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TOTAL INFLUENCE NUMBER OF SOME SPLITTING GRAPHS

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Abstract. Graph labeling is an important area of graph theory. It is used in coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management. In this paper, we study the total influence number as a graph labeling parameter. The total influence number can be viewed as vertex labeling problems concerned with the sum of the labels. We give a general theorem related to the total influence number, and also show how to find a maximum total influence set on various basic splitting graphs.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph, where $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. For notation and terminology not defined here, see [5]. For a vertex $v \in V(G)$, the open neighboorhood $N(v)$ is the set of all vertices adjacent with v. Let S be a vertex subset, $S \subseteq V$, then $\overline{S} = V - S$ denotes the complement of S.

The distance $d(u, v)$ between two vertices u and v in G is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. The diameter $diam(G)$ of a graph G is the maximum distance between two vertices of G. For any vertex u, $d(u, S) = \min_{v \in S} d(u, v)$.

A vertex subset S is called an alternating set if and only if S is either (1) the empty set or (2) a maximal independent set such that $\exists u \in S \ni \forall v \in S$, $d(u, v) = 2k$ for some $k \in \mathbb{Z}$ [9].

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced by Alex Rosa in 1967 [1]. Labeled graphs have applications in many fields. An extensive study on applications of graph labeling carried out by Bloom and Golomb in 1977 [6]. Further, a detailed survey on graph labeling is studied by Gallian [7]. A variety of parameters have been proposed to quantify the graph labeling such as influence and total influence number.

Agah et al. [8] introduced the concept of influence number. Daugherty et al. [9] introduced the total influence number as a natural extention of the influence number. These graph parameters are problems of vertex labeling deal with the sum of the labels. There are many vertex labeling problems which seek to minimize the sum of all of the labels. But, the influence and total influence number have the aim of maximizing the sum.

The concept of the influence number comes from the area of social networks that looks at the level of the influence of a person on another one. For a set of people of S, a person who is not membership of S, is influenced by the closest person in S. But, people in S do not influence themselves. Since the distance between a person in S and his or her closest person is 1, a person in S has an influence of $\frac{1}{2}$ on their friends, an influence of $\frac{1}{4}$ on their friends' friends, and so on.

While in the event of the influence number each vertex in $\overline{S}$ is influenced by the closest vertex in S, in the total influence number each vertex in $\overline{S}$ is influenced by every vertex in S. But, both of them seek to maximize the influence of S to $\overline{S}$. When we think about total influence number in psychology, a person in S is influenced by all people in $\overline{S}$.

The influence number of a vertex subset S is $\eta(S) = \sum_{u \in \overline{S}} \frac{1}{2^{d(u,S)}}$. The influence number of a graph G is $\eta(G) = \max_{S \subseteq V} \eta(S)$. A set S is called η-set if $\eta(S) = \eta(G)$.

The total influence number of a vertex $v \in S$ is $\eta_T(v) = \sum_{u \in \overline{S}} \frac{1}{2^{d(u,v)}}$. The total influence number of a vertex subset S is

$$\eta_T(S) = \sum_{v \in S} \eta_T(v) = \sum_{v \in S} \sum_{u \in \overline{S}} \frac{1}{2^{d(u,v)}}.$$

The total influence number of a graph G is

$$\eta_T(G) = \max_{S \subseteq V} \eta_T(S).$$

A set S is called η_T–set if $\eta_T(S) = \eta_T(G)$.

The aim of this article is to obtain general bound and efficient formulas for the total influence number of some graphs.

The rest of this paper is structured as follows. In section 2, known results on total influence number are given and a general bound is proved. In section 3, definition of the splitting graph is given and exact values for the total influence number of some splitting graphs are determined.

We first give an important theorem which we need in the proof of theorems in Section 3.

Theorem 1.1. *[4] If f is continuous on a closed, bounded set D in $\mathbb{R} \times \mathbb{R}$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

2 Main Results On The Total Influence Number

Theorem 2.1. *[10] Let G be a graph of order n. Then, $\eta_T(G) \leqslant \frac{n^2}{8}$.*

Theorem 2.2. *[9] For any graph $G = (V, E)$, with vertex partitions V_1 and V_2 and a set $S \subseteq V$ let $S_1 = V_1 \cap S$, $S_2 = V_2 \cap S$, $\overline{S} = V - S$, $\overline{S_1} = V_1 - S_1$ and $\overline{S_2} = V_2 - S_2$. Then,*

$$\eta_T(S) = \eta_T(S_1, \overline{S_1}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_2}) + \eta_T(S_1, \overline{S_2}).$$

Theorem 2.3. *[9] For a path P_n $(n > 1)$, a vertex subset S has maximum total influence if and only if it is a non-empty alternating set.*

Corollary 2.4. *[9] The total influence number of a path, P_n, is*

$$\eta_T(P_n) = \begin{cases} \frac{(10)2^{-n}+6n-10}{9} & \text{if } n \text{ is even,} \\ \frac{(8)2^{-n}+6n-10}{9} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.5. *[9] The total influnce number of some graphs is as follows:*

(a) $\eta_T(K_n) = \begin{cases} \frac{n^2}{8} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{8} & \text{if } n \text{ is odd.} \end{cases}$

(b) $\eta_T(K_{1,n}) = \begin{cases} \frac{(n+2)^2}{16} & \text{if } n \text{ is even,} \\ \frac{(n+1)(n+3)}{16} & \text{if } n \text{ is odd.} \end{cases}$

(c) $\eta_T(DS_{n,m}) = \begin{cases} \frac{1}{16}n^2 + \frac{3}{8}n + \frac{1}{16}m^2 + \frac{3}{8}m + \frac{1}{16}nm + \frac{3}{4} & \text{if } n \text{ and } m \text{ are even,} \\ \frac{1}{16}n^2 + \frac{3}{8}n + \frac{1}{16}m^2 + \frac{3}{8}m + \frac{1}{16}nm + \frac{11}{16} & \text{otherwise.} \end{cases}$

(d) $\eta_T(K_{n,m}) = \begin{cases} \frac{mn}{2} & \text{if } \; n \geqslant \frac{m}{2}, \\ \frac{(2n+m)^2}{16} & \text{if } \; n < \frac{m}{2}, m \text{ even}, \\ \frac{(2n+m+1)(2n+m-1)}{16} & \text{if } \; n < \frac{m}{2}, m \text{ odd}. \end{cases}$

Theorem 2.6. *For a graph G of order n, a set S is an η_T–set if and only if $|S|$ and $|\overline{S}|$ must be fairly close. Furthermore,*

$$\begin{cases} \frac{n^2}{2^{diam(G)+2}} \leqslant \eta_T(G) \leqslant \frac{n^2}{8} & \text{if } n \text{ is even} \\ \frac{n^2-1}{2^{diam(G)+2}} \leqslant \eta_T(G) \leqslant \frac{n^2-1}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $|S| = x$ and $S = \{u_1, u_2, ..., u_x\}$, $\overline{S} = \{v_1, v_2, ..., v_{n-x}\}$. Thus,

$$\eta_T(S) = \sum_{u\in S}\sum_{v\in \overline{S}} \frac{1}{2^{d(u,v)}} = \sum_{i=1}^{x}\sum_{j=1}^{n-x} \frac{1}{2^{d(u_i,v_j)}}. \tag{2.1}$$

Since $1 \leqslant d(u_i, v_j) \leqslant diam(G)$ for any $u_i, v_j \in V(G)$,

$$\frac{1}{2^{diam(G)}} \leqslant \frac{1}{2^{d(u_i,v_j)}} \leqslant \frac{1}{2}. \tag{2.2}$$

By (2.1) and (2.2), we say

$$\frac{1}{2^{diam(G)}} x(n-x) \leqslant \eta_T(S) \leqslant \frac{1}{2}x(n-x).$$

By definition of the total influence number,

$$\max_{S\subseteq V} \frac{1}{2^{diam(G)}} x(n-x) \leqslant \max_{S\subseteq V} \eta_T(S) = \eta_T(G) \leqslant \max_{S\subseteq V} \frac{1}{2}x(n-x). \tag{2.3}$$

We have $f(x) = x(n-x)$. By setting $f'(x) = 0$ gives $x = \frac{n}{2}$. Hence, $|S|$ and $|\overline{S}|$ must be fairly close.

