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Conformal anti-invariant submersions from nearly Kähler Manifolds

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Abstract. Akyol and Sahin recently [1], defined conformal anti-invariant submersions from almost Hermitian manifolds and studied the case when ambient manifold is Kähler. In this paper, we discuss conformal anti-invariant submersions from a nearly Kähler manifold onto a Riemannian manifold and derive some results in this respect. We extend the notion of conformal anti-invariant and conformal Lagrangian Riemannian submersion to the case when ambient manifold is nearly Kähler. We also give necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic. Further, we find some decomposition theorems for the total manifold of the submersion and some equivalence conditions.

1 Introduction

The study of Riemannian submersion between Riemannian manifolds was initiated by O'Neill [15] and then Gray [10]. Later such submersions were considered between differentiable manifolds. In 1976, Watson [19] introduced almost Hermitian submersions between almost Hermitian manifolds and showed that in most cases the base manifold and each fibre have the same kind of structure as the total space. We note that almost Hermitian submersions have been extended to the almost contact metric submersions [5], locally conformal Kähler submersions [14], semi-Riemannian submersions and Lorentzian submersions [6], anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds [17]. As we know that Riemannian submersions are related with physics and have their applications in the Yang-Mills theory [18], Kaluza-Klein theory ([4], [11]), supergravity and superstring theories ([12], [13]) etc. Conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds were introduced by Akyol and Sahin [1] and they mainly study conformal anti-invariant submersions from Kähler manifolds onto Riemannian manifolds.

In this paper, we study conformal anti-invariant submersions from nearly Kähler manifolds onto Riemannian manifolds. The paper is organised as follows. The first section is introductory. In the second section, we collect main notions and formulae for other sections.

In section 3, the conformal anti-invariant submersions from nearly Kähler manifolds onto Riemannian manifolds have been studied and the geometry of leaves has been investigated. We also give necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic. Further, in this section we have obtained some equivalence conditions. In section 4, we study certain product structures on the total space of a conformal anti-invariant submersions.

2 Preliminaries

Let M be an even-dimensional differentiable manifold. Let J be a $(1,1)$ tensor field on M such that $J^2 = -I$. Then J is called almost complex structure on M. The manifold M with almost complex structure J is called almost complex manifold. It is well known that almost complex manifold is necessarily orientable. Nijenhuis tensor N of an almost complex structure is defined as:

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY], \quad \text{for all } X,Y \in \Gamma TM.$$

If Nijenhuis tensor field N on an almost complex manifold M is zero, then almost complex manifold M is called complex manifold.

Let g_M be a Riemannian metric on M such that

$$g_M(JX, JY) = g_M(X, Y), \quad \text{for all } X, Y \in \Gamma TM. \tag{2.1}$$

Then g_M is called an almost Hermitian metric on M and manifold M with Hermitian metric g_M is called almost Hermitian manifold. The Riemannian connection ∇ of the almost Hermitian manifold M can be extend to the whole tensor algebra on M, and in this way we obtain tensor fields like $(\nabla_X J)Y$ and that

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y, \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$.

An almost Hermitian manifold (M, g_M, J) is called Kähler manifold if

$$(\nabla_X J)Y = 0, \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$.

An also almost Hermitian manifold (M, g_M, J) is called nearly Kähler manifold [9] if

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$.

Suppose M and N are differentiable manifolds with Riemannian metrics g_M and g_N respectively. Let $\dim M = m$ and $\dim N = n$, where $m > n$. Let $f : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds. We denote the kernel space of f_{*p} by $\ker f_{*p}$ and consider the orthogonal complementary space $(\ker f_{*p})^\perp$ to $\ker f_{*p}$ in T_pM. Then the tangent bundle of M has the following decomposition

$$T_pM = (\ker f_{*p}) \oplus (\ker f_{*p})^\perp. \tag{2.5}$$

We also denote the $range$ of $f_{*f(p)}$ by $rangef_{*f(p)}$ and consider the orthogonal complementary space $(rangef_{*f(p)})^\perp$ to $rangef_{*f(p)}$ in the tangent bundle $T_{f(p)}N$ of N. Thus the tangent bundle $T_{f(p)}N$ of N has the following decomposition

$$TN = (rangef_{*f(p)}) \oplus (rangef_{*f(p)})^\perp. \tag{2.6}$$

As a generalization of Riemannian submersions, horizontally conformal submersions are defined as follows [2]. Let $f : (M, g_M) \to (N, g_N)$ be a differentiable map between Riemannian manifolds, then f is called a horizontally conformal submersion, if there is a positive function λ such that

$$g_M(X, Y) = \frac{1}{\lambda^2} g_N(f_*X, f_*Y), \tag{2.7}$$

for every $X, Y \in \Gamma(\ker f_{*p})^\perp$.

