Conformal anti-invariant submersions from nearly Kähler Manifolds

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Abstract. Akyol and Sahin recently [1], defined conformal anti-invariant submersions from almost Hermitian manifolds and studied the case when ambient manifold is Kähler. In this paper, we discuss conformal anti-invariant submersions from a nearly Kähler manifold onto a Riemannian manifold and derive some results in this respect. We extend the notion of conformal anti-invariant and conformal Lagrangian Riemannian submersion to the case when ambient manifold is nearly Kähler. We also give necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic. Further, we find some decomposition theorems for the total manifold of the submersion and some equivalence conditions.

1 Introduction

The study of Riemannian submersion between Riemannian manifolds was initiated by O’Neill [15] and then Gray [10]. Later such submersions were considered between differentiable manifolds. In 1976, Watson [19] introduced almost Hermitian submersions between almost Hermitian manifolds and showed that in most cases the base manifold and each fibre have the same kind of structure as the total space. We note that almost Hermitian submersions have been extended to the almost contact metric submersions [5], locally conformal Kähler submersions [14], semi-Riemannian submersions and Lorentzian submersions [6], anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds [17]. As we know that Riemannian submersions are related with physics and have their applications in the Yang-Mills theory [18], Kaluza-Klein theory ([4], [11]), supergravity and superstring theories ([12], [13]) etc. Conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds were introduced by Akyol and Sahin [1] and they mainly study conformal anti-invariant submersions from Kähler manifolds onto Riemannian manifolds.

In this paper, we study conformal anti-invariant submersions from nearly Kähler manifolds onto Riemannian manifolds. The paper is organised as follows. The first section is introductory. In the second section, we collect main notions and formulae for other sections.

In section 3, the conformal anti-invariant submersions from nearly Kähler manifolds onto Riemannian manifolds have been studied and the geometry of leaves has been investigated. We also give necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic. Further, in this section we have obtained some equivalence conditions. In section 4, we study certain product structures on the total space of a conformal anti-invariant submersion.

2 Preliminaries

Let $M$ be an even-dimensional differentiable manifold. Let $J$ be a $(1, 1)$ tensor field on $M$ such that $J^2 = -I$. Then $J$ is called almost complex structure on $M$. The manifold $M$ with almost complex structure $J$ is called almost complex manifold. It is well known that almost complex manifold is necessarily orientable. Nijenhuis tensor $N$ of an almost complex structure is defined as:

$$N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY], \quad \text{for all } X, Y \in \Gamma TM.$$

If Nijenhuis tensor field $N$ on an almost complex manifold $M$ is zero, then almost complex manifold $M$ is called complex manifold.

Let $g_M$ be a Riemannian metric on $M$ such that

$$g_M(\mathbf{X}, \mathbf{Y}) = g_M(\mathbf{X}, \mathbf{Y}), \quad \text{for all } \mathbf{X}, \mathbf{Y} \in \Gamma TM. \quad (2.1)$$

Then $g_M$ is called an almost Hermitian metric on $M$ and manifold $M$ with Hermitian metric $g_M$ is called almost Hermitian manifold. The Riemannian connection $\nabla$ of the almost Hermitian manifold $M$ can be extend to the whole tensor algebra on $M$, and in this way we obtain tensor fields like $(\nabla_{\mathbf{X}} \mathbf{Y})$ and that

$$(\nabla_{\mathbf{X}} \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - J\nabla_{\mathbf{X}} \mathbf{Y}, \quad (2.2)$$

for all $\mathbf{X}, \mathbf{Y} \in \Gamma(TM)$.

An almost Hermitian manifold $(M, g_M, J)$ is called Kähler manifold if

$$(\nabla_{\mathbf{X}} \mathbf{Y}) = 0, \quad (2.3)$$

for all $\mathbf{X}, \mathbf{Y} \in \Gamma(TM)$.

An also almost Hermitian manifold $(M, g_M, J)$ is called nearly Kähler manifold [9] if

$$(\nabla_{\mathbf{X}} \mathbf{Y}) + (\nabla_{\mathbf{Y}} \mathbf{X}) = 0, \quad (2.4)$$

for all $\mathbf{X}, \mathbf{Y} \in \Gamma(TM)$.

Suppose $M$ and $N$ are differentiable manifolds with Riemannian metrics $g_M$ and $g_N$ respectively. Let dim $M = m$ and dim $N = n$, where $m > n$. Let $f : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds. We denote the kernel space of $f_{*p}$ by ker $f_{*p}$ and consider the orthogonal complementary space $(\ker f_{*p})^\perp$ to ker $f_{*p}$ in $T_p M$. Then the tangent bundle of $M$ has the following decomposition

$$T_p M = (\ker f_{*p}) \oplus (\ker f_{*p})^\perp. \quad (2.5)$$

We also denote the range of $f_{*f(p)}$ by range $f_{*f(p)}$ and consider the orthogonal complementary space $(\range f_{*f(p)})^\perp$ to range $f_{*f(p)}$ in the tangent bundle $T_{f(p)} N$ of $N$. Thus the tangent bundle $T_{f(p)} N$ of $N$ has the following decomposition

$$T N = (\range f_{*f(p)}) \oplus (\range f_{*f(p)})^\perp. \quad (2.6)$$

As a generalization of Riemannian submersions, horizontally conformal submersions are defined as follows [2]. Let $f : (M, g_M) \to (N, g_N)$ be a differentiable map between Riemannian manifolds, then $f$ is called a horizontally conformal submersion, if there is a positive function $\lambda$ such that

$$g_M(\mathbf{X}, \mathbf{Y}) = \frac{1}{\lambda^2} g_N(f_{*p} \mathbf{X}, f_{*p} \mathbf{Y}), \quad (2.7)$$

for every $\mathbf{X}, \mathbf{Y} \in \Gamma(\ker f_{*p})^\perp$.