If n is even, $x = \frac{n}{2}$. If we substitute $x = \frac{n}{2}$ into the inequality (2.3), the proof is completed for this. In the obtained inequality, the upper bound is equivalent to Theorem 2.1.

If n is odd, we consider $x = \lceil \frac{n}{2} \rceil$ and $x = \lfloor \frac{n}{2} \rfloor$. But, since these are complements of each other, we only consider $x = \lceil \frac{n}{2} \rceil$. By substituting x into the inequality (2.3) the proof is completed. □

3 Total Influence Number of Some Splitting Graphs

Definition 3.1. [11] For a graph G, the splitting graph $S(G)$ of graph G is obtained by adding a new vertex corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.

Let G be a graph of order n and $V(G) = \{v_1, v_2, ..., v_n\}$. For splitting graph of G of order $2n$, let $V(S(G)) = X \cup Y$, where $X = \{v_1, v_2, ..., v_n\}$, $Y = \{v'_1, v'_2, ..., v'_n\}$.

Theorem 3.2. *For a splitting graph of complete graph $S(K_n)$ with $n \geqslant 3$, a set S is an η_T–set if and only if it contains exactly n vertices which are in X or in Y. Furthermore,*

$$\eta_T(S(K_n)) = \frac{n(2n-1)}{4}.$$

Proof. For a vertex subset S, let $x = |X \cap S|$ and $y = y_1 + y_2 = |Y \cap S|$ and $f(x, y_1, y_2) := \eta_T(S)$, where y_1 and y_2 are the number of vertices corresponding to x vertices and not corresponding to x vertices, respectively. This yields the following equation with the bounds $0 \leqslant x \leqslant n$, $0 \leqslant y_1 \leqslant x$ and $0 \leqslant y_2 \leqslant n-x$:

$$\begin{aligned} f(x,y_1,y_2) =& \frac{1}{2}x(n-x) + \frac{1}{2}y_1(n-y_1-y_2) + \frac{1}{2}(x-y_1)(n-1-y_1-y_2) \\ &+ \frac{1}{4}(x-y_1) + \frac{1}{4}(y_1+y_2)(n-y_1-y_2) + \frac{1}{2}y_1(n-x) \\ &+ \frac{1}{2}y_2(n-1-x) + \frac{1}{4}y_2. \end{aligned}$$

Solving the system $f_x(x,y_1,y_2) = 0$, $f_{y_1}(x,y_1,y_2) = 0$, $f_{y_2}(x,y_1,y_2) = 0$ does not give critical points. Thus, we search to the maximum of $f(x,y_1,y_2)$ by looking at the boundaries for x, y_1 and y_2 and we do this search by Theorem 1.1.

Case 1. For $x = 0$, we maximize $f(0,y_1,y_2) = \frac{y_1}{4} - \frac{y_2}{4} - \frac{y_1y_2}{2} + \frac{3ny_1}{4} + \frac{3ny_2}{4} - \frac{y_1^2}{4} - \frac{y_2^2}{4}$ by setting $f_{y_1}(0,y_1,y_2) = 0$, $f_{y_2}(0,y_1,y_2) = 0$ and solving it for y_1 and y_2. Then, we do not find a solution and we must look at the boundaries for y_1 and y_2.

Case 1.1. For $y_1 = 0$ ($y_1 = x = 0$), we maximize $f(0,0,y_2) = -\frac{1}{4}y_2(y_2 - 3n + 1)$. Solving $f_{y_2}(0,0,y_2) = 0$ gives $y_2 = \frac{3n-1}{2} \notin [0,n]$. Thus, we look at the boundaries of y_2 and the function is maximized at $y_2 = n$ and $f(0,0,n) = \frac{2n^2-n}{4}$.

Case 1.2. For $y_2 = 0$ and $y_2 = n - x = n$, we maximize $f(0,y_1,0) = \frac{1}{4}y_1(3n - y_1 + 1)$ and $f(0,y_1,n) = -\frac{y_1^2}{4} + \frac{ny_1}{4} + \frac{y_1}{4} + \frac{n^2}{2} - \frac{n}{4}$, respectively. Since $0 \leqslant y_1 \leqslant x$ and $x = 0$, y_1 takes the unique value $y_1 = 0$. Thus, for the first function, we have $f(0,0,0) = 0$. Since $0 < |S| < 2n$, there is not a maximum value of this function. For the second function, we have $f(0,0,n) = \frac{2n^2-n}{4}$.

Case 2. For $x = n$, we maximize $f(n,y_1,y_2) = -\frac{y_1^2}{4} - \frac{y_1y_2}{2} - \frac{ny_1}{4} + \frac{y_1}{4} - \frac{y_2^2}{4} - \frac{ny_2}{4} - \frac{y_2}{4} + \frac{n^2}{4} - \frac{n}{4}$ by setting $f_{y_1}(n,y_1,y_2) = 0$, $f_{y_2}(n,y_1,y_2) = 0$ and solving for y_1 and y_2. Then we do not find a value and we must examine at the boundaries for y_1 and y_2.

Case 2.1. For $y_1 = 0$ and $y_1 = x = n$, we maximize $f(n,0,y_2) = -\frac{y_2^2}{4} - \frac{ny_2}{4} - \frac{y_2}{4} + \frac{n^2}{2} - \frac{n}{4}$ and $f(n,n,y_2) = -\frac{1}{4}y_2(y_2 + 3n + 1)$ by setting $f_{y_2}(n,0,y_2) = 0$ and $f_{y_2}(n,n,y_2) = 0$, and solving for y_2, respectively. We find $y_2 = -\frac{n+1}{2}$ and $y_2 = -\frac{3n+1}{2}$. But, they are not pozitif integer. Thus, for this case the function is maximized at $y_2 = 0$ and $f(n,0,0) = \frac{2n^2-n}{4}$.

Case 2.2. For $y_2 = 0$ ($y_2 = n - x = 0$), we have $f(n,y_1,0) = -\frac{1}{4}(y_1 - n)(y_1 + 2n - 1)$. Solving $f_{y_1}(n,y_1,0) = 0$ gives $y_1 = \frac{1-n}{2} \notin [0,n]$. The function is maximized at $y_1 = 0$.

