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REFORMULATED RECIPROCAL PRODUCT DEGREE DISTANCE OF STRONG PRODUCT OF GRAPHS

K. Pattabiraman

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Abstract. The reciprocal product degree distance(RDD_*), is defined as $RDD(G) = \sum\limits_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$. The new graph invariant named reformulated reciprocal product degree distance is defined for a connected graph G as $\overline{R}_t^*(G) = \sum\limits_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)+t}$, $t \geq 0$. In this paper, the reformulated reciprocal product degree distance and reciprocal product degree distance of strong product of two graphs are obtained.

1 Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and $d_G(v)$ is the degree of a vertex $v \in V(G)$. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [11]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

Let G be a connected graph. Then the $Wiener\ index$ of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \in V(G)} d_G(u, v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the $Harary\ index$ of G is defined as $H(G) = \frac{1}{2} \sum\limits_{u,\, v \in V(G)} \frac{1}{d_G(u,v)}$. Das et al. [7] proposed the second and third Harary index and they extend it to the generalized version of Harary index, namely, the t-Harary index, which is defined as $\overline{H}_t(G) = \frac{1}{2} \sum\limits_{u,\, v \in V(G)} \frac{1}{d_G(u,v)+t}$, $t \geq 0$. Also they obtained the bounds for t-Harary index of G in terms of Wiener index of G.

Dobrynin and Kochetova [8] and Gutman [10] independently proposed a vertex-degree-weighted version of Wiener index of a connected graph G called *degree distance,* which is defined as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v))d_G(u, v)$. Note that the degree distance is a degree-weight version of the Wiener index.

To strengthen the interactions between nodes in a network is described by their topological distances, it is necessary to consider the weighted versions to measure the centrality of the network with respect to the information flow [6]. Hua and Zhang [12] introduced a new graph invariant named *reciprocal degree distance,* which can be seen as a degree-weight version of Harary index, defined as, $RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}$. Alizadeh et al. [2] has shown that the reciprocal degree distance can be used as an efficient measuring tool in the study of complex networks. Hua and Zhang [12] presented some lower and upper bounds of the reciprocal degree distance in terms of graph invariants such as degree distance, Harary index, first Zagreb index, first Zagreb coindex, pendent vertices, independence number, chromatic number, vertex- and edge-connectivity. They also characterized the extremal cactus graphs with the maximum reciprocal degree distance.

Recently, Li et al. [14] introduced a vertex-degree-weighted version of t-Harary index of a connected graph G called *reformulated reciprocal degree distance,* which is defined as $\overline{R}_t(G) = \frac{1}{2}\sum\limits_{u,v\in V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)+t}$, $t \geq 0$. In view of $\overline{H}_t(G)$, $\overline{R}_t(G)$ is just the additively weighted t-Harary index; while in view of $RDD(G)$ it is also the generalized version of the reciprocal degree distance of a connected graph G. It is natural and interesting to study the mathematical properties of this novel graph index. Li et al. [14] studied the mathematical properties of the reformulated reciprocal degree distance under some edge grafting transformations and extremal properties of the several class of trees. Also they established the sharp upper bound on the maximum reformulated reciprocal degree distance of n-vertex trees with k pendents. Pattabiraman et al. [17, 18, 19] obtained the reformulated reciprocal degree distance of some important classes of graphs.

In these background, the *reformulated reciprocal product degree distance*[15], which is defined as $\overline{R}^*_t(G) = \frac{1}{2}\sum\limits_{u,v\in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)+t}$, $t \geq 0$. In view of $RDD_*(G)$ it is also the generalized version of the reciprocal product degree distance of a connected graph G. The reformulated reciprocal product degree distance of several graph operations are discussed in [15, 16]. In this connection, we have obtained the exact formulae for the reformulated reciprocal product degree distance and reciprocal product degree distance of strong product of graphs.