It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. Suppose that $f$ is a differentiable map between Riemannian manifolds and point $p \in M$. Then, $f$ is called horizontally weakly conformal map at a point $p$ if either (i) $f_{*p} = 0$ or (ii) $f_{*p}$ maps the horizontal space $(\ker f_{*p})^\perp$ conformally onto $T_{f(p)} N$, i.e., $f_{*p}$ is surjective and $f_{*p}$ satisfies the equation (2.7) for $\mathbf{X}, \mathbf{Y}$ vectors tangent to $(\ker f_{*p})^\perp$. If a point $p$ is of type (i) then it is called critical point of $f$. A point $p$ of type (ii) is called regular. The number $\wedge(p)$ is called the square dilation, it is necessarily non-negative. Its square root $\sqrt{\wedge(p)}$ is called the dilation. A horizontally weakly conformal map $f : M \to N$ is said to be horizontally homothetic if the gradient of its dilation $\lambda$ is vertical, i.e., $H(\nabla \lambda) = 0$ at regular points. If a horizontally weakly conformal map $f$ has no critical points, then it is called horizontally conformal submersion [2]. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. We note that horizontal conformal maps were introduced independently by Fuglede [7] and Ishihara [13].
**Definition 2.1.** ([2]) Let $M$ and $N$ are two Riemannian manifolds with Riemannian metrics $g_M$ and $g_N$ respectively. If $f$ is a differentiable map from $(M, g_M)$ to $(N, g_N)$, then $f$ is called horizontally weakly conformal or semi-conformal at $p$ if either

(i) $df_p = 0$, or
(ii) $df_p$ maps the horizontal space $\mathcal{H}_p = (\ker(df_p))^\perp$ conformally onto $T_{f(p)}N$ i.e., $df_p$ is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_N(f_*U, f_*V) = \Lambda(p)g_M(U, V), \quad \text{for } U, V \in \Gamma(\ker f_*)^\perp,$$

where $p \in M$.

Watson introduced the fundamental tensors of a submersion in [15]. It is known that the fundamental tensor play similar role to that of the second fundamental form of an immersion. Define tensors $T$ and $\mathcal{A}$, for vector fields $E, F$ on $M$ by

$$\mathcal{A}_E F = \mathcal{V}\nabla \mathcal{H}_E \mathcal{H} F + \mathcal{H}\nabla \mathcal{H}_E \mathcal{V} F,$$

$$T_E F = \mathcal{H}\nabla \mathcal{V} E F + \mathcal{V}\nabla \mathcal{E} \mathcal{H} F,$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [6], and $\nabla$ is a Riemannian connection on $M$. On the other hand, from equations (2.9) and (2.10), we have

$$\nabla_X Y = T_X Y + \nabla_X Y,$$

$$\nabla_X U = \mathcal{H}\nabla_X U + T_X U,$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X,$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V,$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$, where $\nabla\nabla_X Y = \nabla_X Y$. If $U$ is basic, then $\mathcal{A}_U U = \mathcal{H}\nabla_X U$.

It is seen that for $p \in M$, $X \in \mathcal{V}_p$ and $U \in \mathcal{H}_p$ the linear operators

$$T_X, \mathcal{A}_U : T_p M \rightarrow T_p M,$$

are skew-symmetric, that is

$$g_M(\mathcal{A}_U E, F) = -g_M(E, \mathcal{A}_U F) \quad \text{and} \quad g_M(T_X E, F) = -g_M(E, T_X F),$$

for each $E, F \in T_p M$. We have also defined the restriction of $T$ to the vertical distribution $T|_{\mathcal{V} \times \mathcal{V}}$ is precisely the second fundamental form of the fibres of $f$. Since $T_V$ is skew-symmetric we get: $f$ has totally geodesic fibres if and only if $T \equiv 0$. For the special case when $f$ is horizontally conformal we have the following:

**Proposition 2.2.** ([10](2.1.2)) Let $f : (M, g_M) \rightarrow (N, g_N)$ be a horizontal conformal submersion between Riemannian manifolds with dilation $\lambda$ and $U, V$ be horizontal vectors. Then

$$\mathcal{A}_U V = \frac{1}{2} \{\mathcal{V}[U, V] - \lambda^2 g_M(U, V)\grad_V(\frac{1}{\lambda^2})\}.$$

We know that the skew-symmetric part of $\mathcal{A}|_{\mathcal{H} \times \mathcal{H}}$ measures the obstruction integrability of the horizontal distribution $\mathcal{H}$.

We also recall the concept of harmonic maps between Riemannian manifolds. Let $f : (M, g_M) \rightarrow (N, g_N)$ is a differentiable map between Riemannian manifolds. Then the differential of $f_*$, of $f$ can be observed a section of the bundle $Hom(TM, f^{-1}TN) \rightarrow M$, where $f^{-1}TN$ is the pullback bundle which has fibres $(f^{-1}TN)_p = T_{f(p)}N$ has a connection $\nabla$ induced from
the Riemannian connection $\nabla^M$ and the pullback connection. Then the second fundamental form of $f$ is given by

$$(\nabla f_*)(U,V) = \nabla^f_U f_*(V) - f_*(\nabla^M_U V).$$

(2.17)

Again, we find necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recollection that a differentiable map $f$ between Riemannian manifolds is called totally geodesic if

$$(\nabla f_*)(V,W) = 0, \text{ for all } V,W \in \Gamma(TM).$$

(2.18)

A geometric clarification of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Now, we explain a decomposition theorem related to the concept of twisted product manifold. However, we first define the adjoint map of a map. Let $f : (M, g_M) \rightarrow (N, g_N)$ be a map between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Then the adjoint map $f^*_{\{\}}$ of $f_*$ is characterized $g_M(X, J_{\{\}} f_* Y) = g_N(f_* X, Y)$, $X \in T_p M$, $Y \in T_{f(p)} N$, and $p \in M$. Considering $f^*_h$ at each $p \in M$ as a linear transformation

$$f^*_h : (ker f_*)^p \rightarrow (range f_*)^p,$$

we will denote the adjoint $f^*_h$ by $f^*_{\{\}}$. Let $f^*_{\{\}}$ be the adjoint of $f^*_h : (T^*_p M, g_M(p)) \rightarrow (T^*_{f(p)} N, g_N(p))$. The linear transformation $f^*_{\{\}} : (range f_*)^p \rightarrow (ker f_*)^p$ defined $(f^*_{\{\}})^h = f^*_{\{\}} Y$, where $Y \in (range f_*)_q$, $q = f(p)$, is an isomorphism and $(f^*_{\{\}})^{-1} = (f^*_{\{\}})^h = f^*_h$.