Case 3. For $y_1 = 0$, we maximize $f(x,0,y_2) = \frac{3ny_2}{4} - \frac{x}{4} - \frac{y_2}{4} - xy_2 + nx - \frac{y_2^2}{4} - \frac{x^2}{2}$. Solving the system $f_x(x,0,y_2) = 0$, $f_{y_2}(x,0,y_2) = 0$ gives the solution $x = \frac{2n-1}{4}$ and $y_2 = \frac{n}{2}$. In this case, we look at $x = \lceil \frac{2n-1}{4} \rceil$, $x = \lfloor \frac{2n-1}{4} \rfloor$ and $y_2 = \lceil \frac{n}{2} \rceil$, $y_2 = \lfloor \frac{n}{2} \rfloor$ to determine the maximum integer solution. Consequently, the maximum of the function for these values is

$$\begin{cases} f(\lceil \frac{2n-1}{4} \rceil, 0, \frac{n}{2}) = \frac{7n^2}{16} - \frac{n}{4} & \text{if } n \text{ is even,} \\ f(\lceil \frac{2n-1}{4} \rceil, 0, \lceil \frac{n}{2} \rceil) = f(\lfloor \frac{2n-1}{4} \rfloor, 0, \lceil \frac{n}{2} \rceil) = \frac{7n^2}{16} - \frac{n}{4} + \frac{1}{16} & \text{if } n \text{ is odd.} \end{cases}$$

But, we must examine the maximum of the function at the boundaries for x and y_2.

Case 3.1. Examining at $x = 0$ and $x = n$ are equivalent to Case 1.1 and Case 2.1, respectively.

Case 3.2. For $y_2 = 0$, we maximize $f(x,0,0) = -\frac{1}{4}x(2x - 4n + 1)$ by setting $f_x(x,0,0) = 0$. Then, we find $x = n - \frac{1}{4} \in [0,n]$. Substituting $x = \lceil n - \frac{1}{4} \rceil$, $x = \lfloor n - \frac{1}{4} \rfloor$ into the function gives $f(\lceil n - \frac{1}{4} \rceil, 0, 0) = \frac{2n^2-n}{4}$, $f(\lfloor n - \frac{1}{4} \rfloor, 0, 0) = \frac{2n^2-n-1}{4}$. After doing examination at the boundaries of x, the function is maximized at $x = n$.

Case 3.3. For $y_2 = n - x$, maximizing $f(x,0,n-x) = \frac{n^2}{2} - \frac{nx}{4} - \frac{n}{4} + \frac{x^2}{4}$ gives $x = \frac{n}{2} \in [0,n]$. Then, the maximum of the function for this value is $f(\frac{n}{2}, 0, \frac{n}{2}) = \frac{7n^2}{16} - \frac{n}{4}$ for n is even; $f(\lceil \frac{n}{2} \rceil, 0, n - \lceil \frac{n}{2} \rceil) = \frac{7n^2}{16} - \frac{n}{4} + \frac{1}{16}$ for n is odd. Examining at the boundaries of x, the function is maximized at $x = 0$ and $x = n$.

Case 4. For $y_1 = x$, we maximize $f(x,x,y_2) = \frac{3ny_2}{4} - \frac{y_2}{4} - \frac{3xy_2}{2} + \frac{7nx}{4} - \frac{y_2^2}{4} - \frac{7x^2}{4}$. And solving $f_x(x,x,y_2) = 0$, $f_{y_2}(x,x,y_2) = 0$ gives $x = \frac{2n-3}{4} \in [0,n]$, $y_2 = \frac{7}{8} \notin \mathbb{Z}^+$. Examining at the boundaries gives the maximum of the function as $f(0,0,n) = \frac{2n^2-n}{4}$.

Case 5. For $y_2 = 0$, we maximize $f(x,y_1,0) = \frac{y_1}{4} - \frac{x}{4} + \frac{3ny_1}{4} - xy_1 + nx - \frac{y_1^2}{4} - \frac{x^2}{2}$. Solving $f_x(x,y_1,0) = 0$, $f_{y_1}(x,y_1,0) = 0$ gives $x = \frac{2n+3}{4} \in [0,n]$ and $y_1 = \frac{n-2}{2} \in [0,x]$. We substitute $y_1 = \lceil \frac{n-2}{2} \rceil$, $y_1 = \lfloor \frac{n-2}{2} \rfloor$ and $x = \lceil \frac{2n+3}{4} \rceil$, $x = \lfloor \frac{2n+3}{4} \rfloor$ into the function to determine the maximum integer solution and after examining at the boundaries for x and y_1, we find the maximum value of the function as $f(n,0,0) = \frac{2n^2-n}{4}$.

From all cases, the total influence number of $S(K_n)$ is

$$\eta_T(S(K_n)) = f(n,0,0) = f(0,0,n) = \frac{n(2n-1)}{4}.$$

□

Theorem 3.3. *For a splitting graph of star graph $S(K_{1,n-1})$, a set S is an η_T–set if and only if it contains exactly n vertices such that $(n-1)$ vertices are in X and one vertex is in Y and center vertex of $K_{1,n-1}$ and its corresponding vertex is not in S or one vertex is in X, $(n-1)$ vertices are in Y and both center vertex of $K_{1,n-1}$ and its corresponding vertex are in S. Furthermore,*

$$\eta_T(S(K_{1,n-1})) = \frac{2n^2+4n-3}{8}.$$

Proof. For a vertex subset S, let $x = |X \cap S|$ and $y = |Y \cap S|$. For the vertices $v_i \in X$ and $v_i' \in Y$, where $i \in \{1,2,...,n\}$, we consider three cases depending on center vertex (v_1) and corresponding center vertex's (v_1') membership in S or not in S.
Case 1. Let $S \subseteq X$ and $f(x) := \eta_T(S)$.
Case 1.1. Let $v_1 \in S$. Then, the bound is $1 \leqslant x \leqslant n$ and we have

$$f(x) = \frac{1}{2}(n-x) + \frac{1}{4}(x-1)(n-x) + \frac{1}{2}(n-1) + \frac{1}{4} + \frac{1}{2}(x-1) + \frac{1}{4}(x-1)(n-1).$$

Solving $f'(x) = 0$ gives $x = n$. Thus, $f(n) = \frac{n^2+2n-2}{4}$.
Case 1.2. Let $v_1 \notin S$. Then, the bound is $1 \leqslant x \leqslant n-1$ and we have

$$f(x) = \frac{1}{2}x + \frac{1}{4}x(n-1-x) + \frac{1}{2}x + \frac{1}{2}x(n-1).$$

Solving $f'(x) = 0$ does not give a solution. Thus, we look at the boundaries of x. The function is maximized at $x = n-1$ and $f(n-1) = \frac{n^2-1}{2}$.
Case 2. Let $S \subseteq Y$ and $f(y) := \eta_T(S)$.
Case 2.1. Let $v_1' \in S$. Then, the bound is $1 \leqslant y \leqslant n$ and we have

$$f(y) = \frac{1}{8}(n-y) + \frac{1}{4}(y-1)(n-y) + \frac{1}{2}(n-1) + \frac{1}{4} + \frac{1}{2}(y-1) + \frac{1}{4}(y-1)(n-1).$$

Solving $f'(y) = 0$ gives $y = n + \frac{3}{4}$. For $y = \lfloor n + \frac{3}{4} \rfloor = n$, we have $n + \frac{3}{4} \in [1,n]$. Hence, the function is maximized at $y = n$ and $f(n) = \frac{n^2+2n-2}{4}$.
Case 2.2. Let $v_1' \notin S$. Then, the bound is $1 \leqslant y \leqslant n-1$ and we have

$$f(y) = \frac{1}{8}y + \frac{1}{4}y(n-1-y) + \frac{1}{2}y + \frac{1}{4}y(n-1).$$

Solving $f'(y) = 0$ gives $y = n + \frac{1}{4}$. But, it is out of the range $[1, n-1]$. Thus, we have the maximum of $f(y)$ at the boundary of y and $f(n-1) = \frac{2n^2+n-3}{8}$.
Case 3. Let $S \cap X \neq \emptyset$, $S \cap Y \neq \emptyset$ and $f(x,y) := \eta_T(S)$.