The *first Zagreb index* is defined as $M_1(G) = \sum\limits_{u\in V(G)} d_G(u)^2 = \sum\limits_{uv\in E(G)} (d_G(u) + d_G(v))$. Similarly, the *first Zagreb coindex* is defined as $\overline{M}_1(G) = \sum\limits_{uv\notin E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [9].

2 Strong product of graphs

If $m_0 = m_1 = \ldots = m_{r-1} = s$ in $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$ (the complete multipartite graph with partite sets of sizes $m_0,\ m_1,\ \ldots,\ m_{r-1}$), then we denote it by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S\rangle$ denotes the subgraph of G induced by S. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$, we mean the sum of the distances in G from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum\limits_{s\,\in\, S,\, t\,\in\, T} d_G(s,t)$ and $d^H_G(S,T) = \sum\limits_{s\,\in\, S,\, t\,\in\, T} \frac{1}{d_G(s,t)+t}$, $t \geq 0$.

Let G be a simple connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$, $r \geq 2$, be the complete multiparite graph with partite sets $V_0,\ V_1,\ \ldots,\ V_{r-1}$ and let $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, the vertex set of $G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$ is written as $V(G) \times V(K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}) = \bigcup\limits_{\substack{i\,=\,0\\ j\,=\,0}}^{\substack{r-1\\ n-1}} B_{ij}$. Let $\mathscr{B} = \{B_{ij}\}_{\substack{i\,=\,0,1,\ldots,\,n-1\\ j\,=\,0,1,\ldots,\,r-1}}$. Let $X_i = \bigcup\limits_{j\,=\,0}^{r-1} B_{ij}$ and $Y_j = \bigcup\limits_{i\,=\,0}^{n-1} B_{ij}$; we call X_i and Y_j as *layer* and *column* of $G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$, respectively. If we denote $V(B_{ij}) = \{x_{i1}, x_{i2}, \ldots, x_{im_j}\}$ and $V(B_{kp}) = \{x_{k1}, x_{k2}, \ldots, x_{k\,m_p}\}$, then $x_{i\ell}$ and $x_{k\ell}, 1 \leq \ell \leq j$, are called the *corresponding vertices* of B_{ij} and B_{kp}. Further, if $v_iv_k \in E(G)$, then the induced subgraph $\langle B_{ij} \bigcup B_{kp}\rangle$ of $G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or, m_p independent edges joining the corresponding vertices of B_{ij} and B_{kj} according as $j \neq p$ or $j = p$, respectively.

The following remark is follows from the structure of the graph $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$.

Remark 2.1. Let n_0 and q be the number of vertices and edges of $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$. Then the sums $\sum_{\substack{j,\,p\,=\,0\\ j\,\neq\, p}}^{r-1} m_jm_p = 2q$, $\sum\limits_{j=0}^{r-1} m_j^2 = n_0^2 - 2q, \sum_{\substack{j,\,p\,=\,0\\ j\,\neq\, p}}^{r-1} m_j^2m_p = n_0q - 3t = \sum_{\substack{j,\,p\,=\,0\\ j\,\neq\, p}}^{r-1} m_jm_p^2$, $\sum\limits_{j=0}^{r-1} m_j^3 = n_0^3 - 3n_0q + 3t$ and $\sum\limits_{j=0}^{r-1} m_j^4 = n_0^4 - 4n_0^2q + 2q^2 + 4n_0t - 4\tau$, where t and τ are the number of triangles and $K_4^{'s}$ in $K_{m_0,\, m_1,\, \dots,\, m_{r-1}}$.$\square$

The proof of the following lemma follows easily from the properties and structure of $G \boxtimes K_{m_0,\, m_1,\, \dots,\, m_{r-1}}$.

