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VOLUME OF A PENTAHEDRON REVISITED

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Abstract. The volume of a pentahedron is calculated in terms of dihedral angles, sides, face angles of component tetrahedrons (3-simplex) using an exact computation technique (known as signed simplex decomposition). Here the signed simplex decomposition means an established technique to subdivide a polyhedra (a simplicial complex or more strictly CW complex) to component tetrahedral (3-simplex) which is used in computer graphics where meshes with pyramids, prisms or hexahedra must be subdivided into tetrahedral to use efficient algorithms for volume-rendering, iso-contouring and particle advect. Moreover we attempt to give a tensorial notation of the volume of the pentahedron.

1 Introduction

A simplex is generalization of the notion of tetrahedron to arbitrary dimension. A simplicial complex δ is a finite collection of K-simplices such that (i) $\delta \in K$ and $r \leq \delta$ implies $r \in K$ and (ii) $\delta, v \in K$ implies $\delta \cap v = \phi$. The Cayley-Menger determinant gives the volume of simplex in j dimension. If S is a j-simplex in E^N with vertices v_0, v_1, v_2,...,v_j and $B = (B_{ik})$ denotes the $(i+1)\times(j+1)$ matrix given by $V_j = (-1)^{j+1}/2j.(j!)^2.det(\rho)$, where ρ is the$(j+2)\times(j+2)$ matrix obtained from B. Here det(ρ) is the Cayley-Menger determinant. But according to the definition we cannot apply the volume formula, of a simplex in j dimension to pentahedron directly because pentahedron is not a simplex (more specifically it is a simplicial complex). So we are deploying an established technique of exact volume computation where a pentahedron is decomposed into simplices (tetrahedra) to obtain a conformal mesh and an algebraic summation is obtained over the individual tetrahedrons. This technique find its root in [2], where he obtained that any generalized polyhedron with five vertices is a root of the polynomial equation of the form

$$Q(V) = V^4 + a_1(l)V^2 + a_2(l) = 0, \tag{1.1}$$

where l denotes the set of squares of the edge lengths of the polyhedron and a_1 and a_2 are some polynomials in l with rational coefficients. Sabitov deduced the above formula by decomposing a pentahedron V into two tetrahedrons (V_1 and V_2) and also stated the fact that the volume of a generalised pentahedron is given by

$$V = V_1 + \varepsilon V_2, \tag{1.2}$$

which is utilised in section 3. He obtained several cases which are valid for all polyhedral with five vertices independent of their actual configuration in R^3 which are discussed in detail in [2]. We are considering the first case only for this paper.

2 Volume of pentahedron (pyramid)

2.1 Volume of pentahedron (pyramid) in terms of coordinate of vertices

A theorem by H.B. Newson states that "Six times the volume of a polyhedron is equal to the sum of all determinants corresponding to the triangles being described in the same direction" [5]. Let V be the volume of the pentahedron. Let the vertices of the pentahedron be numbered 1, 2, 3, 4, 5 and if the corresponding coordinate be written as $(x_1, y_1, z_1\ )$ etc. Then

$$V=1\backslash 6\Bigg[\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}+\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}+\begin{vmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{vmatrix}+$$
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_5 & y_5 & z_5 \\ x_2 & y_2 & z_2 \end{vmatrix}+\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}+\begin{vmatrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{vmatrix}\Bigg]$$

The last two determinants are obtained by dividing the base of the pentahedron into two coplanar triangles

3 Volume of pentahedron(pyramid) in other forms

The generalized oriented volume of a pentahedron(pyramid) is a root of Sabitov's polynomial for pyramid. It may involve degenerate faces, self –intersections. As a special case, let us now consider a simple pentahedron(pyramid) as a polyhedron which is devoid of degenerate faces, self intersection and self-superposition. From now on, a generalised pentahedron (a pentahedron having generalized volume) would be mentioned specifically.

3.1 Method outline

Let v_1, v_2, v_3, v_4, v_5 be the vertices of the pentahedron(pyramid) and v_1, v_2, v_3 and v_4 be the vertices counterclockwise of a quadrilateral face. The face will be cut with the diagonal v_1 to v_3 if $(v_1, v_3) < (v_2, v_4)$ or will cut the diagonal v_2 to v_4 if $(v_1, v_3) < (v_2, v_4)$ [3]. The notation $(v_r, v_l) < (v_q, v_s)$ means the smallest identifier of the vertices v_r and v_l is smaller than the smallest identifier of the vertices v_q and v_s. Instead of using identifiers we may metricize the topological space generated by inducing a metric or a quasimetric $d(v_r, v_l)$ and $d(v_q, v_s)$ and replacing (v_r, v_l) and (v_q, v_s). However a computational issue arises for comparable suitability between identifier associated with each vertex and metric. Now the quadrilateral faces are split into triangular face that ensures conformity across the mesh, elements (i.e pyramids) must be tetrahedralised according to triangular meshes. But if we want to generate numerous meshes,the simplest method should be to insert a vertex at each element (pentahedron or pyramid in this case) and to build tetrahedra by connecting to all triangular face. This process has drawbacks, well shaped meshes are not generated and computational issues regarding the new vertex arises. So to remove the drawbacks, we mesh the pyramid according to the following figures.