Lastly, we recollection the subsequent lemma from [2].

Lemma 2.3. Let $(M, g_M)$ and $(N, g_N)$ are two Riemannian manifolds. If $f : M \rightarrow N$ horizontally conformal submersion between Riemannian manifolds, then for any horizontal vector fields $U, V$ and vertical vector fields $X, Y$ we have

$$(i)\nabla df(U, V) = U(ln\lambda)df(V) + V(ln\lambda)df(U) - g_M(U, V)df(\nabla g\ln\lambda);$$

$$(ii)\nabla df(X, Y) = -df(A^h_X Y);$$

$$(iii)\nabla df(U, X) = -df(\nabla^H U)_X = -df((A^H)_X U).$$

where $(A^H)_X$ is the adjoint of $(A^H)$ characterized by

$$\langle (A^H)_X ; E, F \rangle = \langle E, A^H_0 F \rangle \quad (f \text{ for } E, F \in \Gamma(TM)).$$

3 Conformal anti-invariant submersions

Definition 3.1. Let $(M, g_M, J)$ be a nearly Kähler manifold and $(N, g_N)$ be a Riemannian manifold. A horizontal conformal submersion $f : (M, g_M, J) \rightarrow (N, g_N)$ with dilation $\lambda$ is called anti-invariant submersion if the distribution $ker f_*$ is anti-invariant submersion with respect to $J$, i.e., $J(ker f_*) \subseteq (ker f_*)^\perp$.

Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. From above definition we have $J(ker f_*)^\perp \cap (ker f_*) \neq \{0\}$ and denote the complementary orthogonal distribution to $J(ker f_*)$ in $(ker f_*)^\perp$ by $\mu$. Then

$$(ker f_*)^\perp = J(ker f_*) \oplus \mu.$$  

(3.1)

It is clear to see that $\mu$ is invariant distribution of $(ker f_*)^\perp$, under the complex structure $J$. Thus, for $X \in \Gamma(ker f_*)^\perp$, we have

$$JX = BX + CX,$$

(3.2)

where $BX \in \Gamma(ker f_*)$ and $CX \in \Gamma(\mu)$.

Further, since $f_*(ker f_*)^\perp = TN$ and $f$ is a Riemannian submersion, using equation (3.2) it can be shown that $\frac{1}{2}g_N(f_* JX, f_* CX) = 0$, for any $X \in \Gamma(ker f_*)^\perp$ and $V \in \Gamma(ker f_*)$ which implies that $TN = f_*(J(ker f_*) \oplus f_*(\mu))$. Now, we prove following.
Lemma 3.2. Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(N, g_N)$. Then
\[ g_M(X, Y) = 0, \] (3.3)

and
\[ g_M(\nabla_X Y, F) = -2g_M(CY, JA_X V) + g_M(CY, T_V BX) + g_M(CY, A_C X V), \] (3.4)

for $X, Y \in \Gamma(\ker f_\ast) \perp$ and $V \in \Gamma(\ker f_\ast)$.

Proof. For $X, Y \in \Gamma(\ker f_\ast) \perp$ and $V \in \Gamma(\ker f_\ast)$, since $BY \in \Gamma(\ker f_\ast)$, $CX \in \Gamma(\mu)$ and $\phi V \in \Gamma(\ker f_\ast) \perp$. Using equations (2.1) and (3.2), we have
\[ g_M(X, Y) = 0. \]
Again, using equations (2.4), (3.3), (2.12) and (2.13), we have
\[ g_M(\nabla_X Y, F) = -2g_M(CY, JA_X V) + g_M(CY, T_V BX) + g_M(CY, A_C X V), \]

which completes the proof. □

Note: Whenever it is need we have assumed the horizontal vector field to be basic. For any arbitrary tangent vector fields $Z$ and $W$ on $M$, we get
\[ (\nabla_Z J)W = Q_Z W + P_Z W, \] (3.5)

where $Q_Z W$ and $P_Z W$ denote the vertical and horizontal part of $(\nabla_Z J)W$, respectively. For a Kähler manifold $M$, we have
\[ P = Q = 0, \forall Z, W \in \Gamma(TM). \] (3.6)

If $M$ is a nearly Kähler manifold, then it can be easily seen that both $Q$ and $P$ are anti-symmetric in $Z$ and $W$ i.e.,
\[ Q_Z W = -Q_W Z, \quad P_Z W = -P_W Z. \] (3.7)

Lemma 3.3. Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then
\[ g_M(\nabla_X Y, F) = -2g_M(CY, JA_X V) - g_M(CY, P_X V), \] (3.8)

for $X, Y \in \Gamma(\ker f_\ast) \perp$ and $V \in \Gamma(\ker f_\ast)$.

Proof. For $X, Y \in \Gamma(\ker f_\ast) \perp$ and $V \in \Gamma(\ker f_\ast)$, using equations (3.3), (2.4), (2.13) and (3.5), we have
\[ g_M(\nabla_X Y, F) = -g_M(CY, \nabla_X F), \]
\[ = -g_M(CY, J \nabla_X V) - g_M(CY, T_V BX) - g_M(CY, A_C X V), \]

From Lemmas 3.2 and 3.3, we have the following result.

Lemma 3.4. Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(N, g_N)$. Then
\[ g_M(\nabla_X Y, F) = g_M(CY, J A_X V) - g_M(CY, J \nabla_X V) - g_M(CY, T_V BX) - g_M(CY, A_C X V), \] (3.9)

for $X, Y \in \Gamma(\ker f_\ast) \perp$ and $V \in \Gamma(\ker f_\ast)$. 