Lemma 2.2. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0,\, m_1,\, \dots,\, m_{r-1}}$, where $r \geq 2$. Then*
(i) If $v_iv_k \in E(G)$ and $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kj}$, then

$$d_{G'}(x_{it}, x_{k\ell}) = \begin{cases} 1, & \text{if } t = \ell, \\ 2, & \text{if } t \neq \ell, \end{cases}$$

and if $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ j \neq p$, then $d_{G'}(x_{it}, x_{k\ell}) = 1$.
(ii) If $v_iv_k \notin E(G)$, then for any two vertices $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp}$, $d_{G'}(x_{it}, x_{k\ell}) = d_G(v_i, v_k)$.
(iii) For any two distinct vertices in B_{ij}, their distance is 2.$\square$

The proof of the following lemma follows easily from Lemma 2.2. The lemma is used in the proof of the main theorems of this section.

Lemma 2.3. *Let G be a connected graph and let B_{ij} in $G' = G \boxtimes K_{m_0,\, m_1,\, \dots,\, m_{r-1}}$. Then the degree of a vertex $(v_i, u_j) \in B_{ij}$ in G' is $d_{G'}((v_i, u_j)) = d_G(v_i) + (n_0 - m_j) + d_G(v_i)(n_0 - m_j)$, where $n_0 = \sum\limits_{j=0}^{r-1} m_j$.*$\square$

Lemma 2.4. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0,\, m_1,\, \dots,\, m_{r-1}}$, where $r \geq 2$.*
(i) If $v_iv_k \in E(G)$, then

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_jm_p}{1+t}, & \text{if } j \neq p, \\ \frac{m_j}{1+t} + \frac{m_j(m_j-1)}{2+t}, & \text{if } j = p, \end{cases}$$

(ii) If $i \neq k$ and $v_iv_k \notin E(G)$, then $d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_jm_p}{d_G(v_i,v_k)+t}, & if\ j \neq p, \\ \frac{m_j^2}{d_G(v_i,v_k)+t}, & if\ j = p. \end{cases}$

(iii) $d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} \frac{m_jm_p}{1+t}, & if\ j \neq p, \\ \frac{m_j(m_j-1)}{2+t}, & if\ j = p. \end{cases}$ $\square$

Noe we obtain the reformulated reciprocal product degree distance of $G \boxtimes K_{m_0,\, m_1,\, \dots,\, m_{r-1}}$.

Theorem 2.5. *Let G be a connected graph with n vertices and m edges. Then* $\overline{R}^*_t(G \boxtimes K_{m_0,\, m_1,\, \dots,\, m_{r-1}}) = (n_0^2 + 4q^2 + 4n_0q)\overline{R}^*_t(G) + (4q^2 + 2n_0q)\overline{R}_t(G) + 4q^2\overline{H}_t(G) + \frac{1}{1+t}\Big[\Big(2q^2 + 2qn_0 + 2n_0t + 2q + 4\tau + 6t\Big)\frac{M_1(G)}{2} + m\Big(2qn_0 + 4n_0t - 4q^2 + 6t + 8\tau\Big) + \frac{n}{2}\Big(2n_0t + 2q^2 + 4\tau\Big)\Big] + \frac{1}{(1+t)(2+t)}\Big[M_2(G)\Big(-2q^2 + 2n_0t + 4\tau + 3n_0^3 - 10n_0q + 18t - n_0^2 + 6q + n_0\Big) + M_1(G)\Big(-2q^2 + 4\tau + 2n_0t + 6t + 2q\Big) + m\Big(-2q^2 + 4\tau + 2n_0t + n_0q + 3t\Big)\Big] + \frac{1}{2+t}\Big[\frac{M_1(G)}{2}\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big) + 2m\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big) + \frac{n}{2}\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big)\Big].$

Proof. Let $G' = G \boxtimes K_{m_0, m_1, \dots, m_{r-1}}$. Clearly,

$$\begin{aligned}
\overline{R}_t^*(G') &= \frac{1}{2} \sum_{B_{ij}, B_{kp} \in \mathscr{B}} d_{G'}(B_{ij}) d_{G'}(B_{kp}) d_{G'}^H(B_{ij}, B_{kp}) \\
&= \frac{1}{2} \Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{ip}) d_{G'}^H(B_{ij}, B_{ip}) \\
&\quad + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{j=0}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{kj}) d_{G'}^H(B_{ij}, B_{kj}) \\
&\quad + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{kp}) d_{G'}^H(B_{ij}, B_{kp}) \\
&\quad + \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{ij}) d_{G'}^H(B_{ij}, B_{ij}) \Bigg).
\end{aligned}$$

We consider the four sums $S_1, \dots, S_4$ as follows.