For the vertices v_1 and v_1', we consider two subcases: firstly $v_1, v_1' \in S$ and secondly $v_1 \in S$, $v_1' \notin S$. Since the complements of these cases cover each of the four combinations of set

membership, these two cases are comprehensive.
Case 3.1. Let $v_1, v_1' \in S$. For this case, we have

$$f(x,y) = \frac{1}{2}(n-x) + \frac{1}{2}(n-y) + \frac{1}{4}(x-1)(n-x) + \frac{1}{4}(x-1)(n-y) + \frac{1}{8}(n-y) + \frac{1}{4}(y-1)(n-y) + \frac{1}{2}(n-x) + \frac{1}{4}(y-1)(n-x)$$

with the bounds $1 \leqslant x \leqslant n$ and $1 \leqslant y \leqslant n$.

Solving the system $f_x(x,y) = 0, f_y(x,y) = 0$ does not give a solution. Therefore, we must seek the maximum of $f(x,y)$ at the boundaries for x and y.
Case 3.1.1. For $x = 1$, we maximize $f(1,y) = \frac{9n}{8} - \frac{5y}{8} + \frac{ny}{2} - \frac{y^2}{4} - \frac{3}{4}$ by setting $f_y(1,y) = 0$ and solving for y. We find $y = n - \frac{5}{4}$ and look at $y = \lceil n - \frac{5}{4} \rceil, y = \lfloor n - \frac{5}{4} \rfloor$ for integer solution. By considering the boundaries of y, the function is maximized at $y = \lfloor n - \frac{5}{4} \rfloor = n-1$ and $f(1, n-1) = \frac{2n^2+4n-3}{8}$.
Case 3.1.2. For $x = n$, we maximize $f(n,y) = \frac{1}{8}(n-y)(2n+2y+1)$. Solving $f_y(n,y) = 0$ gives $y = -\frac{1}{4} \notin \mathbb{Z}^+$. By looking at the boundaries of y, we have $f(n,0) = \frac{2n^2+n}{8}$.
Case 3.1.3. For $y = 1$, we maximize $f(x,1) = \frac{9n}{8} - x + \frac{nx}{2} - \frac{x^2}{4} - \frac{3}{8}$ by setting $f_x(x,1) = 0$ and solving for x. We find $x = n-2$. Then, the function is maximized at $x = n-2$ and $f(n-2, 1) = \frac{2n^2+n+5}{8}$.
Case 3.1.4. For $y = n$, solving $f_x(x,n) = 0$ gives $x = -1 \notin [1,n]$. Then, searching at boundaries of x gives $f(1,n) = \frac{2n^2+4n-6}{8}$.
Case 3.2. Let $v_1 \in S$ and $v_1' \notin S$. For this case, the bounds are $1 \leqslant x \leqslant n, 1 \leqslant y \leqslant n-1$ and the function is as follows:

$$f(x,y) = \frac{1}{2}(n-x) + \frac{1}{4}(x-1)(n-x) + \frac{1}{2}(n-1-y) + \frac{1}{4} + \frac{1}{2}(x-1) + \frac{1}{4}(x-1)(n-1-y) + \frac{1}{4}y(n-1-y) + \frac{1}{8}y + \frac{1}{4}y(n-x).$$

Solving the system $f_x(x,y) = 0, f_y(x,y) = 0$ does not give a solution. Therefore, we must search the maximum of $f(x,y)$ at the boundaries for x and y.

We examine along $x = 1, x = n, y = 1, y = n-1$ similarly to above and we find $f(1, n-2) = \frac{n^2}{4} + \frac{n}{8}$ as the maximum value of the function.

From all cases, consequently we have

$$\eta_T(S(K_{1,n-1})) = f(1, n-1) = f(n-1, 1) = \frac{2n^2+4n-3}{8}.$$

□

Theorem 3.4. *Total influence number of a splitting graph of path, $S(P_n)$, with $n \geqslant 4$ is*

$$\eta_T(S(P_n)) = \begin{cases} \frac{(8)2^{-n}+6n-10}{3} + \frac{n-1}{8} + \frac{4+4^{\lfloor \frac{n}{2} \rfloor}(3\lfloor \frac{n}{2} \rfloor - 4)}{9(2^{n-1})} & \text{if } n \text{ is odd,} \\ \frac{(10)2^{-n}+6n-10}{3} + \frac{n-1}{8} + \frac{20+2^n(3n-11)}{9(2^{n+1})} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Vertex set of $S(P_n)$ can be partitioned into vertex set of two paths. Let $V(S(P_n)) = V(P_n^{(1)}) \cup V(P_n^{(2)})$ and for a vertex subset $S \subseteq V(S(P_n))$, let $S_1 = X \cap S$, $S_2 = Y \cap S$, $\overline{S_1} = X - S_1$, $\overline{S_2} = Y - S_2$. Thus, by Theorem 2.2

$$\eta_T(S(P_n)) = \eta_T(S_1, \overline{S_1}) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_2})$$

and considering P_n and the total influence set of P_n gives

$$\eta_T(S(P_n)) = 3\eta_T(P_n) + \eta_T(S_2, \overline{S_2}).$$

Let n be even and $V(P_n^{(1)}) = \{v_1, v_2', v_3, v_4', ..., v_{n-1}, v_n'\}$, $V(P_n^{(2)}) = \{v_1', v_2, v_3', v_4, ..., v_{n-1}', v_n\}$. By Theorem 2.3, $S_1 = \{v_1, v_3, v_5, ..., v_{n-3}, v_{n-1}\}$, $S_2 = \{v_1', v_3', v_5', ..., v_{n-3}', v_{n-1}'\}$.

$$\begin{aligned}\eta_T(S_2, \overline{S_2}) &= 2\frac{1}{2^3} + \sum_{i=1}^{\frac{n}{2}-1}\left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + ... + \frac{1}{2^{n-(2i-1)}}\right) \\ &+ \sum_{i=2}^{\frac{n}{2}-1}\left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + ... + \frac{1}{2^{n-(2i-1)}}\right) \\ &= 2\sum_{i=1}^{\frac{n}{2}-1}\left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + ... + \frac{1}{2^{n-(2i-1)}}\right) - \sum_{i=2}^{\frac{n}{2}-1}\frac{1}{2^{2i+1}} \\ &= \frac{n-1}{2^3} + \frac{1}{2^{n+1}}\sum_{i=1}^{\frac{n}{2}-1}(2i-1)2^{2i}.\end{aligned}$$

Thus, we have

$$\eta_T(S(P_n)) = 3\eta_T(P_n) + \frac{n-1}{8} + \frac{1}{2^{n+1}}\sum_{i=1}^{\frac{n}{2}-1}(2i-1)2^{2i}. \tag{3.1}$$

It is easy to see that $\sum_{i=1}^{\frac{n}{2}-1}(2i-1)2^{2i} = \frac{20+2^n(3n-11)}{9}$. By substituting this formula into (3.1) and using Corollary 2.4, this case is proved.