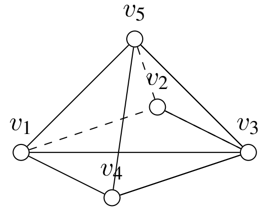

Fig 1

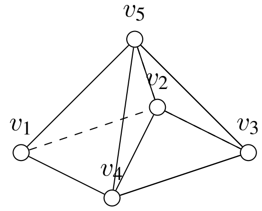

Fig 2

Now we have the following table:

	If	Tetrahedron
Fig 1	$(v_1, v_3) < (v_2, v_4)$ or $d(v_1, v_3) < d(v_2, v_4)$	Tet $v_1v_2v_3v_5$, Tet $v_1v_3v_4v_5$
Fig 2	$(v_1, v_3) > (v_2, v_4)$ or $d(v_1, v_3) > d(v_2, v_4)$	Tet $v_2v_3v_4v_5$, Tet $v_2v_1v_4v_5$

3.2 Volume of a pyramid in terms of r_i, R and θ_{ij} of the component tetrahedrons(meshes)

Let us suppose x_1, x_2, x_3, and x_4 are the vertices of one of the two component tetrahedrons and x_5, x_6, x_7, and x_4 are the vertices of the other. F_1, F_2, F_3, and F_4 are the faces of one tetrahedron given by $x_i \in F_i$ for $i = 0, 1, 2, 3$ and F_4, F_5, F_6, and F_7 are the four faces of the other tetrahedron given by $x_m \in F_m$ for $m = 0, 1, 2, 3$. Let r_i be the inradii of the corresponding faces F_i for $i = 0, 1, 2, 3$ and θ_{ik} are the face angles of the face F_i and point x_k, $k \in \{0, 1, 2, 3\} \setminus \{i\}$ and r_m be the inradii of the corresponding faces F_m for m=4, 5, 6, 7 and θ_{ml} are the face angles of the face F_m and point x_l, $l \in \{4, 5, 6, 7\} \setminus \{m\}$. Also a_{ij} are the edge length of the tetrahedron, $a_{ij} = [x_i, x_j]$ for $i, j \in \{0, 1, 2, 3\}$, $i \neq j$ and $a_{ij} = a_{ji}$, $\forall i, j$ and $a_{ml} = [x_m, x_l]$ for $m, l \in \{0, 1, 2, 3\}$, $m \neq l$ and $a_{ml} = a_{lm}$ $\forall m, l$. From equation 2 of section 1 and [1], the volume of a generalized oriented pyramid is

$$\boxed{V = r_3^2 r_1^2/24R_1D_1^2\sqrt{(-M_1)} + \varepsilon(r_7^2 r_5^2/24R_2D_2^2\sqrt{(-M_2)}),}$$

where $M_1 = A^4 + B^4 + C^4 - 2A^2B^2 - 2C^2B^2 + 2A^2C^2$ and $M_2 = D^4 + E^4 + F^4 - 2D^2E^2 - 2E^2F^2 + 2D^2F^2$,

For ε=+1, V is the volume of the simple pyramid.

A=1\8(sin θ_{32} sin θ_{21} sin θ_{10}),
B=1\8(sin θ_{31} sin θ_{20} sin θ_{12}),
C=1\8(sin θ_{30} sin θ_{21} sin θ_{10}),
D_1=1\8(sin $\theta_{32/2}$ sin $\theta_{30/2}$ sin $\theta_{31/2}$sin $\theta_{21/2}$ sin $\theta_{13/2}$ sin $\theta_{12/2}$cos $\theta_{21/2}$)
D=1\8(sin θ_{76} sin θ_{65} sin θ_{54}),
E=1\8(sin θ_{75} sin θ_{64} sin θ_{54}),
F=1\8(sin θ_{74} sin θ_{65} sin θ_{56}),
D_2=1\8(sin $\theta_{76/2}$ sin $\theta_{74/2}$ sin $\theta_{75/2}$sin $\theta_{65/2}$ sin $\theta_{57/2}$ sin $\theta_{56/2}$cos $\theta_{65/2}$),

where ε is an invariant ϵ {+1, -1}and R_1 and R_2 are the radius of the sphere circumscribing the terahedrons. For ϵ =+1, V is the volume of the simple pyramid.

4 Volume of a pyramid in tensor notation.

The volume of an n-dimensional simplex is given by

$$V(n) = 1/2^{n/2}.n!((T^{i1......................in}X_{i1}.........X_{in})^{1/2},$$

where $X = [a_1^2 \ldots\ldots\ldots\ldots a_{n(n+1)/2}^2]$ is a vector containing the squares of the edge lengths $\forall\Gamma\epsilon L_n(K_{n+1})$ the tensor entry corresponding to is given by T^Γ $= \{-1^{\#loops(T)} \times 2^{\#components}(T)/n!$ if valency(Γ) <= 2,otherwise 0 }. The maximal valency constraint implies that each Γ is just a disjoint union of paths and circuits.

Corollary:[4] The volume of an arbitrary tetrahedron is

$$V = (T^{ijk}X_iX_jX_k)^{1/2}/12\sqrt{2},$$

where T^(Γ)={-2/3(Γ $\cong K_2 + P_1$),-1/3(Γ $\cong K_3$),-1/3(Γ $\cong P_3$), 0(otherwise)}.

Using equation 2 of section 1, the volume of the generalized arbitrary pyramid is $\boxed{[(\sqrt{T^{ijk}X_iX_jX_k}) + \epsilon(\sqrt{T^{mnp}A_mB_nC_p}).}$ So we have the following theorem:

Theorem 4.1. The volume of the generalized arbitrary pyramid is $[(\sqrt{T^{ijk}X_iX_jX_k}) + \epsilon(\sqrt{T^{mnp}A_mB_nC_p})]$

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