First we compute $S_1 = \sum\limits_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{ip}) d_{G'}^H(B_{ij}, B_{ip})$. For that first we find the following.

By Lemma 2.3, we have

$$\begin{aligned}
T_1' &= d_{G'}(B_{ij}) d_{G'}(B_{ip}) \\
&= \Big(d_G(v_i)(n_0 - m_j + 1) + (n_0 - m_j) \Big) \Big(d_G(v_i)(n_0 - m_p + 1) + (n_0 - m_p) \Big) \\
&= \Big((n_0 + 1)^2 - (n_0 + 1)m_j - (n_0 + 1)m_p + m_j m_p \Big) d_G^2(v_i) \\
&\quad + \Big(2n_0(n_0 + 1) - (2n_0 + 1)m_j - (2n_0 + 1)m_p + 2m_j m_p \Big) d_G(v_i) \\
&\quad + \Big(n_0^2 - n_0 m_p - n_0 m_j + m_j m_p \Big).
\end{aligned}$$

From Lemma 2.4, we have $d_{G'}^H(B_{ij}, B_{ip}) = \frac{m_j m_p}{1+t}$. Thus

$$\begin{aligned}
& T_1' d_{G'}^H(B_{ij}, B_{ip}) = T_1' \frac{m_j m_p}{1+t} \\
&= \frac{1}{1+t} \Big[\Big((n_0 + 1)^2 m_j m_p - (n_0 + 1) m_j^2 m_p - (n_0 + 1) m_j m_p^2 + m_j^2 m_p^2 \Big) d_G^2(v_i) \\
&\quad + \Big(2n_0(n_0 + 1) m_j m_p - (2n_0 + 1) m_j^2 m_p - (2n_0 + 1) m_j m_p^2 + 2m_j^2 m_p^2 \Big) d_G(v_i) \\
&\quad + \Big(n_0^2 m_j m_p - n_0 m_j^2 m_p - n_0 m_j m_p^2 + m_j^2 m_p^2 \Big) \Big].
\end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned}
T_1 &= \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} T_1' d_{G'}^H(B_{ij}, B_{ip}) \\
&= \frac{1}{1+t} \Big[\Big(2q^2 + 2qn_0 + 2n_0 t + 2q + 4\tau + 6t \Big) d_G^2(v_i) \\
&\quad + \Big(2qn_0 + 4n_0 t - 4q^2 + 6t + 8\tau \Big) d_G(v_i) \\
&\quad + \Big(2n_0 t + 2q^2 + 4\tau \Big) \Big].
\end{aligned}$$

From the definition of the first Zagreb index, we have

$$\begin{aligned}
S_1 &= \sum_{i=0}^{n-1} T_1 \\
&= \frac{1}{1+t}\Big[\Big(2q^2+2qn_0+2n_0t+2q+4\tau+6t\Big)M_1(G) \\
&\quad +2m\Big(2qn_0+4n_0t-4q^2+6t+8\tau\Big) \\
&\quad +n\Big(2n_0t+2q^2+4\tau\Big)\Big].
\end{aligned}$$

Next we compute $S_2 = \sum\limits_{j=0}^{r-1} \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d^H_{G'}(B_{ij},B_{kj})$. For that first we find T_2'.