Let n be odd and $V(P_n^{(1)}) = \{v_1, v_2', v_3, v_4', ..., v_{n-1}', v_n\}$, $V(P_n^{(2)}) = \{v_1', v_2, v_3', v_4, ..., v_{n-1}, v_n'\}$. By Theorem 2.3, $S_1 = \{v_1, v_3, v_5, ..., v_{n-2}, v_n\}$, $S_2 = \{v_1', v_3', v_5', ..., v_{n-2}', v_n'\}$.

$$\begin{aligned}\eta_T(S_2, \overline{S_2}) &= \frac{1}{2^3} + \frac{1}{2^3} + 2\sum_{i=2}^{\lfloor\frac{n}{2}\rfloor-1}\left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + ... + \frac{1}{2^{n-2i}}\right) \\ &= \frac{n-1}{2^3} + 2\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor-1} i\frac{1}{2^{n-2i}} = \frac{n-1}{2^3} + \frac{1}{2^{n-1}}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor-1} i2^{2i}.\end{aligned}$$

Then, we have

$$\begin{aligned}\eta_T(S(P_n)) &= 3\eta_T(P_n) + \frac{n-1}{8} + \frac{1}{2^{n-1}}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor-1} i2^{2i} \\ &= 3\eta_T(P_n) + \frac{n-1}{8} + \frac{4+4^{\lfloor\frac{n}{2}\rfloor}(3\lfloor\frac{n}{2}\rfloor-4)}{9(2^{n-1})}\end{aligned}$$

By Corollary 2.4, the proof is completed. □

Theorem 3.5. *Total influence number of a splitting graph of cycle, $S(C_n)$, with $n \geqslant 12$ is*

$$\eta_T(S(C_n)) = \begin{cases} \eta_T(S(P_{\lceil\frac{n-2}{2}\rceil})) + \eta_T(S(P_{\lfloor\frac{n-2}{2}\rfloor})) - \frac{72\lfloor\frac{n}{4}\rfloor+120}{9(4^{\lfloor\frac{n}{4}\rfloor})} + \frac{299}{24} & \text{if } n \text{ is odd,} \\ \begin{cases} 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor\frac{n}{4}\rfloor+128}{9(4^{\lfloor\frac{n}{4}\rfloor})} + \frac{2n}{2^{\frac{n}{2}}} + \frac{229}{18} & \text{if } \frac{n}{2} \text{ is odd} \\ 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor\frac{n}{4}\rfloor+128}{9(4^{\lfloor\frac{n}{4}\rfloor})} + \frac{229}{18} & \text{if } \frac{n}{2} \text{ is even} \end{cases} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $V(S(C_n)) = X \cup Y = (X_1 \cup X_2) \cup (Y_1 \cup Y_2)$, where $X_1 = \{v_1, v_2, ..., v_{\lfloor\frac{n}{2}\rfloor}\}$, $X_2 = \{v_{\lfloor\frac{n}{2}\rfloor+1}, v_{\lfloor\frac{n}{2}\rfloor+2}, ..., v_n\}$ and $Y_1 = \{v_1', v_2', ..., v_{\lfloor\frac{n}{2}\rfloor}'\}$, $Y_2 = \{v_{\lfloor\frac{n}{2}\rfloor+1}', v_{\lfloor\frac{n}{2}\rfloor+2}', ..., v_n'\}$.

Vertex set of $S(C_n)$ can be partitioned as $V(S(C_n)) = V(S(P_{n-2})) \cup \{v_1, v_n, v_1', v_n'\}$, where $V(S(P_{n-2})) = V(S(P_{\lfloor\frac{n-2}{2}\rfloor})) \cup V(S(P_{\lceil\frac{n-2}{2}\rceil}))$. For a subset $S \subseteq V(S(P_{n-2}))$, $S_1, S_3 \subseteq$

$V(S(P_{\lfloor \frac{n-2}{2} \rfloor}))$ and $S_2, S_4 \subseteq V(S(P_{\lceil \frac{n-2}{2} \rceil}))$, let $S_1 = X_1 \cap S$, $S_2 = X_2 \cap S$, $S_3 = Y_1 \cap S$, $S_4 = Y_2 \cap S$, $\overline{S_1} = X_1 - S_1$, $\overline{S_2} = X_2 - S_2$, $\overline{S_3} = Y_1 - S_3$, $\overline{S_4} = Y_2 - S_4$, $S = S_1 \cup S_2 \cup S_3 \cup S_4$. By Theorem 2.3, if n is even, $S_1 = \{v_2, v_4, v_6, ..., v_{\frac{n-2}{2}}\}$, $S_2 = \{v_{\frac{n+2}{2}}, v_{\frac{n+2}{2}+2}, v_{\frac{n+2}{2}+4}, ..., v_{n-2}\}$, $S_3 = \{v'_2, v'_4, v'_6, ..., v'_{\frac{n-2}{2}}\}$, $S_4 = \{v'_{\frac{n+2}{2}}, v'_{\frac{n+2}{2}+2}, v'_{\frac{n+2}{2}+4}, ..., v'_{n-2}\}$ and if n is odd, $S_1 = \{v_2, v_4, v_6, ..., v_{\lfloor \frac{n}{2} \rfloor}\}$, $S_2 = \{v_{\lfloor \frac{n}{2} \rfloor+2}, v_{\lfloor \frac{n}{2} \rfloor+4}, v_{\lfloor \frac{n}{2} \rfloor+6}, ..., v_{n-1}\}$, $S_3 = \{v'_2, v'_4, v'_6, ..., v'_{\lfloor \frac{n}{2} \rfloor}\}$, $S_4 = \{v'_{\lfloor \frac{n}{2} \rfloor+2}, v'_{\lfloor \frac{n}{2} \rfloor+4}, v'_{\lfloor \frac{n}{2} \rfloor+6}, ..., v'_{n-1}\}$.

Let S' be a total influence set of $S(C_n)$, where $S' \subseteq V(S(C_n))$ $(S \subseteq S')$. We consider three cases: when n is odd; when n is even and $\frac{n}{2}$ is odd; when n is even and $\frac{n}{2}$ is even. For each cases, we examine the total influence number of $S(C_n)$ depending on v_1, v_n, v'_1, v'_n's membership in S' or not in S'.

For abbreviation, we use d instead of $diam(S(C_n))$.

Case 1. Let n be even and $\frac{n}{2}$ be odd.

- Let $S = S'$. Then,

$$\begin{aligned}\eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3}) \\ &+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(u_1, S) \\ &+ \eta_T(u_n, S) + \eta_T(v_1, S) + \eta_T(v_n, S).\end{aligned} \tag{3.2}$$