Theorem 3.5. Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(N, g_N)$. Then the following assertions are equivalent to each other

(i) $(\ker f_\ast)^\perp$ is integrable,

(ii) $\frac{1}{\lambda N}(\nabla_X^f CX - \nabla_X^f CY, f_\ast JV) = g_M(A_X BY - A_Y BX, J V) - g_M(\nabla_X J X, CY) = g_M(H grad ln \lambda, CY)g_M(X, JV) + g_M(H grad ln \lambda, CX)g_M(Y, JV) - 2g_M(H grad ln \lambda, CY)g_M(X, JV) - 2g_M(H grad ln \lambda, CY)g_M(Y, JV)$

for $X, Y \in \Gamma(\ker f_\ast)^\perp$ and $V \in \Gamma(\ker f_\ast)$. 

Proof. For $X, Y \in \Gamma(\ker f_\ast)^\perp$ and $V \in \Gamma(\ker f_\ast)$, since $JV \in \Gamma(\ker f_\ast)^\perp$ and $JV \in \Gamma J(\ker f_\ast) \oplus \mu$. Using equations (2.1) and (2.4), we get

$g_M([X, Y], V) = g_M(\nabla_X J Y, JV) - g_M(\nabla_Y J X, JV) - 2g_M((\nabla_X J)Y, JV)$.

Again, using equations (3.2), (2.13), (2.14) and (3.5), we get

$g_M([X, Y], V) = g_M(A_X BY - A_Y BX, J V) + g_M(\nabla_X CY, JV) - g_M(\nabla_Y CX, JV) - 2g_M(P_X Y, JV)$.

Since $f$ is conformal submersion, using Lemma 2.3(i) and equation (3.3) we have

$g_M([X, Y], V) = g_M(A_X BY - A_Y BX, J V) - \frac{1}{\lambda N} g_M(\nabla_X^f CX - \nabla_X^f CY, f_\ast JV) - g_M(H grad ln \lambda, CY)g_M(X, JV) + g_M(H grad ln \lambda, CX)g_M(Y, JV) - 2g_M(H grad ln \lambda, CY)g_M(X, JV) - 2g_M(P_X Y, JV)$

which proves $(i) \Leftrightarrow (ii)$. \qed

Theorem 3.6. Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then any two conditions below given imply the third:

(i) $(\ker f_\ast)^\perp$ is integrable,

(ii) $f$ is horizontally homothetic,

(iii) $\frac{1}{\lambda N}(\nabla_X^f CX - \nabla_X^f CY, f_\ast JV) = g_M(A_X BY - A_Y BX, J V) - 2g_M(P_X Y, JV)$, for $X, Y \in \Gamma(\ker f_\ast)^\perp$ and $V \in \Gamma(\ker f_\ast)$.

Proof. For $X, Y \in \Gamma(\ker f_\ast)^\perp$ and $V \in \Gamma(\ker f_\ast)$, from Theorem (3.5), we have

$g_M([X, Y], V) = g_M(A_X BY - A_Y BX, J V) - \frac{1}{\lambda N} g_M(\nabla_X^f CX - \nabla_X^f CY, f_\ast JV) - g_M(H grad ln \lambda, CY)g_M(X, JV) + g_M(H grad ln \lambda, CX)g_M(Y, JV) - 2g_M(H grad ln \lambda, CY)g_M(X, JV) - 2g_M(P_X Y, JV)$.

Now, if using (i) and (ii), we have

$\frac{1}{\lambda N} g_M(\nabla_X^f CX - \nabla_X^f CY, f_\ast JV) = g_M(A_X BY - A_Y BX, J V) - 2g_M(P_X Y, JV)$.

Similarly, one can obtain the other assertions. \qed

We say that a conformal anti-invariant submersion is a conformal Lagrangian submersion if $J(\Gamma ker f_\ast) = \Gamma(\ker f_\ast)^\perp$.

Corollary 3.7. Let $f$ be a conformal Lagrangian submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the following assertions are equivalent to each other:
(i) $(\ker f_*)^\perp$ is integrable.
(ii) $A_X JY = A_Y JX + 2P_X Y$,
(iii) $(\nabla f_*)(Y, JX) = (\nabla f_*)(X, JY) + 2P_X Y$, for $X, Y \in \Gamma(\ker f_*)^\perp$.

**Proof.** For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, since $JV \in \Gamma(\ker f_*)^\perp$ and $JY \in \Gamma(\ker f_*)$, from Theorem 3.8, we have

$$g_M([X, Y], V) = g_M(A_X BY - A_Y BX, JV) - \frac{1}{X}g_N(\nabla_X f_* CX - \nabla_X f_* CY, f_* JV) - g_M(\mathcal{H}_{\text{gradln}}\lambda, CY)g_M(X, JV) + g_M(\mathcal{H}_{\text{gradln}}\lambda, CX)g_M(Y, JV) - 2g_M(\mathcal{H}_{\text{gradln}}\lambda, JV)g_M(CX, Y) - 2g_M(P_X Y, JV).$$

Since $f$ is a conformal Lagrangian submersion, we have

$$g_M([X, Y], V) = g_M(A_X BY - A_Y BX, JV) - 2g_M(P_X Y, JV),$$

which proves (i) $\Leftrightarrow$ (ii). On the other hand, we have

$$g_M(A_X BY - A_Y BX, JV) - 2g_M(P_X Y, JV) = \frac{1}{X}g_N(f_* A_X BY - f_* A_Y BX, f_* JV) - 2g_M(P_X Y, JV).$$

Now, using equations (2.13) and (2.17), we have

$$g_M(A_X BY - A_Y BX, JV) - 2g_M(P_X Y, JV) = \frac{1}{X}g_N((\nabla f_*)(Y, BX) - (\nabla f_*)(X, BY), f_* JV) - 2g_M(P_X Y, JV),$$

which proves (ii) $\Leftrightarrow$ (iii). □

**Theorem 3.8.** Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the following assertions are equivalent to each other:

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on $M$,
(ii) $-\frac{1}{X}g_N(\nabla_X f_* CY, f_* JV) = g_M(A_X BY, JV) + g_M(\mathcal{H}_{\text{gradln}}\lambda, CY)g_M(X, JV) - g_M(\mathcal{H}_{\text{gradln}}\lambda, JV)g_M(X, CY) - g_M(P_X Y, JV)$, for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

**Proof.** For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$. Using equations (2.1), (3.2) and (3.5), we get

$$g_M(\nabla_X Y, V) = g_M(\nabla_X BY, JV) + g_M(\nabla_X CY, JV) - g_M(P_X Y, JV).$$

Since $f$ is conformal anti-invariant submersion, using equations (2.17), (3.3) and Lemma 2.3(i), we get

$$g_M(\nabla_X Y, V) = g_M(A_X BY, JV) - g_M(\mathcal{H}_{\text{gradln}}\lambda, CY)g_M(X, JV) + g_M(\mathcal{H}_{\text{gradln}}\lambda, JV)g_M(X, CY) + \frac{1}{X}g_N(\nabla_X f_* CY, f_* JV) - g_M(P_X Y, JV),$$

which proves (i) $\Leftrightarrow$ (ii). □

**Theorem 3.9.** Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then any two conditions below imply the third:

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on $M$,
(ii) $f$ is horizontally homothetic,
(iii) $\frac{1}{X}g_N(\nabla_X f_* CY, f_* JV) - g_M(P_X Y, JV) = g_M(A_X BY, JV)$, for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$. 


Proof. For $X, Y \in \Gamma(\ker f_\ast)^\perp$ and $V \in \Gamma(\ker f_\ast)$, from above Theorem (3.8), we get
\[
g_M(\nabla_X Y, V) = g_M(A_X BY, JV) - g_M(\mathcal{H}_{\text{gradln}}\lambda, CY)g_M(\nabla_X, JV) + g_M(\mathcal{H}_{\text{gradln}}\lambda, JV)g_M(X, CY) + \frac{1}{X^2}g_N(\nabla^f_X f_{\ast} CY, f_{\ast} JV) - g_M(P_X Y, JV).
\]
Now, if we using (i) and (ii), then we obtain:
\[
\frac{1}{X^2}g_N(\nabla^f_X f_{\ast} CY, f_{\ast} JV) - g_M(P_X Y, JV) = g_M(A_X BY, JV).
\]
Similarly, one can obtain the other assertions. 

Corollary 3.10. Let $f$ be a conformal Lagrangian submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the following assertions are equivalent to each other:

(i) $(\ker f_\ast)^\perp$ defines a totally geodesic foliation on $M$,
(ii) $A_X JY = P_X Y$,
(iii) $(\nabla f_\ast)(X, JY) = -f_\ast(P_X Y)$, for $X, Y \in \Gamma(\ker f_\ast)^\perp$.

Proof. For $X, Y \in \Gamma(\ker f_\ast)^\perp$ and $V \in \Gamma(\ker f_\ast)$, since $JV \in \Gamma(\ker f_\ast)^\perp$ and $JY \in \Gamma J(\ker f_\ast)$. From Theorem (3.8), we have
\[
g_M(\nabla_X Y, V) = g_M(A_X BY, JV) - g_M(\mathcal{H}_{\text{gradln}}\lambda, CY)g_M(\nabla_X, JV) + g_M(\mathcal{H}_{\text{gradln}}\lambda, JV)g_M(X, CY) + \frac{1}{X^2}g_N(\nabla^f_X f_{\ast} CY, f_{\ast} JV) - g_M(P_X Y, JV),
\]
which shows (i) $\iff$ (ii). On the other hand, since $f$ is conformal submersion and using equations (2.13) and (2.17), we have
\[
(\nabla f_\ast)(X, BY) = -f_\ast(P_X Y),
\]
which tells that (ii) $\iff$ (iii). 

Theorem 3.11. Let $f$ be a conformal anti-invariant submersion from a nearly Kähler manifold $(M, g_M, J)$ onto a Riemannian manifold $(N, g_N)$. Then the following assertions to each other:

(i) $(\ker f_\ast)$ defines a totally geodesic foliation on $M$,
(ii) $\frac{1}{X^2}g_N(\nabla f_{\ast}, JW f_{\ast}, JV, f_{\ast} CX) = g_M(\nabla_{\nabla_J W} f_{\ast}, BX) + g_M(\mathcal{H}_{\text{gradln}}\lambda, J\mathcal{C} X)g_M(JW, JV) + g_M(P_{\mathcal{V} W}, J\mathcal{C} X) + g_M(\mathcal{V}_{\nabla_J W}, J\mathcal{C} X) - g_M(P_{\mathcal{V} W}, CX) - g_M(Q_{\mathcal{V} W}, BX)$, for $V, W \in \Gamma(\ker f_\ast)$ and $X \in \Gamma(\ker f_\ast)^\perp$.

Proof. For $V, W \in \Gamma(\ker f_\ast)$ and $X \in \Gamma(\ker f_\ast)^\perp$, using equations (2.1) and (2.12), we have
\[
g_M(\nabla_V W, X) = g_M(\nabla_{\nabla_J W} f_{\ast}, BX) + g_M(\mathcal{H}_{\nabla_J W} f_{\ast}, CX) - g_M((\nabla_{\nabla_J W}) f_{\ast}, JW, JX),
\]
\[
= g_M(\nabla_{\nabla_J W} f_{\ast}, BX) + g_M(\mathcal{H}_{\nabla_J W} f_{\ast}, CX) - g_M(\nabla_{\nabla_J W} f_{\ast}, JW, JX).
\]
Since $\nabla$ is torsion free and $[V, JW] \in \Gamma(\ker f_\ast)$, we obtain
\[
g_M(\nabla_V W, X) = g_M(\nabla_{\nabla_J W} f_{\ast}, BX) + g_M(\nabla_{\nabla_J W} f_{\ast}, CX) - g_M((\nabla_{\nabla_J W}) f_{\ast}, JW, JX),
\]
\[
- g_M(P_{\mathcal{V} W}, CX) - g_M(Q_{\mathcal{V} W}, BX).
\]
Hence we have used that \( \mu \) is invariant. Since \( f \) is a conformal submersion and using equation (2.17) and Lemma 2.3(i), we have

\[
g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + g_M(\mathcal{H}_{\text{gradln}} \lambda, JCX)g_M(JW, JV) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX).
\]

\[\square\]

**Theorem 3.12.** Let \( f \) be a conformal anti-invariant submersion from a nearly Kähler manifold \((M, g_M, J)\) onto a Riemannian manifold \((N, g_N)\). Then any two conditions below imply the three:

(i) \((\ker f_\ast)\) defines a totally geodesic foliation on \(M\),

(ii) \(\lambda\) is a constant on \(\nabla(\mu\)).