By Lemma 2.3, we have

$$\begin{aligned}
&T_2' = d_{G'}(B_{ij})d_{G'}(B_{kj}) \\
=\ & \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)\Big(d_G(v_k)(n_0-m_j+1)+(n_0-m_j)\Big) \\
=\ & (n_0-m_j+1)^2 d_G(v_i)d_G(v_k)+(n_0-m_j)(n_0-m_j+1)(d_G(v_i)+d_G(v_k))+(n_0-m_j)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
S_2 &= \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} T_2' d^H_{G'}(B_{ij},B_{kj}) \\
&= \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2' d^H_{G'}(B_{ij},B_{kj}) + \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} T_2' d^H_{G'}(B_{ij},B_{kj})
\end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned}
&S_2 = \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2'\Big(\frac{m_j}{1+t}+\frac{m_j(m_j-1)}{2+t}\Big) + \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} T_2' \frac{m_j^2}{d_G(v_i,v_k)+t}, \\
=\ & \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2'\Big(\frac{m_j}{1+t}+\frac{m_j(m_j-1)}{2+t}+\frac{m_j^2}{1+t}-\frac{m_j^2}{1+t}\Big) + \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} T_2' \frac{m_j^2}{d_G(v_i,v_k)+t} \\
=\ & \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2' \frac{m_j-m_j^2}{(1+t)(2+t)} + \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} T_2' \frac{m_j^2}{d_G(v_i,v_k)+t} \\
=\ & S_2'+S_2'', \qquad (2.1)
\end{aligned}$$

where S_2' and S_2'' are the sums of the terms of the above expression, in order.

Now we calculate S_2'. For that first we find the following.

$$\begin{aligned}
&T_2'\Big(m_j-m_j^2\Big) = \Big[\Big(-m_j^4+(2n_0+3)m_j^3-(n_0^2+4n_0+3)m_j^2+(n_0+1)^2m_j\Big) \\
&d_G(v_i)d_G(v_k)+\Big(-m_j^4+(2n_0+2)m_j^3-(n_0^2+3n_0+1)m_j^2+(n_0^2+n_0)m_j\Big) \\
&(d_G(v_i)+d_G(v_k))+\Big(-m_j^4+(2n_0+1)m_j^3-(n_0^2+2n_0)m_j^2+n_0^2m_j\Big)\Big].
\end{aligned}$$

By Remark 2.1, we have

$$
\begin{aligned}
T_2'' &= \sum_{j=0}^{r-1} T_2'\Big(m_j - m_j^2\Big)\\
&= \Big[\Big(-2q^2+2n_0t+4\tau+3n_0^3-10n_0q+18t-n_0^2+6q+n_0\Big)d_G(v_i)d_G(v_k)\\
&\quad +\Big(-2q^2+4\tau+2n_0t+6t+2q\Big)(d_G(v_i)+d_G(v_k))\\
&\quad +\Big(-2q^2+4\tau+2n_0t+n_0q+3t\Big)\Big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
S_2' &= \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} \frac{T_2''}{(1+t)(2+t)}\\
&= \frac{1}{(1+t)(2+t)}\Big[2M_2(G)\Big(-2q^2+2n_0t+4\tau+3n_0^3-10n_0q+18t-n_0^2+6q+n_0\Big)\\
&\quad +2M_1(G)\Big(-2q^2+4\tau+2n_0t+6t+2q\Big)+2m\Big(-2q^2+4\tau+2n_0t+n_0q+3t\Big)\Big].
\end{aligned}
$$

Next we calculate S_2''. For that we need the following.

$$
\begin{aligned}
T_2'm_j^2 &= \Big(m_j^4-(2n_0+2)m_j^3+(n_0+1)^2m_j^2\Big)d_G(v_i)d_G(v_k)\\
&\quad +\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+n_0)m_j^2\Big)(d_G(v_i)+d_G(v_k))\\
&\quad +\Big(m_j^4-2n_0m_j^3+n_0^2m_j^2\Big).
\end{aligned}
$$