It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. Suppose that f is a differentiable map between Riemannian manifolds and point $p \in M$. Then, f is called horizontally weakly conformal map at a point p if either (i) $f_{*p} = 0$ or (ii) f_{*p} maps the horizontal space $(\ker f_*)^\perp$ conformally onto $T_{f(p)}N$, $i.e.$, f_{*p} is surjective and f_{*p} satisfies the equation (2.7) for X, Y vectors tangent to $(\ker f_*)^\perp$. If a point p is of type (i) then it is called critical point of f. A point p of type (ii) is called regular. The number $\wedge(p)$ is called the square dilation, it is necessarily non-negative. Its square root $\lambda(p) = \sqrt{\wedge(p)}$ is called the dilation. A horizontally weakly conformal map $f : M \to N$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e., $\mathcal{H}\ (grad\lambda) = 0$ at regular points. If a horizontally weakly conformal map f has no critical points, then it is called horizontally conformal submersion [2]. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. We note that horizontal conformal maps were introduced independently by Fuglede [7] and Ishihara [13].

Definition 2.1. ([2]) Let M and N are two Riemannian manifolds with Riemannian metrics g_M and g_N respectively. If f is a differentiable map from (M, g_M) to (N, g_N), then f is called horizontally weakly conformal or semi-conformal at p if either

(i) $df_p = 0$, or
(ii) df_p maps the horizontal space $\mathcal{H}_p = (\ker(df_p))^\perp$ conformally onto $T_{f(p)}N$ $i.e.,$ df_p is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_N(f_*U, f_*V) = \Lambda(p) g_M(U, V), \qquad \text{for } U, V \in \Gamma(\ker f_*)^\perp, \tag{2.8}$$

where $p \in M$.

Watson introduced the fundamental tensors of a submersion in [15]. It is known that the fundamental tensor play similar role to that of the second fundamental form of an immersion. Define tensors $\mathcal{T}$ and $\mathcal{A}$, for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.9}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.10}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [6], and ∇ is a Riemannian connection on M. On the other hand, from equations (2.9) and (2.10), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.11}$$

$$\nabla_X U = \mathcal{H}\nabla_X U + \mathcal{T}_X U, \tag{2.12}$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X, \tag{2.13}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{2.14}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$, where $\mathcal{V}\nabla_X Y = \widehat{\nabla}_X Y$. If U is basic, then $\mathcal{A}_X U = \mathcal{H}\nabla_X U$.

It is seen that for $p \in M$, $X \in \mathcal{V}_p$ and $U \in \mathcal{H}_p$ the linear operators

$$\mathcal{T}_X, \mathcal{A}_U : T_pM \to T_pM,$$

are skew-symmetric, that is

$$g_M(\mathcal{A}_U E, F) = -g_M(E, \mathcal{A}_U F) \text{ and } g_M(\mathcal{T}_X E, F) = -g_M(E, \mathcal{T}_X F)\ , \tag{2.15}$$

for each $E, F \in T_pM$. We have also defined the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}|_{\mathcal{V}\times\mathcal{V}}$ is precisely the second fundamental form of the fibres of f. Since $\mathcal{T}_V$ is skew-symmetric we get: f has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$. For the special case when f is horizontally conformal we have the following:

Proposition 2.2. ([10](2.1.2)) *Let $f : (M, g_M) \to (N, g_N)$ be a horizontal conformal submersion between Riemannian manifolds with dilation λ and U, V be horizontal vectors. Then*

$$\mathcal{A}_U V = \frac{1}{2}\{\mathcal{V}[U, V] - \lambda^2 g_M(U, V) grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.16}$$

We know that the skew-symmetric part of $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ measures the obstruction integrability of the horizontal distribution $\mathcal{H}$.

We also recall the concept of harmonic maps between Riemannian manifolds. Let $f : (M, g_M) \to (N, g_N)$ is a differentiable map between Riemannian manifolds. Then the differential of f_* of f can be observed a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the pullback bundle which has fibres $(f^{-1}TN)_p = T_{f(p)}N$ has a connection ∇ induced from the Riemannian connection ∇^M and the pullback connection. Then the second fundamental form of f is given by

$$(\nabla f_*)(U,V) = \nabla^f_U f_*(V) - f_*(\nabla^M_U V). \tag{2.17}$$

Again, we find necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recollection that a differentiable map f between Riemannian manifolds is called totally geodesic if

$$(\nabla f_*)(V,W) = 0, \text{ for all } V,W \in \Gamma(TM). \tag{2.18}$$

A geometric clarification of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Now, we explain a decomposition theorem related to the concept of twisted product manifold. However, we first define the adjoint map of a map. Let $f : (M, g_M) \to (N, g_N)$ be a map between Riemannian manifolds (M, g_M) and (N, g_N). Then the adjoint map ${}^*f_*$ of f_* is characterized $g_M(X,{}^* f_{*p}Y) = g_N(f_{*p}X, Y)$ by $X \in T_pM$, $Y \in T_{f(p)}N$ and $p \in M$. Considering f^h_* at each $p \in M$ as a linear transformation

$$f^h_{*p} : ((\ker f_*)^\perp_{(p)}, g_{M(p)((\ker f_*)^\perp_p)}) \to (range f_{*(q)}, g_{N(q)(range f_*)(q)}),$$

we will denote the adjoint $f^h_{*(p)}$ by ${}^*f^h_{*(p)}$. Let $f^h_{*(p)}$ be the adjoint of $f^h_{*(p)} : (T_pM, g_{M(p)}) \to (T_{(q)}N, g_{N(q)})$. The linear transformation $({}^*f_{*p})^h : (range f_{*(p)}) \to (\ker f_*)^\perp_{(p)}$ defined $({}^*f_{*(p)})^h Y ={}^* f^h_{*(p)}Y$, where $Y \in (ranrge f_{*(p)}), q = f(p)$, is an isomorphism and $(f^h_{*(p)})^{-1} = ({}^*f_{*p})^h ={}^* f^h_{*(p)}$.