By definition of the total influence number, we find following equalities:

$$\begin{aligned}\eta_T(S_1, \overline{S_2}) &= 2\frac{1}{2^3} + 4\frac{1}{2^5} + 6\frac{1}{2^7} + ... + (d-3)\frac{1}{2^{d-2}} + \frac{d-1}{2^{d+1}} \\ &= \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} 2i\frac{1}{2^{2i+1}} + (d-1)\frac{1}{2^{d+1}}.\end{aligned}$$

$$\begin{aligned}\eta_T(S_2, \overline{S_1}) &= \frac{1}{2} + 2\frac{1}{2^3} + 4\frac{1}{2^5} + 6\frac{1}{2^7} + ... + (d-3)\frac{1}{2^{d-2}} + \frac{d-3}{2^{d+1}} \\ &= \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} 2i\frac{1}{2^{2i+1}} + (d-3)\frac{1}{2^{d+1}} + \frac{1}{2}.\end{aligned}$$

$$\eta_T(S_4, \overline{S_3}) = \frac{1}{2^3} + 2\frac{1}{2^3} + 4\frac{1}{2^5} + 6\frac{1}{2^7} + ... + (d-3)\frac{1}{2^{d-2}} + \frac{d-3}{2^{d+1}} = \eta_T(S_2, \overline{S_1}) - \frac{3}{8}.$$

$$\eta_T(S_1, \overline{S_4}) = \eta_T(S_3, \overline{S_2}) = \eta_T(S_3, \overline{S_4}) = \eta_T(S_1, \overline{S_2}).$$

$$\eta_T(S_2, \overline{S_3}) = \eta_T(S_4, \overline{S_1}) = \eta_T(S_2, \overline{S_1}).$$

$$\eta_T(v_1, S) = 2\frac{1}{2} + 4\frac{1}{2^3} + 4\frac{1}{2^5} + 4\frac{1}{2^7} + ... + 4\frac{1}{2^{d-2}} + 2\frac{1}{2^d} = 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{2^{2i+1}} + \frac{2}{2^d} + 1.$$

$$\eta_T(v'_1, S) = 2\frac{1}{2} + 4\frac{1}{2^3} + 4\frac{1}{2^5} + 4\frac{1}{2^7} + ... + 4\frac{1}{2^{d-2}} + 2\frac{1}{2^d} = \eta_T(v_1, S) - \frac{3}{8}.$$

$$\eta_T(v_n, S) = \eta_T(v'_n, S) = 4\frac{1}{2^2} + 4\frac{1}{2^4} + 4\frac{1}{2^6} + ... + 4\frac{1}{2^{d-1}} = 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}}.$$

It is obvious that, $diam(S(C_n)) = d = \frac{n}{2}$ if n is even. Thus, we have

$$\begin{cases} \eta_T(S_1, \overline{S_2}) = \frac{4(4^{\lfloor \frac{n}{4} \rfloor} - 3\lfloor \frac{n}{4} \rfloor - 1)}{9(4^{\lfloor \frac{n}{4} \rfloor})} + (\frac{n}{2} - 1)\frac{1}{2^{\frac{n}{2}+1}} \\ \eta_T(S_2, \overline{S_1}) = \frac{4(4^{\lfloor \frac{n}{4} \rfloor} - 3\lfloor \frac{n}{4} \rfloor - 1)}{9(4^{\lfloor \frac{n}{4} \rfloor})} + (\frac{n}{2} - 3)\frac{1}{2^{\frac{n}{2}+1}} \\ \eta_T(S_4, \overline{S_3}) = \frac{4(4^{\lfloor \frac{n}{4} \rfloor} - 3\lfloor \frac{n}{4} \rfloor - 1)}{9(4^{\lfloor \frac{n}{4} \rfloor})} + (\frac{n}{2} - 3)\frac{1}{2^{\frac{n}{2}+1}} - \frac{3}{8} \\ \eta_T(S_1, \overline{S_4}) = \eta_T(S_3, \overline{S_2}) = \eta_T(S_3, \overline{S_4}) = \eta_T(S_1, \overline{S_2}) \\ \eta_T(S_2, \overline{S_3}) = \eta_T(S_4, \overline{S_1}) = \eta_T(S_2, \overline{S_1}). \end{cases} \tag{3.3}$$

$$\begin{cases} \eta_T(v_1, S) = \frac{2(4^{\lfloor \frac{n}{4} \rfloor}-4)}{3(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2}{2^{\frac{n}{2}}} + 1 \\ \eta_T(v_1', S) = \frac{2(4^{\lfloor \frac{n}{4} \rfloor}-4)}{3(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2}{2^{\frac{n}{2}}} + \frac{5}{8} \\ \eta_T(v_n, S) = \eta_T(v_n', S) = \frac{4(4^{\lfloor \frac{n}{4} \rfloor}-1)}{3(4^{\lfloor \frac{n}{4} \rfloor})}. \end{cases} \tag{3.4}$$

Substituting (3.3) and (3.4) into the equality (3.2), we get

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 200}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-4}{2^{\frac{n}{2}}} + \frac{197}{18}.$$

- Let $S' = \{v_1\} \cup S$. Then,

$$\begin{aligned} \eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3}) \\ &+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(v_1, \overline{S'}) \\ &+ \eta_T(v_n, S) + \eta_T(v_1', S) + \eta_T(v_n', S). \end{aligned}$$

Since

$$\begin{aligned} \eta_T(v_1, \overline{S'}) &= 2\frac{1}{2} + \frac{1}{2^2} + 4\frac{1}{2^2} + 4\frac{1}{2^4} + ... + 4\frac{1}{2^{d-1}} \\ &= \frac{5}{4} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}} = \frac{4(4^{\lfloor \frac{n}{4} \rfloor} - 1)}{3(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{5}{4}, \end{aligned} \tag{3.5}$$

substituting (3.3), (3.4) and (3.5) into the above equality gives

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 92}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-6}{2^{\frac{n}{2}}} + \frac{211}{18}.$$

- Let $S' = \{v_1'\} \cup S$. This case is equavilent to Case 2.
- Let $S' = \{v_n\} \cup S$. Then,

$$\begin{aligned} \eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3}) \\ &+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(v_1, S) \\ &+ \eta_T(v_n, \overline{S'}) + \eta_T(v_1', S) + \eta_T(v_n', S). \end{aligned}$$

Since

$$\begin{aligned} \eta_T(v_n, \overline{S'}) &= \frac{1}{2^2} + 2\frac{1}{2^d} + 4\frac{1}{2} + 4\frac{1}{2^3} + ... + 4\frac{1}{2^{d-2}} \\ &= \frac{9}{4} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{2^{2i+1}} = \frac{2(4^{\lfloor \frac{n}{4} \rfloor} - 4)}{3(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2}{2^{\frac{n}{2}}} + \frac{9}{4}, \end{aligned} \tag{3.6}$$

substituting (3.3), (3.4) and (3.6) into the above equality gives

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 116}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-2}{2^{\frac{n}{2}}} + \frac{223}{18}.$$

- Let $S' = \{v_n'\} \cup S$. Then,

$$\begin{aligned} \eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3}) \\ &+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(v_1, S) \\ &+ \eta_T(v_n, S) + \eta_T(v_1', S) + \eta_T(v_n', \overline{S'}). \end{aligned}$$

$$\eta_T(v'_n, \overline{S'}) = 2\frac{1}{2} + \frac{1}{2^2} + 2\frac{1}{2^3} + \frac{2}{2^d} + 4\frac{1}{2^3} + 4\frac{1}{2^5} + ... + 4\frac{1}{2^{d-2}}$$
$$= \frac{3}{2} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{2^{2i+1}} = \frac{2(4^{\lfloor \frac{n}{4} \rfloor} - 4)}{3(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2}{2^{\frac{n}{2}}} + \frac{3}{2}. \tag{3.7}$$

Substituting (3.3), (3.4) and (3.7) into the above equality, we get

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 116}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-2}{2^{\frac{n}{2}}} + \frac{419}{36}.$$