By Remark 2.1, we have

$$
\begin{aligned}
T_2 &= \sum_{j=0}^{r-1} T_2'm_j^2\\
&= \Big(2q^2-4\tau-2n_0t-6t+2n_0q-2q+n_0^2\Big)d_G(v_i)d_G(v_k)\\
&\quad +\Big(2q^2-4\tau-2n_0t-3t+n_0q\Big)(d_G(v_i)+d_G(v_k))\\
&\quad +\Big(2q^2-4\tau-2n_0t\Big).
\end{aligned}
$$

From the definitions of $\overline{R}_t^*$,$\overline{R}_t$ and $\overline{H}_t$, we obtain

$$
\begin{aligned}
S_2'' &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} \frac{T_2}{d_G(v_i,v_k)+t}\\
&= 2\Big(2q^2-4\tau-2n_0t-6t+2n_0q-2q+n_0^2\Big)\overline{R}_t^*(G)\\
&\quad +2\Big(2q^2-4\tau-2n_0t-3t+n_0q\Big)\overline{R}_t(G)\\
&\quad +2\Big(2q^2-4\tau-2n_0t\Big)\overline{H}_t(G).
\end{aligned}
$$

Now we calculate $A_3 = \sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum\limits_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^{H}(B_{ij},B_{kp})$. For that first we compute T_3'. By Lemma 2.3, we have

$$\begin{aligned} T_3' &= d_{G'}(B_{ij})d_{G'}(B_{kp}) \\ &= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)\Big(d_G(v_k)(n_0-m_p+1)+(n_0-m_p)\Big) \\ &= d_G(v_i)d_G(v_k)(n_0-m_j+1)(n_0-m_p+1)+d_G(v_i)(n_0-m_j+1)(n_0-m_p) \\ &\quad +d_G(v_k)(n_0-m_p+1)(n_0-m_j)+(n_0-m_j)(n_0-m_p). \end{aligned}$$

Since the distance between B_{ij} and B_{kp} is $\frac{m_jm_p}{d_G(v_i,v_k)+t}$. Thus

$$\begin{aligned} T_3'm_jm_p &= d_G(v_i)d_G(v_k)\Big((n_0^2+2n_0+1)m_jm_p-(n_0+1)m_j^2m_p-(n_0+1)m_jm_p^2+m_j^2m_p^2\Big) \\ &\quad +d_G(v_i)\Big((n_0^2+n_0)m_jm_p-(n_0+1)m_jm_p^2-n_0m_j^2m_p+m_j^2m_p^2\Big) \\ &\quad +d_G(v_k)\Big((n_0^2+n_0)m_jm_p-n_0m_jm_p^2-(n_0+1)m_j^2m_p+m_j^2m_p^2\Big) \\ &\quad +\Big(n_0^2m_jm_p-n_0m_jm_p^2-n_0m_j^2m_p+m_j^2m_p^2\Big). \end{aligned}$$

By Remark 2.1, we obtain

$$\begin{aligned} T_3 &= \sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1} T_3'm_jm_p \\ &= d_G(v_i)d_G(v_k)\Big(2n_0q+2n_0t+2q+2q^2+6t+4\tau\Big) \\ &\quad +(d_G(v_i)+d_G(v_k))\Big(qn_0+2n_0t+3t+2q^2+4\tau\Big) \\ &\quad +\Big(2n_0t+2q^2+4\tau\Big). \end{aligned}$$

Hence

$$\begin{aligned} S_3 &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} \frac{T_3}{d_G(v_i,v_k)+t} \\ &= 2\overline{R}_t^*(G)\Big(2n_0q+2n_0t+2q+2q^2+6t+4\tau\Big) \\ &\quad +2\overline{R}_t(G)\Big(qn_0+2n_0t+3t+2q^2+4\tau\Big) \\ &\quad +2\overline{H}_t(G)\Big(2n_0t+2q^2+4\tau\Big). \end{aligned}$$

Finally, we obtain $S_4=\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ij})d_{G'}^H(B_{ij},B_{ij})$. For that first we calculate T_4'. By Lemma 2.3, we have

$$\begin{aligned} T_4' &= d_{G'}(B_{ij})d_{G'}(B_{ij}) \\ &= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)^2 \\ &= d_G^2(v_i)(n_0-m_j+1)^2+2d_G(v_i)(n_0-m_j)(n_0-m_j+1)+(n_0-m_j)^2. \end{aligned}$$