Lastly, we recollection the subsequent lemma from [2].

Lemma 2.3. *Let (M, g_M) and (N, g_N) are two Riemannian manifolds. If $f : M \to N$ horizontally conformal submersion between Riemannian manifolds, then for any horizontal vector fields U, V and vertical vector fields X, Y we have*

$$\begin{aligned}
(i)\nabla df(U,V) &= U(ln\lambda)df(V) + V(ln\lambda)df(U) - g_M(U,V)df(\mathcal{H}gradln\lambda);\\
(ii)\nabla df(X,Y) &= -df(\mathcal{A}^{\mathcal{V}}_X Y);\\
(iii)\nabla df(U,X) &= -df(\nabla^M_U X) = -df((\mathcal{A}^{\mathcal{H}})^*_U X).
\end{aligned}$$

where $(\mathcal{A}^{\mathcal{H}})^*_X$ is the adjoint of $(\mathcal{A}^{\mathcal{H}}_X)$ characterized by

$$\langle(\mathcal{A}^{\mathcal{H}})^*_U E, F\rangle = \langle E, \mathcal{A}^{\mathcal{H}}_U F\rangle \qquad (for\ E, F \in \Gamma(TM)).$$

3 Conformal anti-invariant submersions

Definition 3.1. Let (M, g_M, J) be a nearly Kähler manifold and (N, g_N) be a Riemannian manifold. A horizontal conformal submersion $f : (M, g_M, J) \to (N, g_N)$ with dilation λ is called anti-invariant submersion if the distribution $\ker f_*$ is anti-invariant submersion with respect to J, i.e., $J(\ker f_*) \subseteq (\ker f_*)^\perp$.

Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). From above definition we have $J(\ker f_*)^\perp \cap (\ker f_*) \neq \{0\}$ and denote the complementary orthogonal distribution to $J(\ker f_*)$ in $(\ker f_*)^\perp$ by μ. Then

$$(\ker f_*)^\perp = J(\ker f_*) \oplus \mu. \tag{3.1}$$

It is clear to see that μ is invariant distribution of $(\ker f_*)^\perp$, under the complex structure J. Thus, for $X \in \Gamma(\ker f_*)^\perp$, we have

$$JX = BX + CX, \tag{3.2}$$

where $BX \in \Gamma(\ker f_*)$ and $CX \in \Gamma(\mu)$.

Further, since $f_*((\ker_*)^\perp) = TN$ and f is a Riemannian submersion, using equation (3.2) it can be shown that $\frac{1}{\lambda^2} g_N(f_*JV, f_*CX) = 0$, for any $X \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$ which implies that $TN = f_*(J(\ker f_*)) \oplus f_*(\mu)$. Now, we prove following.

Lemma 3.2. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then*

$$g_M(CY, JV) = 0, \tag{3.3}$$

and

$$g_M(\nabla_X CY, JV) = -2g_M(CY, J\mathcal{A}_X V) + g_M(CY, \mathcal{T}_V BX) + g_M(CY, \mathcal{A}_{CX} V), \tag{3.4}$$

for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, since $BY \in \Gamma(\ker f_*)$, $CX \in \Gamma(\mu)$ and $\phi V \in \Gamma(\ker f_*)^\perp$. Using equations (2.1) and (3.2), we have

$$g_M(CY, JV) = 0.$$

Again, using equations (2.4), (3.3), (2.12) and (2.13), we have

$$g_M(\nabla_X CY, JV) = -2g_M(CY, J\mathcal{A}_X V) + g_M(CY, \mathcal{T}_V BX) + g_M(CY, \mathcal{A}_{CX} V),$$

which completes the proof. □

Note: Whenever it is need we have assumed the horizontal vector field to be basic.
For any arbitrary tangent vector fields Z and W on M, we get

$$(\nabla_Z J)W = Q_Z W + P_Z W, \tag{3.5}$$

where $Q_Z W$ and $P_Z W$ denote the vertical and horizontal part of $(\nabla_Z J)W$, respectively.
For a Kähler manifold M, we have

$$P = Q = 0, \forall Z, W \in \Gamma(TM). \tag{3.6}$$

If M is a nearly Kähler manifold, then it can be easily seen that both Q and P are anti-symmetric in Z and W i.e.,

$$Q_Z W = -Q_W Z, \;\; P_Z W = -P_W Z. \tag{3.7}$$

Lemma 3.3. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then*

$$g_M(\nabla_X CY, JV) = -g_M(CY, J\mathcal{A}_X V) - g_M(CY, P_X V), \tag{3.8}$$

for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, using equations (3.3), (2.4), (2.13) and (3.5), we have

$$\begin{aligned} g_M(\nabla_X CY, JV) &= -g_M(CY, \nabla_X JV), \\ &= -g_M(CY, J\nabla_X V) - g_M(CY, P_X V). \end{aligned}$$

□

From Lemmas 3.2 and 3.3, we have the following result.

