x_1^2 - dy_1^2 &= (db - ka)^2 - d(a - kb)^2 = b^2d(d - k^2) + a^2(k^2 - d) = b^2d - a^2 \\ &= -(a^2 - db^2) = -4. \end{aligned}$$

Thus $x_1 + y_1\sqrt{d}$ is a positive solution of the equation $x^2 - (k^2+1)y^2 = -4$, which is impossible by Corollary 3.19. Therefore if $a + b\sqrt{d}$ is any solutions of the equation $x^2 - dy^2 = 4$, then $a$ and $b$

are even integers and thus $\frac{a}{2}+\frac{b}{2}\sqrt{d}$ is a solution to the equation $x^2-dy^2=1$. Then it follows that the fundamental solution of the equation $x^2-dy^2=4$ is $4k^2+2+4k\sqrt{d}$ by Corollary 3.10. Thus by Theorem 2.5, it follows that all positive integer solutions of the equation $x^2-(k^2+1)y^2=4$ are given by

$$x_n+y_n\sqrt{k^2+1}=2\left(\frac{4k^2+2+4k\sqrt{k^2+1}}{2}\right)^n=2\left(\frac{4k^2+2+2k\sqrt{(2k)^2+4}}{2}\right)^n.$$

Let $\alpha=\frac{2k+\sqrt{(2k)^2+4}}{2}$ and $\beta=\frac{2k-\sqrt{(2k)^2+4}}{2}$. Then

$$\alpha^2=\left(\frac{2k+\sqrt{(2k)^2+4}}{2}\right)^2=\frac{4k^2+2+2k\sqrt{(2k)^2+4}}{2}$$

and

$$\beta^2=\left(\frac{2k-\sqrt{(2k)^2+4}}{2}\right)^2=\frac{4k^2+2-2k\sqrt{(2k)^2+4}}{2}.$$

Thus it follows that $x_n+y_n\sqrt{k^2+1}=x_n+\frac{y_n}{2}\sqrt{(2k)^2+4}=2\alpha^{2n}$ and $x_n-\frac{y_n}{2}\sqrt{(2k)^2+4}=2\beta^{2n}$. Then it is seen that

$$x_n=\alpha^{2n}+\beta^{2n}=V_{2n}(2k,1)$$

and

$$y_n=2\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{(2k)^2+4}}=2\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}=2U_{2n}(2k,1),$$

by (1.1). □

It can be shown that if $k>2$, then the continued fraction expansion of $\sqrt{k^2-k}$ is $[k-1,\overline{1,2(k-1)}]$ (see [2], page 234). Therefore we can give the following corollary easily.

**Corollary 3.22.** *Let $k>2$. Then the equation $x^2-(k^2-k)y^2=-1$ has no positive integer solutions.*

**Corollary 3.23.** *Let $k\geq 2$ and $k\neq 3$. Then the equation $x^2-(k^2-1)y^2=-4$ has no positive integer solutions.*

*Proof.* Assume that $k$ is even. Then $k^2-1$ is odd and the proof follows from Theorem 3.8 and Corollary 3.16.

Assume that $k$ is odd. Then $k^2-1$ is even. Now assume that $a^2-(k^2-1)b^2=-4$ for some positive integers $a$ and $b$. Then $a$ is even and this implies that

$$(a/2)^2-[\left(k^2-1\right)/4]b^2=-1.$$

This is impossible by Corollary 3.22, since

$$\left(k^2-1\right)/4=((k+1)/2)^2-(k+1)/2.$$

□

Continued fraction expansion of $\sqrt{5}$ is $\left[2,\overline{4}\right]$. Then the period length of the continued fraction expansion of $\sqrt{5}$ is 1. Therefore the fundamental solution to the equation $x^2-5y^2=1$ is $9+4\sqrt{5}$ and the fundamental solution to the equation $x^2-5y^2=-1$ is $2+\sqrt{5}$ by Lemma 2.2. Therefore, by using (1.3), we can give the following corollaries easily.

**Corollary 3.24.** *All positive integer solutions of the equation $x^2-5y^2=1$ are given by*

$$(x,y)=\left(\frac{L_{6n}}{2},\frac{F_{6n}}{2}\right)$$

with $n \geq 1$.

**Corollary 3.25.** *All positive integer solutions of the equation* $x^2 - 5y^2 = -1$ *are given by*

$$(x, y) = \left( \frac{L_{6n-3}}{2}, \frac{F_{6n-3}}{2} \right)$$

with $n \geq 1$.

It can be seen that fundamental solutions of the equations $x^2 - 5y^2 = -4$ and $x^2 - 5y^2 = 4$ are $1 + \sqrt{5}$ and $3 + \sqrt{5}$, respectively. Thus we can give following corollaries.

**Corollary 3.26.** *All positive integer solutions of the equation* $x^2 - 5y^2 = 4$ *are given by*

$$(x, y) = (L_{2n}, F_{2n})$$

with $n \geq 1$.

**Corollary 3.27.** *All positive integer solutions of the equation* $x^2 - 5y^2 = -4$ *are given by*

$$(x, y) = (L_{2n-1}, F_{2n-1})$$

with $n \geq 1$.

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