From Lemma 2.4, the distance between B_{ij} and B_{ij} is $\frac{m_j(m_j-1)}{2+t}$. Thus

$$\begin{aligned} T_4'm_j(m_j-1) &= d_G^2(v_i)\Big(m_j^4-(2n_0+3)m_j^3+((n_0+1)^2+2)m_j^2-(n_0+1)^2m_j\Big) \\ &\quad +2d_G(v_i)\Big(m_j^4-(2n_0+2)m_j^3+(n_0^2+3n_0+1)m_j^2-(n_0^2+n_0)m_j\Big) \\ &\quad +\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+2n_0)m_j^2-n_0^2m_j\Big). \end{aligned}$$

By Remark 2.1, we obtain

$$\begin{aligned}
T_4 &= \sum_{j=0}^{r-1} T_4' m_j(m_j-1)\\
&= d_G^2(v_i)\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big)\\
&\quad +2d_G(v_i)\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big)\\
&\quad +\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
S_4 &= \sum_{i=0}^{n-1} \frac{T_4}{2+t}\\
&= \frac{1}{2+t}\Big[M_1(G)\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big)\\
&\quad +4m\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big)\\
&\quad +n\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big)\Big].
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\overline{R}_t^*(G') = \frac{1}{2}\Big(S_1 + S_2 + S_3 + S_4\Big)\\
&= (n_0^2 + 4q^2 + 4n_0q)\overline{R}_t^*(G) + (4q^2 + 2n_0q)\overline{R}_t(G) + 4q^2\overline{H}_t(G)\\
&+\frac{1}{1+t}\Big[\Big(2q^2 + 2qn_0 + 2n_0t + 2q + 4\tau + 6t\Big)\frac{M_1(G)}{2} + m\Big(2qn_0 + 4n_0t - 4q^2 + 6t + 8\tau\Big)\\
&+\frac{n}{2}\Big(2n_0t + 2q^2 + 4\tau\Big)\Big] + \frac{1}{(1+t)(2+t)}\Big[M_2(G)\Big(-2q^2 + 2n_0t + 4\tau + 3n_0^3 - 10n_0q\\
&+18t - n_0^2 + 6q + n_0\Big) + M_1(G)\Big(-2q^2 + 4\tau + 2n_0t + 6t + 2q\Big) + m\Big(-2q^2 + 4\tau + 2n_0t\\
&+n_0q + 3t\Big)\Big] + \frac{1}{2+t}\Big[\frac{M_1(G)}{2}\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big)\\
&+2m\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big) + \frac{n}{2}\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big)\Big].
\end{aligned}$$

If $t = 0$, in Theorem 2.5, we obtain the reciprocal product degree distance of $G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$.

Corollary 2.6. *Let G be a connected graph with n vertices and m edges. Then* $RDD_*(G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}) = (4q^2 + n_0^2 + 4n_0q)RDD_*(G) + 4q^2H(G) + (4q^2 + 2n_0q)RDD(G) + \frac{n}{2}(4q^2 - n_0q - 3t) + \frac{M_1(G)}{2}\Big[4n_0^2q + 2n_0t + 3t + 7n_0q - n_0 - 3n_0^2 - 2n_0^3 - 2q + 4\tau\Big] + m\Big[\frac{5n_0q}{2} + n_0t - q^2 - \frac{9t}{2} - 4q + 2\tau\Big] - \frac{M_2(G)}{2}\Big[2q^2 - 2n_0t - 3n_0^3 + 10n_0q + n_0^2 - 18t - 6q - n_0 - 4\tau\Big]$ $r \geq 2$.

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Author information

K. Pattabiraman, Department of Mathematics, Annamalai University, Annamalainagar 608 002, India.
E-mail: `pramank@gmail.com`

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