Lemma 3.4. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then*

$$\begin{aligned} & g_M(\nabla_X CY, JV) \\ = \;& g_M(CY, J\mathcal{A}_X V) - g_M(CY, J\nabla_X V) - g_M(CY, \mathcal{T}_V BX) - g_M(CY, \mathcal{A}_{CX} V), \end{aligned} \tag{3.9}$$

for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Theorem 3.5. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other*

(i) $(\ker f_*)^\perp$ is integrable,
(ii) $\frac{1}{\lambda^2} g_N(\nabla^f_Y f_*CX - \nabla^f_X f_*CY, f_*JV)$
$= g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - g_M(\mathcal{H}gradln\lambda, CY) g_M(X, JV)$
$+ g_M(\mathcal{H}gradln\lambda, CX) g_M(Y, JV) - 2g_M(\mathcal{H}gradln\lambda, JV) g_M(CX, Y) - 2g_M(P_X Y, JV)$,
for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, since $JV \in \Gamma(\ker f_*)^\perp$ and $JY \in \Gamma J(\ker f_*) \oplus \mu$. Using equations (2.1) and (2.4), we get

$$g_M([X, Y], V) = g_M(\nabla_X JY, JV) - g_M(\nabla_Y JX, JV) - 2g_M((\nabla_X J)Y, JV).$$

Again, using equations (3.2), (2.13), (2.14) and (3.5), we get

$$\begin{aligned} g_M([X, Y], V) &= g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) + g_M(\nabla_X CY, JV) \\ &\quad - g_M(\nabla_Y CX, JV) - 2g_M(P_X Y, JV). \end{aligned}$$

Since f is conformal submersion, using Lemma 2.3(i) and equation (3.3) we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - \frac{1}{\lambda^2} g_N(\nabla^f_Y f_*CX - \nabla^f_X f_*CY, f_*JV) \\ &\quad - g_M(\mathcal{H}gradln\lambda, CY) g_M(X, JV) + g_M(\mathcal{H}gradln\lambda, CX) g_M(Y, JV) \\ &\quad - 2g_M(\mathcal{H}gradln\lambda, JV) g_M(CX, Y) - 2g_M(P_X Y, JV), \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. □

Theorem 3.6. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then any two conditions below given imply the third:*

(i) $(\ker f_*)^\perp$ is integrable,
(ii) f is horizontally homothetic,
(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_Y f_*CX - \nabla^f_X f_*CY, f_*JV) = g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - 2g_M(P_X Y, JV)$,
for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, from Theorem (3.5), we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - \frac{1}{\lambda^2} g_N(\nabla^f_Y f_*CX - \nabla^f_X f_*CY, f_*JV) \\ &\quad - g_M(\mathcal{H}gradln\lambda, CY) g_M(X, JV) + g_M(\mathcal{H}gradln\lambda, CX) g_M(Y, JV) \\ &\quad - 2g_M(\mathcal{H}gradln\lambda, JV) g_M(CX, Y) - 2g_M(P_X Y, JV), \end{aligned}$$

Now, if using (i) and (ii), we have

$$\frac{1}{\lambda^2} g_N(\nabla^f_Y f_*CX - \nabla^f_X f_*CY, f_*JV) = g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - 2g_M(P_X Y, JV).$$

Similarly, one can obtain the other assertions. □

We say that a conformal anti-invariant submersion is a conformal Lagrangian submersion if $J(\Gamma \ker f_*) = \Gamma(\ker f_*)^\perp$.

Corollary 3.7. *Let f be a conformal Lagrangian submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^\perp$ is integrable,
(ii) $\mathcal{A}_X JY = \mathcal{A}_Y JX + 2P_X Y$,
(iii) $(\nabla f_*)(Y, JX) = (\nabla f_*)(X, JY) + 2P_X Y$, for $X, Y \in \Gamma(\ker f_*)^\perp$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$. Since $JV \in \Gamma(\ker f_*)^\perp$ and $JY \in \Gamma(\ker f_*)$, from Theorem (3.5), we have

$$\begin{aligned} g_M([X,Y],V) &= g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - \frac{1}{\lambda^2} g_N(\nabla^f_Y f_* CX - \nabla^f_X f_* CY, f_* JV) \\ &\quad -g_M(\mathcal{H} gradln\lambda, CY) g_M(X, JV) + g_M(\mathcal{H} gradln\lambda, CX) g_M(Y, JV) \\ &\quad -2g_M(\mathcal{H} gradln\lambda, JV) g_M(CX, Y) - 2g_M(P_X Y, JV). \end{aligned}$$

Since f is a conformal Lagrangian submersion, we have

$$g_M([X,Y],V) = g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - 2g_M(P_X Y, JV),$$

which proves $(i) \Leftrightarrow (ii)$. On the other hand, we have

$$\begin{aligned} & g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - 2g_M(P_X Y, JV) \\ = \; & \frac{1}{\lambda^2} g_N(f_* \mathcal{A}_X BY - f_* \mathcal{A}_Y BX, f_* JV) - 2g_M(P_X Y, JV). \end{aligned}$$