(iii) \(-\frac{1}{\lambda}g_N(\nabla_{f_\ast JW} f_\ast JV, f_\ast JCX)\)

\[= g_M(\nabla_V JW, BX) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX).\]

Proof. For \(V, W \in \Gamma(\ker f_\ast)\) and \(X \in \Gamma(\ker f_\ast)^\perp\), from Theorem (3.11) we have

\[
g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + g_M(\mathcal{H}_{\text{gradln}} \lambda, JCX)g_M(JW, JV) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX).
\]

Now, if we have (i) and (ii), then we obtain

\[-\frac{1}{\lambda}g_N(\nabla_{f_\ast JW} f_\ast JV, f_\ast JCX)\]

\[= g_N(\nabla_V JW, BX) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX).\]

Similarly, one can obtain the other assertions. \(\square\)

**Corollary 3.13.** Let \( f \) be a conformal Lagrangian submersion from a nearly Kähler manifold \((M, g_M, J)\) onto a Riemannian manifold \((N, g_N)\). Then the following assertions are equivalent to each other:

(i) \((\ker f_\ast)\) defines a totally geodesic foliation on \(M\),

(ii) \(\nabla_{\nabla V W} = Q_V W\), for \(V, W \in \Gamma(\ker f_\ast)\).

Proof. For \(V, W \in \Gamma(\ker f_\ast)\) and \(X \in \Gamma(\ker f_\ast)^\perp\), from Theorem (3.11), we have

\[
g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + g_M(\mathcal{H}_{\text{gradln}} \lambda, JCX)g_M(JW, JV) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX).
\]

Since \( f \) is conformal Lagrangian submersion, we get

\[g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) - g_M(Q_V W, BX),\]

which shows \((i) \Leftrightarrow (ii)\). \(\square\)

Now we obtain necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recall that a differentiable map \( f \) between Riemannian manifolds is called totally geodesic if

\[(\nabla f_\ast)(X, Y) = 0, \text{ for all } X, Y \in \Gamma(TM).\]

A geometric interpretation of totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.
Theorem 3.14. Let \( f \) be a conformal anti-invariant submersion from a nearly Kähler manifold \((M, g_M, J)\) onto a Riemannian manifold \((N, g_N)\). Then \( f \) is a totally geodesic map if and only if
\[
-\nabla^f_X f = f_* (J(A_X Y_1 + \nabla_X Y_2 + A_X CY_2 - Q_X Y) + C(\nabla_X Y_1 + A_X Y_2 + \nabla_X CY_2 - P_X Y)),
\]
for any \( X \in \Gamma(\ker f_*) \) and \( Y \in \Gamma(TM) \) where \( Y_1 \in \Gamma(\ker f_*) \) and \( Y_2 \in \Gamma(\ker f_*)^\perp \).

Proof. Using equations (2.17), (2.2) and (2.4), we have
\[
(\nabla f_*)(X, Y) = \nabla^f_X f_* Y + f_* (J\nabla_X Y - J(\nabla_X Y)),
\]
for any \( X \in \Gamma(\ker f_*) \) and \( Y \in \Gamma(TM) \). Then from equations (2.12), (2.13) and (3.2), we get
\[
(\nabla f_*)(X, Y) = \nabla^f_X f_* Y + f_* (J(A_X Y_1 + B\nabla_X Y_1 + C\nabla_X Y_1 + BAX Y_1 + CAX Y_2 + J\nabla_X Y_2 + JAX Y_2
+ B\nabla_X CY_2 + C\nabla_X CY_2 - JQ_X Y - JQ_X Y - JQ_X Y),
\]
for any \( Y_1 \in \Gamma(\ker f_*) \) and \( Y_2 \in \Gamma(\ker f_*)^\perp \). Thus taking into account the vertical parts, we find
\[
(\nabla f_*)(X, Y) = \nabla^f_X f_* Y + f_* (J(A_X Y_1 + \nabla_X Y_2 + A_X CY_2 - Q_X Y) + C(\nabla_X Y_1
+ C(BAX Y_2 + C\nabla_X CY_2 - JQ_X Y)).
\]
Thus \((\nabla f_*)(X, Y) = 0\) if and only if the equation (3.10) is satisfied.

Definition 3.15. Let \( f \) be conformal anti-invariant submersion from a nearly Kähler manifold \((M, g_M, J)\) onto a Riemannian manifold \((N, g_N)\). Then \( f \) is called a \((J \ker f_*, \mu)\)–totally geodesic map if
\[
(\nabla f_*)(JU, X) = 0,
\]
for \( U \in \Gamma(\ker f_*) \) and \( X \in \Gamma(\ker f_*)^\perp \).

In the sequel we show that this notion has an important effect on the character of the conformal submersion.

Theorem 3.16. Let \( f \) be conformal anti-invariant submersion from a nearly Kähler manifold \((M, g_M, J)\) to a Riemannian manifold \((N, g_N)\). Then \( f \) is called a \((J \ker f_*, \mu)\)–totally geodesic map and only if \( f \) is horizontally homothetic map.