- Let $S' = \{v_1, v'_1\} \cup S$. Then,

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3})$$
$$+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(v_1, \overline{S'})$$
$$+ \eta_T(v_n, S) + \eta_T(v'_1, \overline{S'}) + \eta_T(v'_n, S).$$

$$\begin{cases} \eta_T(v_1, \overline{S'}) = 1 + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}} = 1 + \frac{4(4^{\lfloor \frac{n}{4} \rfloor}-1)}{3(4^{\lfloor \frac{n}{4} \rfloor})} \\ \eta_T(v'_1, \overline{S'}) = \frac{5}{8} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}} = \frac{5}{8} + \frac{4(4^{\lfloor \frac{n}{4} \rfloor}-1)}{3(4^{\lfloor \frac{n}{4} \rfloor})}. \end{cases} \tag{3.8}$$

Substituting (3.3), (3.4) and (3.8) into the above equality, we get

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 80}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-8}{2^{\frac{n}{2}}} + \frac{437}{36}.$$

- Let $S' = \{v_1, v_n\} \cup S$. Then,

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3})$$
$$+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(v_1, \overline{S'})$$
$$+ \eta_T(v_n, \overline{S'}) + \eta_T(v'_1, S) + \eta_T(v'_n, S).$$

$$\begin{cases} \eta_T(v_1, \overline{S'}) = \frac{3}{4} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}} = \frac{3}{4} + \frac{4(4^{\lfloor \frac{n}{4} \rfloor}-1)}{3(4^{\lfloor \frac{n}{4} \rfloor})} \\ \eta_T(v_n, \overline{S'}) = \frac{7}{4} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{2^{2i+1}} + \frac{2}{2^d} = \frac{7}{4} + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor \frac{n}{4} \rfloor}-4)}{3(4^{\lfloor \frac{n}{4} \rfloor})} \end{cases} \tag{3.9}$$

Substituting (3.3), (3.4) and (3.9) into the above equality, we get

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 200}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-4}{2^{\frac{n}{2}}} + \frac{923}{72}.$$

- Let $S' = \{v_1, v'_n\} \cup S$. Then,

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3})$$
$$+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3}) + \eta_T(v_1, \overline{S'})$$
$$+ \eta_T(v_n, S) + \eta_T(v'_1, S) + \eta_T(v'_n, \overline{S'}).$$

$$\begin{cases} \eta_T(v_1, \overline{S'}) = \frac{3}{4} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}} = \frac{3}{4} + \frac{4(4^{\lfloor \frac{n}{4} \rfloor}-1)}{3(4^{\lfloor \frac{n}{4} \rfloor})} \\ \eta_T(v'_n, \overline{S'}) = 1 + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{2^{2i+1}} = 1 + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor \frac{n}{4} \rfloor}-4)}{3(4^{\lfloor \frac{n}{4} \rfloor})}. \end{cases} \tag{3.10}$$

Substituting (3.3), (3.4) and (3.10) into the above equality, we get

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 200}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-4}{2^{\frac{n}{2}}} + \frac{421}{36}.$$

- Let $S' = \{v_n, v'_n\} \cup S$. Then,

$$\begin{aligned}\eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1,\overline{S_2}) + \eta_T(S_1,\overline{S_4}) + \eta_T(S_2,\overline{S_1}) + \eta_T(S_2,\overline{S_3}) \\ &+ \eta_T(S_3,\overline{S_2}) + \eta_T(S_3,\overline{S_4}) + \eta_T(S_4,\overline{S_1}) + \eta_T(S_4,\overline{S_3}) + \eta_T(v_1,S) \\ &+ \eta_T(v_n,\overline{S'}) + \eta_T(v'_1,S) + \eta_T(v'_n,\overline{S'}).\end{aligned}$$

$$\begin{cases}\eta_T(v_n,\overline{S'}) = 2 + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor-1}\frac{1}{2^{2i+1}} = 2 + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor\frac{n}{4}\rfloor}-4)}{3(4^{\lfloor\frac{n}{4}\rfloor})} \\ \eta_T(v'_n,\overline{S'}) = \frac{5}{4} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor-1}\frac{1}{2^{2i+1}} = \frac{5}{4} + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor\frac{n}{4}\rfloor}-4)}{3(4^{\lfloor\frac{n}{4}\rfloor})}.\end{cases} \tag{3.11}$$

We substitute (3.3), (3.4) and (3.11) into the above equality and we have

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor\frac{n}{4}\rfloor + 128}{9(4^{\lfloor\frac{n}{4}\rfloor})} + \frac{2n}{2^{\frac{n}{2}}} + \frac{229}{18}.$$

- Let $S' = \{v'_1, v'_n\} \cup S$. Then,

$$\begin{aligned}\eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1,\overline{S_2}) + \eta_T(S_1,\overline{S_4}) + \eta_T(S_2,\overline{S_1}) + \eta_T(S_2,\overline{S_3}) \\ &+ \eta_T(S_3,\overline{S_2}) + \eta_T(S_3,\overline{S_4}) + \eta_T(S_4,\overline{S_1}) + \eta_T(S_4,\overline{S_3}) + \eta_T(v_1,S) \\ &+ \eta_T(v_n,S) + \eta_T(v'_1,\overline{S'}) + \eta_T(v'_n,\overline{S'}).\end{aligned}$$

$$\begin{cases}\eta_T(v'_1,\overline{S'}) = \frac{3}{4} + 4\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor}\frac{1}{2^{2i}} = \frac{3}{4} + \frac{4(4^{\lfloor\frac{n}{4}\rfloor}-1)}{3(4^{\lfloor\frac{n}{4}\rfloor})} \\ \eta_T(v'_n,\overline{S'}) = \frac{11}{8} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor-1}\frac{1}{2^{2i+1}} = \frac{11}{8} + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor\frac{n}{4}\rfloor}-4)}{3(4^{\lfloor\frac{n}{4}\rfloor})}.\end{cases} \tag{3.12}$$

Substituting (3.3),(3.4) and (3.12) into the above equality gives

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor\frac{n}{4}\rfloor + 200}{9(4^{\lfloor\frac{n}{4}\rfloor})} + \frac{2n-4}{2^{\frac{n}{2}}} + \frac{224}{18}.$$

- Let $S' = \{v'_1, v_n\} \cup S$. Then,

$$\begin{aligned}\eta_T(S(C_n)) =& 2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1,\overline{S_2}) + \eta_T(S_1,\overline{S_4}) + \eta_T(S_2,\overline{S_1}) + \eta_T(S_2,\overline{S_3}) \\ &+ \eta_T(S_3,\overline{S_2}) + \eta_T(S_3,\overline{S_4}) + \eta_T(S_4,\overline{S_1}) + \eta_T(S_4,\overline{S_3}) + \eta_T(v_1,S) \\ &+ \eta_T(v_n,\overline{S'}) + \eta_T(v'_1,\overline{S'}) + \eta_T(v'_n,S).\end{aligned}$$

$$\begin{cases}\eta_T(v'_1,\overline{S'}) = \frac{3}{8} + 4\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor}\frac{1}{2^{2i}} = \frac{3}{8} + \frac{4(4^{\lfloor\frac{n}{4}\rfloor}-1)}{3(4^{\lfloor\frac{n}{4}\rfloor})} \\ \eta_T(v_n,\overline{S'}) = \frac{7}{4} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor-1}\frac{1}{2^{2i+1}} = \frac{7}{4} + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor\frac{n}{4}\rfloor}-4)}{3(4^{\lfloor\frac{n}{4}\rfloor})}.\end{cases} \tag{3.13}$$

Substituting (3.3), (3.4) and (3.13) into the above equality, we get

$$\eta_T(S(C_n)) = 2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor\frac{n}{4}\rfloor + 104}{9(4^{\lfloor\frac{n}{4}\rfloor})} + \frac{2n-4}{2^{\frac{n}{2}}} + \frac{443}{36}.$$

- Let $S' = \{v_1, v_n, v'_1\} \cup S$ and $S' = \{v_1, v'_1, v'_n\} \cup S$. Then, these cases are equivalent to being $S' = \{v_1\} \cup S$.