Now, using equations (2.13) and (2.17), we have

$$\begin{aligned} & g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, JV) - 2g_M(P_X Y, JV) \\ = \; & \frac{1}{\lambda^2} g_N((\nabla f_*)(Y, BX) - (\nabla f_*)(X, BY), f_* JV) - 2g_M(P_X Y, JV), \end{aligned}$$

which proves $(ii) \Leftrightarrow (iii)$. □

Theorem 3.8. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $-\frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) = g_M(\mathcal{A}_X BY, JV) + g_M(\mathcal{H} gradln\lambda, CY) g_M(X, JV)$ $-g_M(\mathcal{H} gradln\lambda, JV) g_M(X, CY) - g_M(P_X Y, JV)$, for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$. Using equations (2.1), (3.2) and (3.5), we get

$$g_M(\nabla_X Y, V) = g_M(\nabla_X BY, JV) + g_M(\nabla_X CY, JV) - g_M(P_X Y, JV).$$

Since f is conformal anti-invariant submersion, using equations (2.17), (3.3) and Lemma 2.3(i), we get

$$\begin{aligned} & g_M(\nabla_X Y, V) \\ = \; & g_M(\mathcal{A}_X BY, JV) - g_M(\mathcal{H} gradln\lambda, CY) g_M(X, JV) \\ & + g_M(\mathcal{H} gradln\lambda, JV) g_M(X, CY) + \frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) - g_M(P_X Y, JV), \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. □

Theorem 3.9. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then any two conditions below imply the third:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) f is horizontally homothetic,
(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) - g_M(P_X Y, JV) = g_M(\mathcal{A}_X BY, JV)$, for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, from above Theorem (3.8), we get

$$\begin{aligned} & g_M(\nabla_X Y, V) \\ = \; & g_M(\mathcal{A}_X BY, JV) - g_M(\mathcal{H} gradln\lambda, CY) g_M(X, JV) \\ & + g_M(\mathcal{H} gradln\lambda, JV) g_M(X, CY) + \frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) - g_M(P_X Y, JV). \end{aligned}$$

Now, if we using (i) and (ii), then we obtain:

$$\frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) - g_M(P_X Y, JV) = g_M(\mathcal{A}_X BY, JV).$$

Similarly, one can obtain the other assertions. □

Corollary 3.10. *Let f be a conformal Lagrangian submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $\mathcal{A}_X JY = P_X Y$,
(iii) $(\nabla f_*)(X, JY) = -f_*(P_X Y)$, for $X, Y \in \Gamma(\ker f_*)^\perp$.

Proof. For $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$, since $JV \in \Gamma(\ker f_*)^\perp$ and $JY \in \Gamma J(\ker f_*)$. From Theorem (3.8), we have

$$\begin{aligned} & g_M(\nabla_X Y, V) \\ = \; & g_M(\mathcal{A}_X BY, JV) - g_M(\mathcal{H} gradln\lambda, CY) g_M(X, JV) \\ & + g_M(\mathcal{H} gradln\lambda, JV) g_M(X, CY) + \frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) - g_M(P_X Y, JV), \end{aligned}$$

Since f is conformal Lagrangian submersion, we have

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY, JV) - g_M(P_X Y, JV)$$

which shows $(i) \Leftrightarrow (ii)$. On the other hand, since f is conformal submersion and using equations (2.13) and (2.17), we have

$$(\nabla f_*)(X, BY) = -f_*(P_X Y),$$

which tells that $(ii) \Leftrightarrow (iii)$. □

Theorem 3.11. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the following assertions to each other:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,
$(ii) -\frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) = g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H} gradln\lambda, JCX) g_M(JW, JV)$ $+ g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX)$, for $V, W \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$.

Proof. For $V, W \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, using equations (2.1) and (2.12), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H} \nabla_V JW, CX) - g_M((\nabla_V J) W, JX), \\ &= g_M(\mathcal{T}_V JW, BX) + g_M([V, JW] + \nabla_{JW} V, CX) - g_M((\nabla_V J) W, JX). \end{aligned}$$

Since ∇ is torsion free and $[V, JW] \in \Gamma(\ker f_*)$, we obtain

$$g_M(\nabla_V W, X) = g_M(\mathcal{T}_V JW, BX) + g_M(\nabla_{JW} V, CX) - g_M((\nabla_V J) W, JX),$$

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V JW, BX) + g_M(\nabla_{JW} JV, JCX) - g_M((\nabla_{JW} J) V, JCX) \\ & \quad - g_M(P_V W, CX) - g_M(Q_V W, BX). \end{aligned}$$

Hence we have used that μ is invariant. Since f is a conformal submersion and using equation (2.17) and Lemma 2.3(i), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H} gradln\lambda, JCX) g_M(JW, JV) \\ &+ \frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) + g_M(P_V JW, JCX) \\ &+ g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX). \end{aligned}$$

□

Theorem 3.12. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then any two conditions below imply the three:*

(i) (ker f_*) defines a totally geodesic foliation on M,
(ii) λ is a constant on $\Gamma(\mu)$.
(iii) $-\frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX)$
$= g_M(\mathcal{T}_V JW, BX) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX)$
$-g_M(Q_V W, BX)$, for $V, W \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$.

Proof. For $V, W \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, from Theorem (3.11) we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H} gradln\lambda, JCX) g_M(JW, JV) \\ &+ \frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) + g_M(P_V JW, JCX) \\ &+ g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX). \end{aligned}$$

Now, if we have (i) and (ii), then we obtain

$$\begin{aligned} & -\frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) \\ = \; & g_M(\mathcal{T}_V JW, BX) + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) \\ & - g_M(P_V W, CX) - g_M(Q_V W, BX). \end{aligned}$$

Similarly, one can obtain the other assertions. □

Corollary 3.13. *Let f be a conformal Lagrangian submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

(i) (ker f_*) defines a totally geodesic foliation on M,
(ii) $\mathcal{T}_V JW = Q_V W$, for $V, W \in \Gamma(\ker f_*)$.