Proof. For \( U \in \Gamma(\ker f_*) \) and \( X \in \Gamma(\mu) \), from Lemma 2.3(i), we have
\[
(\nabla f_*)(JU, X) = JU(\ln \lambda) f_* X + X(\ln \lambda) f_* JU - g_M(JU, X) f_*(H \grad \ln \lambda).
\]
From above equation if \( f \) is a horizontally homothetic, then \((\nabla f_*)(JU, X) = 0\).

Conversely if \((\nabla f_*)(JU, X) = 0\), we get
\[
JU(\ln \lambda) f_* X + X(\ln \lambda) f_* JU = 0.
\]
(3.11)
Taking inner product in (3.11) with \( f_* JU \) and since \( f \) is a conformal submersion, we get
\[
g_M(H \grad \ln \lambda, X) g_M(JU, JU) = 0.
\]
Above equation implies that \( \lambda \) is a constant on \( \Gamma(\mu) \). On the other hand, taking inner product in (3.11) with \( f_* X \) and since \( f \) is a conformal submersion, we get
\[
g_M(H \grad \ln \lambda, X) g_M(X, X) = 0.
\]
Above equation implies that \( \lambda \) is a constant on \( \Gamma(\ker f_*)^\perp \).

Theorem 3.17. Let \( f \) be conformal anti-invariant submersion from a nearly Kähler manifold \((M, g_M, J)\) onto a Riemannian manifold \((N, g_N)\). Then \( f \) is a totally geodesic map if and only if
Using equations (2.17), (2.2) and (2.4), we have
\[(\nabla f_*)(U, V) = f_*(J(\nabla_U JV) - J(\nabla_U J)V).\]

Using equations (2.12), (3.2) and (3.5), we have
\[(\nabla f_*)(U, V) = f_*(JT_U JV + CH\nabla_U JV - JQ_U V - CP_V V).\]

From above equation \((\nabla f_*)(U, V) = 0\) if and only if
\[f_*(JT_U JV + CH\nabla_U JV - JQ_U V - CP_V V) = 0.\]
This implies \(T_U JV = Q_U V\) and \(H\nabla_U JV - P_V V \in \Gamma(J(\ker f_*).\)

On the other hand, from Lemma 2.3(i), we have
\[\nabla f_*(X, Y) = X(ln\lambda)f_*Y + Y(ln\lambda)f_*X - g_M(X, Y)f_*(H\text{gradln}\lambda),\]
for any \(X, Y \in \Gamma(\mu).\) It is obvious that \(f\) is horizontally homothetic if it follows that \((\nabla f_*)(X, Y) = 0.\) Conversely, if \((\nabla f_*)(X, Y) = 0,\) taking \(Y = JX\) in above equation, we get
\[X(ln\lambda)f_*JX + JX(ln\lambda)f_*X = 0.\] \((3.12)\)
Taking inner product in (3.12) with \(f_*JX,\) we get
\[g_M(H\text{gradln}\lambda, X)g_M(JX, JX) = 0.\] \((3.13)\)
From above equation \(\lambda\) is a constant on \(\Gamma(\mu).\) On the other hand, for \(U, V \in \Gamma(\ker f_*,\) from Lemma 2.3(i), we have
\[\nabla f_*(U, V) = JU(ln\lambda)f_*JV + JV(ln\lambda)f_*JU - g_M(JU, JV)f_*(H\text{gradln}\lambda).\]
Again if \(f\) is horizontally homothetic, then \((\nabla f_*)(U, V) = 0.\) Conversely, if \((\nabla f_*)(U, V) = 0,\) putting \(U\) instead of \(V\) in above equation, we derive
\[2JU(ln\lambda)f_*JU - g_M(JU, JU)f_*(H\text{gradln}\lambda) = 0.\]
Taking inner product in (3.12) with and since \(f\) is a conformal submersion, we have
\[g_M(JU, JU)g_M(H\text{gradln}\lambda, JU) = 0.\]
From above equation, \(\lambda\) is a constant on \(\Gamma(\ker f_*).\) Thus \(\lambda\) is a constant on \(\Gamma(\ker f_*)^\perp.\)

Now, for \(X \in \Gamma(\ker f_*)^\perp\) and \(V \in \Gamma J(\ker f_*),\) from equations (2.2), (2.4), and (2.17), we have
\[\nabla f_*(X, V) = f_*(J\nabla_V JX - J(\nabla X J)V).\]
Using equations (3.2), (2.11), (2.12) and (3.5), we have
\[\nabla f_*(X, V) = f_*(CT_V BX + J\nabla_V BX + CH\nabla_V CX + JT_V CX - CP_V X - JQ_V X).
\] Thus \((\nabla f_*)(X, V) = 0\) if and only if
\[f_*(CT_V BX + J\nabla_V BX + CH\nabla_V CX + JT_V CX - CP_V X - JQ_V X) = 0.\]
4 Total manifold as product manifold

In this section, we obtain some decomposition theorems for a conformal anti-invariant submersion from a nearly Kähler manifold \( (M, g_M, J) \) onto a Riemannian manifold \( (N, g_N) \).

**Definition 4.1.** [16] Let \( g_B \) be a Riemannian metric tensor on the manifold \( B = M \times N \) and assume that the canonical foliations \( D_M \) and \( D_N \) intersect perpendicularly everywhere. The \( g_B \) is a metric tensor of

(i) a twisted product if and only if \( D_M \) is totally geodesic foliation and \( D_N \) is totally umbilical foliation,

(ii) a usually product of Riemannian manifolds if and only if \( D_M \) and \( D_N \) are totally geodesic foliations,

(iii) a warped product if and only if \( D_M \) is totally geodesic foliation and \( D_N \) is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel.

We note in this case, from [3] we have

\[
\nabla_X U = X(\ln F) U,
\]

for \( X \in \Gamma(TM) \) and \( U \in \Gamma(TN) \), where \( \nabla \) is the Riemannian connection on \( M \times N \).

We have the following decomposition theorem for a conformal anti-invariant submersion which follows from Theorems (3.8) and (3.11) in term of the second fundamental form of such submersions.