Proof. For $V, W \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, from Theorem (3.11), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H} gradln\lambda, JCX) g_M(JW, JV) \\ &+ \frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) + g_M(P_V JW, JCX) \\ &+ g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX). \end{aligned}$$

Since f is conformal Lagrangian submersion, we get

$$g_M(\nabla_V W, X) = g_M(\mathcal{T}_V JW, BX) - g_M(Q_V W, BX),$$

which shows $(i) \Leftrightarrow (ii)$. □

Now we obtain necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recall that a differentiable map f between Riemannian manifolds is called totally geodesic if

$$(\nabla f_*)(X, Y) = 0, \text{ for all } X, Y \in \Gamma(TM).$$

A geometric interpretation of totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 3.14. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then f is a totally geodesic map if and only if*

$$\begin{aligned} -\nabla^f_X f_*Y &= f_*(J(\mathcal{A}_X JY_1 + \mathcal{V}\nabla_X BY_2 + \mathcal{A}_X CY_2 - Q_X Y) \\ &\quad +C(\mathcal{H}\nabla_X JY_1 + \mathcal{A}_X BY_2 + \mathcal{H}\nabla_X CY_2 - P_X Y)), \end{aligned} \tag{3.10}$$

for any $X \in \Gamma(\ker f_*)$ and $Y \in \Gamma(TM)$ where $Y_1 \in \Gamma(\ker f_*)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$.

Proof. Using equations $(2.17), (2.2)$ and (2.4), we have

$$(\nabla f_*)(X,Y) = \nabla^f_X f_*Y + f_*(J\nabla_X JY - J(\nabla_X J)Y),$$

for any $X \in \Gamma(\ker f_*)$ and $Y \in \Gamma(TM)$. Then from equations $(2.12), (2.13)$ and (3.2), we get

$$\begin{aligned} (\nabla f_*)(X,Y) &= \nabla^f_X f_*Y + f_*(J\mathcal{A}_X JY_1 + B\mathcal{H}\nabla_X JY_1 + C\mathcal{H}\nabla_X JY_1 \\ &\quad +B\mathcal{A}_X BY_1 + C\mathcal{A}_X BY_2 + J\mathcal{V}\nabla_X BY_2 + J\mathcal{A}_X CY_2 \\ &\quad +B\mathcal{H}\nabla_X CY_2 + C\mathcal{H}\nabla_X CY_2 - JP_X Y - JQ_X Y), \end{aligned}$$

for any $Y_1 \in \Gamma(\ker f_*)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$. Thus taking into account the vertical parts, we find

$$\begin{aligned} (\nabla f_*)(X,Y) &= \nabla^f_X f_*Y + f_*(J(\mathcal{A}_X JY_1 + \mathcal{V}\nabla_X BY_2 + \mathcal{A}_X CY_2 - Q_X Y) + C(\mathcal{H}\nabla_X JY_1 \\ &\quad +C(\mathcal{A}_X BY_2 + C\mathcal{H}\nabla_X CY_2 - P_X Y)). \end{aligned}$$

Thus $(\nabla f_*)(X,Y) = 0$ if and only if the equation (3.10) is satisfied. □

Definition 3.15. Let f be conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then f is called a $(J\ker f_*, \mu)-$totally geodesic map if

$$(\nabla f_*)(JU, X) = 0,$$

for $U \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$.

In the squeal we show that this notion has an important effect on the character of the conformal submersion.

Theorem 3.16. *Let f be conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then f is called a $(J\ker f_*, \mu)-$totally geodesic map if and only if f is horizontally homothetic map.*

Proof. For $U \in \Gamma(\ker f_*)$ and for $X \in \Gamma(\mu)$, from Lemma 2.3(i), we have

$$(\nabla f_*)(JU, X) = JU(ln\lambda)f_*X + X(ln\lambda)f_*JU - g_M(JU, X)f_*(\mathcal{H}gradln\lambda).$$

From above equation if f is a horizontally homothetic, then $(\nabla f_*)(JU, X) = 0$.

Conversely if $(\nabla f_*)(JU, X) = 0$, we get

$$JU(ln\lambda)f_*X + X(ln\lambda)f_*JU = 0. \tag{3.11}$$

Taking inner product in (3.11) with f_*JU and since f is a conformal submersion, we get

$$g_M(\mathcal{H}gradln\lambda, X)g_M(JU, JU) = 0.$$

Above equation implies that λ is a constant on $\Gamma(\mu)$. On the other hand, taking inner product in (3.11) with f_*X and since f is a conformal submersion, we get

$$g_M(\mathcal{H}gradln\lambda, X)g_M(X, X) = 0.$$

Above equation implies that λ is a constant on $\Gamma J(\ker f_*)$. Then λ is a constant on $\Gamma(\ker f_*)^\perp$. □

Theorem 3.17. *Let f be conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then f is a totally geodesic map if and only if*

(i) $\mathcal{T}_U JV = Q_U V$ and $\mathcal{H}\nabla_U JV - P_U V \in \Gamma J(\ker f_*)$,
(ii) f is horizontally homothetic,
(iii) $\widehat{\nabla}_V BX + \mathcal{T}_V CX - Q_V X = 0$ and $\mathcal{T}_V BX + \mathcal{H}\nabla_V CX - P_V X \in \Gamma J(\ker f_*)$,
for $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$.

Proof. For any $U, V \in \Gamma(\ker f_*)$, from using equations $(2.17), (2.2)$ and (2.4), we have

$$(\nabla f_*)(U, V) = f_*(J(\nabla_U JV) - J(\nabla_U J)V).$$

Using equations $(2.12), (3.2)$ and (3.5), we have

$$(\nabla f_*)(U, V) = f_*(J\mathcal{T}_U JV + C\mathcal{H}\nabla_U JV) - JQ_U V - CP_U V).$$

From above equation $(\nabla f_*)(U, V) = 0$ if and only if

$$f_*(J\mathcal{T}_U JV + C\mathcal{H}\nabla_U JV - JQ_U V - CP_U V) = 0.$$

This implies $T_U JV = Q_U V$ and $\mathcal{H}\nabla_U JV - P_U V \in \Gamma J(\ker f_*)$. On the other hand, from Lemma 2.3(i), we have

$$(\nabla f_*)(X, Y) = X(ln\lambda) f_* Y + Y(ln\lambda) f_* X - g_M(X, Y) f_*(\mathcal{H} gradln\lambda),$$

for any $X, Y \in \Gamma(\mu)$. It is obvious that f is horizontally homothetic it follows that $(\nabla f_*)(X, Y) = 0$. Conversely, if $(\nabla f_*)(X, Y) = 0$, taking $Y = JX$ in above equation, we get

$$X(ln\lambda) f_* JX + JX(ln\lambda) f_* X = 0. \tag{3.12}$$

Taking inner product in (3.12) with $f_* JX$, we get

$$g_M(\mathcal{H} gradln\lambda, X) g_M(JX, JX) = 0. \tag{3.13}$$

From above equation λ is a constant on $\Gamma(\mu)$. On the other hand, for $U, V \in \Gamma(\ker f_*)$, from Lemma 2.3$(i)$, we have

$$(\nabla f_*)(U, V) = JU(ln\lambda) f_* JV + JV(ln\lambda) f_* JU - g_M(JU, JV) f_*(\mathcal{H} gradln\lambda).$$

Again if f is horizontally homothetic, then $(\nabla f_*)(U, V) = 0$. Conversely, if $(\nabla f_*)(U, V) = 0$, putting U instead of V in above equation, we derive

$$2JU(ln\lambda) f_* JU - g_M(JU, JU) f_*(\mathcal{H} gradln\lambda) = 0.$$

Taking inner product in (3.12) with and since f is a conformal submersion, we have

$$g_M(JU, JU) g_M(\mathcal{H} gradln\lambda, JU) = 0.$$

From above equation, λ is a constant on $\Gamma J(\ker f_*)$. Thus λ is a constant on $\Gamma(\ker f_*)^\perp$.

Now, for $X \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma J(\ker f_*)$, from equations $(2.2), (2.4)$, and (2.17), we have

$$(\nabla f_*)(X, V) = f_*(J\nabla_V JX - J(\nabla_X J)V).$$

Using equations $(3.2), (2.11), (2.12)$ and (3.5), we have

$$(\nabla f_*)(X, V) = f_*(C\mathcal{T}_V BX + J\widehat{\nabla}_V BX + C\mathcal{H}\nabla_V CX + J\mathcal{T}_V CX - CP_V X - JQ_V X).$$

Thus $(\nabla f_*)(X, V) = 0$ if and only if

$$f_*(C\mathcal{T}_V BX + J\widehat{\nabla}_V BX + C\mathcal{H}\nabla_V CX + J\mathcal{T}_V CX - CP_V X - JQ_V X) = 0.$$

□

4 Total manifold as product manifold

In this section, we obtain some decomposition theorems for a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N).

Definition 4.1. [16] Let g_B be a Riemannian metric tensor on the manifold $B = M \times N$ and assume that the canonical foliations D_M and D_N intersect perpendicularly everywhere. The g_B is a metric tensor of

(i) a twisted product if and only if D_M is totally geodesic foliation and D_N is totally umbilical foliation,

(ii) a usually product of Riemannian manifolds if and only if D_M and D_N are totally geodesic foliations,

(iii) a warped product if and only if D_M is totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel,

We note in this case, from [3] we have

$$\nabla_X U = X(\ln F)U,$$

for $X \in \Gamma(TM)$ and $U \in \Gamma(TN)$, where ∇ is the Riemannian connection on $M \times N$.

We have the following decomposition theorem for a conformal anti-invariant submersion which follows from Theorems (3.8) and (3.11) in term of the second fundamental form of such submersions.