**Theorem 4.2.** Let \( f \) be a conformal anti-invariant submersion from a nearly Kähler manifold \( (M, g_M, J) \) onto a Riemannian manifold \( (N, g_N) \). Then \( M \) is a locally product manifold if and only if

\[
-\frac{1}{X^2} g_N(\nabla^f_X f_* CY, f_* JV) = g_M(A_X BY, J V) + g_M(\nabla \ln \lambda, CY) g_M(X, J V) - g_M(\nabla \ln \lambda, J V) g_M(X, CY) - g_M(P_X Y, J V),
\]

and

\[
-\frac{1}{X^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) = g_M(\nabla_{f_* JW} BX, f_* JCX) + g_M(\nabla \ln \lambda, JCX) g_M(JW, J V) + g_M(P_J W, JCX) + g_M(Q_J W, JCX) - g_M(P_J W, CX) - g_M(Q_J W, BX),
\]

for \( X, Y \in \Gamma(\ker f_*) \) and \( V \in \Gamma(\ker f_*) \).

From Corollaries (3.10) and (3.13), we have the following theorem.

**Theorem 4.3.** Let \( f \) be a conformal Lagrangian submersion from a nearly Kähler manifold \( (M, g_M, J) \) onto a Riemannian manifold \( (N, g_N) \). Then \( M \) is a locally product manifold if and only if \( A_X J Y = P_X Y \) and \( \nabla_{f_* JW} BX = Q_J W \), for \( V, W \in \Gamma(\ker f_*) \) and \( X, Y \in \Gamma(\ker f_*) \).

Again we obtain a decomposition theorem which is the related to the notation of twisted product manifold.

**Theorem 4.4.** Let \( f \) be a conformal anti-invariant submersion from a nearly Kähler manifold \( (M, g_M, J) \) onto a Riemannian manifold \( (N, g_N) \). Then \( M \) is a locally twisted product manifold of the form \( M(\ker f_*) \times M(\ker f_*) \) if and only if

\[
-\frac{1}{X^2} g_N(\nabla^f_{JW} f_* JV, f_* JCX) = g_M(\nabla_{JW} BX, f_* JCX) + g_M(\nabla \ln \lambda, JCX) g_M(JW, J V) + g_M(P_J W, JCX) + g_M(Q_J W, JCX) - g_M(P_J W, CX) - g_M(Q_J W, BX), \tag{4.1}
\]
and
\[ g_M(X, Y)H = -BA_X BY + CY (ln \lambda) BX - H \text{gradln} \lambda g_M(X, CY) \] (4.2)
\[ -BP_X Y - Jf_s(\nabla^g_X f_s CY), \]
for \( V, W \in \Gamma(\ker f_s) \) and \( X, Y \in \Gamma(\ker f_s)^\perp \), where \( M_{(\ker f_s)} \) and \( M_{(\ker f_s)^\perp} \) are integral manifold of the distribution \( \Gamma(\ker f_s)^\perp \) and \( \Gamma(\ker f_s) \) and \( H \) is the mean curvature vector field of \( M_{(\ker f_s)^\perp} \).

**Proof.** For \( V, W \in \Gamma(\ker f_s) \) and \( X \in \Gamma(\ker f_s)^\perp \), using equations (2.1), (2.2), (2.4), (3.2), (2.12) and (3.5), we have
\[ g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + g_M(\text{H} \nabla_V JW, CX) - g_M(P_V W, CX) - g_M(Q_V W, BX), \]
\[ g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + g_M([V, JW] + \nabla JW V, CX) - g_M(P_V W, CX) - g_M(Q_V W, BX). \]

Since \( \nabla \) is torsion free and \( [V, JW] \in \Gamma(\ker f_s) \), we obtain
\[ g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + g_M(\nabla JW V, CX) - g_M((\nabla JW) V, CX) - g_M(P_V W, CX) - g_M(Q_V W, BX). \]

Since \( f \) is conformal submersion, using equations (2.1), (2.2), (2.4), (2.17) and (3.3) and Lemma 2.3(i), we have
\[ g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) + \frac{1}{\lambda} g_N(\nabla^g_{f_s} f_s J V, f_s JCX) + g_M(JW, JV) g_M(\text{H} \text{gradln} \lambda, JCX) \]
\[ g_M(\nabla_V W, X) = g_M(\nabla_V JW, BX) - g_M(P_V W, CX) - g_M(Q_V W, BX). \]

Thus it follows that \( M_{(\ker f_s)} \) is totally geodesic if and only if the equation (4.1) is satisfied. On the other hand, for \( V \in \Gamma(\ker f_s) \) and \( X, Y \in \Gamma(\ker f_s)^\perp \), using equations (2.1), (2.2), (2.4), (2.13), (2.14) and (3.5), we have
\[ g_M(\nabla_X Y, V) = g_M(A_X BY, JV) + g_M(\text{H} \nabla_X CY, JV) - g_M(P_X Y, JV). \]

Since \( f \) is conformal submersion, using equations (2.17) and (3.3) and Lemma 2.3(i), we have
\[ g_M(\nabla_X Y, V) = g_M(A_X BY, JV) + \frac{1}{\lambda} g_N(\nabla^f_X f_s CY, f_s JV) - g_M(X, JV) g_M(\text{H} \text{gradln} \lambda, CY) \]
\[ + g_M(X, CY) g_M(\text{H} \text{gradln} \lambda, JV) - g_M(P_X Y, JV), \]
using that conclude that \( M_{(\ker f_s)^\perp} \) is totally umbilical if and only if the equation (4.2) is satisfied. \( \square \)

In a similar way with Theorem (5.4) ([1]), we obtain following theorem:

**Theorem 4.5.** Let \( f \) be a conformal anti-invariant submersion from nearly Kähler manifold \( (M, g_M, J) \) to a Riemannian manifold \( (N, g_N) \) with \( \text{rank}(\ker f_s) > 1 \). If \( M \) is a locally warped product manifold of the form \( M_{(\ker f_s)} \times M_{(\ker f_s)^\perp} \), with either \( f \) is horizontally homothetic submersion or the fibres are one dimensional.
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