Theorem 4.2. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then M is a locally product manifold if and only if*

$$\begin{aligned} -\frac{1}{\lambda^2} g_N(\nabla^f_X f_* CY, f_* JV) &= g_M(\mathcal{A}_X BY, JV) + g_M(\mathcal{H}gradln\lambda, CY) g_M(X, JV) \\ &\quad - g_M(\mathcal{H}gradln\lambda, JV) g_M(X, CY) - g_M(P_X Y, JV), \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{\lambda^2} g_N(\nabla_{f_* JW} f_* JV, f_* JCX) \\ = \; & g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H}gradln\lambda, JCX) g_M(JW, JV) \\ & + g_M(P_V JW, JCX) + g_M(Q_V JW, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX), \end{aligned}$$

for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$.

From Corollaries (3.10) and (3.13), we have the following theorem.

Theorem 4.3. *Let f be a conformal Lagrangian submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then M is a locally product manifold if and only if $\mathcal{A}_X JY = P_X Y$, and $\mathcal{T}_V JW = Q_V W$, for $V, W \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$.*

Again we obtain a decomposition theorem which is the related to the notation of twisted product manifold.

Theorem 4.4. *Let f be a conformal anti-invariant submersion from a nearly Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then M is a locally twisted product manifold of the from $M_{(\ker f_*)} \times M_{(\ker f_*)^\perp}$ if and only if*

$$\begin{aligned} & -\frac{1}{\lambda^2} g_N(\nabla^f_{JW} f_* JV, f_* JCX) \qquad (4.1) \\ = \; & g_M(\mathcal{T}_V JW, BX) + g_M(JW, JV) g_M(\mathcal{H}gradln\lambda, JCX) \\ & - g_M(P_{JW} V, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX), \end{aligned}$$

and

$$g_M(X,Y)H = -B\mathcal{A}_X BY + CY(ln\lambda)BX - \mathcal{H}gradln\lambda g_M(X,CY) \quad (4.2)$$
$$-BP_X Y - Jf_*(\nabla^f_X f_* CY),$$

for $V,W \in \Gamma(\ker f_*)$ *and* $X,Y \in \Gamma(\ker f_*)^\perp$, *where* $M_{(\ker f_*)^\perp}$ *and* $M_{(\ker f_*)}$ *are integral manifold of the distribution* $\Gamma(\ker f_*)^\perp$ *and* $\Gamma(\ker f_*)$ *and* H *is the mean curvature vector field of* $M_{(\ker f_*)^\perp}$.

Proof. For $V,W \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, using equations $(2.1), (2.2), (2.4), (3.2), (2.12)$ and (3.5), we have

$$g_M(\nabla_V W, X)$$
$$= g_M(\mathcal{T}_V JW, BX) + g_M(\mathcal{H}\nabla_V JW, CX) - g_M(P_V W, CX) - g_M(Q_V W, BX),$$

$$g_M(\nabla_V W, X)$$
$$= g_M(\mathcal{T}_V JW, BX) + g_M([V, JW] + \nabla_{JW} V, CX) - g_M(P_V W, CX) - g_M(Q_V W, BX).$$

Since ∇ is torsion free and $[V, JW] \in \Gamma(\ker f_*)$, we obtain

$$g_M(\nabla_V W, X)$$
$$= g_M(\mathcal{T}_V JW, BX) + g_M(\nabla_{JW} JV, JCX) - g_M((\nabla_{JW} J)V, JCX)$$
$$-g_M(P_V W, CX) - g_M(Q_V W, BX).$$

Since f is conformal submersion, using equations $(2.1), (2.2), (2.4), (2.17)$ and (3.3) and Lemma $2.3(i)$, we have

$$g_M(\nabla_V W, X)$$
$$= g_M(\mathcal{T}_V JW, BX) + \frac{1}{\lambda} g_N(\nabla^f_{JW} f_* JV, f_* JCX) + g_M(JW, JV) g_M(\mathcal{H}gradln\lambda, JCX)$$
$$g_M(P_{JW} V, JCX) - g_M(P_V W, CX) - g_M(Q_V W, BX).$$

Thus it follows that $M_{(\ker f_*)}$ is totally geodesic if and only if the equation (4.1) is satisfied. On the other hand, for $V \in \Gamma(\ker f_*)$ and $X,Y \in \Gamma(\ker f_*)^\perp$, using equations $(2.1), (2.2), (2.4), (2.13), (2.14)$ and (3.5), we have

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY, JV) + g_M(\mathcal{H}\nabla_X CY, JV) - g_M(P_X Y, JV).$$

Since f is conformal submersion, using equations (2.17) and (3.3) and Lemma $2.3(i)$, we have

$$g_M(\nabla_X Y, V)$$
$$= g_M(\mathcal{A}_X BY, JV) + \frac{1}{\lambda} g_N(\nabla^f_X f_* CY, f_* JV) - g_M(X, JV) g_M(\mathcal{H}gradln\lambda, CY)$$
$$+g_M(X, CY) g_M(\mathcal{H}gradln\lambda, JV) - g_M(P_X Y, JV),$$

using that conclude that $M_{(\ker f_*)^\perp}$ is totally umbilical if and only if the equation (4.2) is satisfied. □

In a similar way with Theorem (5.4) ([1]), we obtain following theorem:

Theorem 4.5. *Let* f *be a conformal anti-invariant submersion from nearly Kähler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N) *with* $rank(\ker f_*) > 1$. *If* M *is a locally warped product manifold of the form* $M_{(\ker f_*)} \times M_{(\ker f_*)^\perp}$, *with either f is horizontally homothetic submersion or the fibres are one dimensional.*

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