- Let $S' = \{v_1, v_n, v'_n\} \cup S$. Thus, we have

$$\begin{aligned}\eta_T(S(C_n)) =&2\eta_T(S(P_{\frac{n-2}{2}})) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_1, \overline{S_4}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_3})\\ &+ \eta_T(S_3, \overline{S_2}) + \eta_T(S_3, \overline{S_4}) + \eta_T(S_4, \overline{S_1}) + \eta_T(S_4, \overline{S_3})\\ &+ \eta_T(v_1, \overline{S'}) + \eta_T(v_n, \overline{S'}) + \eta_T(v_1', S) + \eta_T(v_n', \overline{S'}).\end{aligned}$$

$$\begin{cases}\eta_T(v_1, \overline{S'}) = \frac{1}{4} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2^{2i}} = \frac{1}{4} + \frac{4(4^{\lfloor \frac{n}{4} \rfloor}-1)}{3(4^{\lfloor \frac{n}{4} \rfloor})}\\ \eta_T(v_n, \overline{S'}) = \frac{6}{4} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor-1} \frac{1}{2^{2i+1}} = \frac{6}{4} + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor \frac{n}{4} \rfloor}-4)}{3(4^{\lfloor \frac{n}{4} \rfloor})}\\ \eta_T(v_n', \overline{S'}) = \frac{3}{4} + \frac{2}{2^d} + 4\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor-1} \frac{1}{2^{2i+1}} = \frac{3}{4} + \frac{2}{2^{\frac{n}{2}}} + \frac{2(4^{\lfloor \frac{n}{4} \rfloor}-4)}{3(4^{\lfloor \frac{n}{4} \rfloor})}\end{cases} \tag{3.14}$$

Substituting (3.3), (3.4) and (3.14) into the above equality, we get

$$\eta_T(S(C_n)) =2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 116}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n-4}{2^{\frac{n}{2}}} + \frac{419}{36}.$$

- Let $S' = \{v_n, v_1', v_n'\} \cup S$. This case is equivalent to being $S' = \{v_n\} \cup S$.
- Let $S' = \{v_1, v_n, v_1', v_n'\} \cup S$. This case is equivalent to being $S' = S$.

Analysing all sets gives

$$\eta_T(S(C_n)) =2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 128}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{2n}{2^{\frac{n}{2}}} + \frac{229}{18} \quad \text{for } n \geqslant 10.$$

Case 2. Let n be even and $\frac{n}{2}$ be even. When we prove similarly to the proof of Case 1, we have $S' = \{v_n, v_n'\} \cup S$ and

$$\eta_T(S(C_n)) =2\eta_T(S(P_{\frac{n-2}{2}})) - \frac{96\lfloor \frac{n}{4} \rfloor + 128}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{229}{18} \quad \text{for } n \geqslant 12.$$

Case 3. Let n be odd. Examining all cases of S' similar to the proof of Case 1 gives $S' = \{v_1\} \cup S$ or $S' = \{v_1'\} \cup S$ or $S' = \{v_n\} \cup S$ or $S' = \{v_n'\} \cup S$ and

$$\eta_T(S(C_n)) =\eta_T(S(P_{\lceil \frac{n-2}{2} \rceil})) + \eta_T(S(P_{\lfloor \frac{n-2}{2} \rfloor})) - \frac{72\lfloor \frac{n}{4} \rfloor + 120}{9(4^{\lfloor \frac{n}{4} \rfloor})} + \frac{299}{24} \quad \text{for } n \geqslant 13.$$

Consequently, comparing the results gives the theorem as stated. □

Theorem 3.6. *For the graph $S(K_{n,m})$ with $n \leqslant m$ and $n, m \geqslant 4$, a set S is an η_T-set if and only if, for $k \in \mathbb{Z}^+$,*

$$(x_1, x_2, y_1, y_2) = \begin{cases}\begin{cases}(0, m, 0, \frac{5n}{4}) & n = 4k,\ n \text{ even}\\ (0, m, 0, \lceil \frac{5n}{4} \rceil) \text{ or } (0, m, 0, \lfloor \frac{5n}{4} \rfloor) & n \neq 4k,\ n \text{ even}\\ (0, m, 0, \lfloor \frac{5n}{4} \rfloor) & n = 4k+1,\ n \text{ odd}\\ (0, m, 0, \lceil \frac{5n}{4} \rceil) & n \neq 4k+1,\ n \text{ odd}\end{cases} & \text{if } m \geqslant \frac{5n}{4}\\ (0, m, 0, m) \text{ or } (n, 0, n, 0) & \text{otherwise}\end{cases}$$

or S is the complement of one of these sets.

Furthermore,

$$\eta_T(S(K_{n,m})) = \begin{cases}\begin{cases}\begin{cases}\frac{m^2}{4} + \frac{25n^2}{64} + mn & n = 4k\\ \frac{m^2}{4} + \frac{25n^2}{64} + mn - \frac{1}{16} & n \neq 4k\end{cases} & n \text{ even}\\ \begin{cases}\frac{m^2}{4} + \frac{25n^2}{64} + mn - \frac{1}{64} & n = 4k+1\\ \frac{m^2}{4} + \frac{25n^2}{64} + mn - \frac{1}{64} & n \neq 4k+1\end{cases} & n \text{ odd}\end{cases} & \text{if } m \geqslant \frac{5n}{4},\\ \frac{13mn}{8} & \text{otherwise.}\end{cases}$$

Proof. Let $V(S(K_{n,m})) = X \cup Y = (X_1 \cup X_2) \cup (Y_1 \cup Y_2)$, where $X_1 = \{v_1, v_2, ..., v_n\}$, $X_2 = \{v_{n+1}, v_{n+2}, ..., v_{n+m}\}$ and $Y_1 = \{v_1', v_2', ..., v_n'\}$, $Y_2 = \{v_{n+1}', v_{n+2}', ..., v_{n+m}'\}$. For a vertex subset S, let $x_1 = |X_1 \cap S|$, $x_2 = |X_2 \cap S|$, $y_1 = |Y_1 \cap S|$, $y_2 = |Y_2 \cap S|$ and $f(x_1, x_2, y_1, y_2) := \eta_T(S)$. Then, bounds are $0 \leqslant x_1 \leqslant n$, $0 \leqslant x_2 \leqslant m$ and $0 \leqslant y_1 \leqslant n$, $0 \leqslant y_2 \leqslant m$. Using the definitions of x_1, x_2, y_1, y_2 above, we have

$$\begin{aligned} f(x_1,x_2,y_1,y_2) =& \frac{1}{2}x_1(m-x_2) + \frac{1}{2}x_2(n-x_1) + \frac{1}{4}x_1(n-x_1) + \frac{1}{4}x_2(m-x_2) \\ &+ \frac{1}{4}x_1(n-y_1) + \frac{1}{2}x_1(m-y_2) + \frac{1}{4}x_2(m-y_2) + \frac{1}{2}x_2(n-y_1) \\ &+ \frac{1}{4}y_1(n-y_1) + \frac{1}{4}y_2(m-y_2) + \frac{1}{8}y_1(m-y_2) + \frac{1}{8}y_2(n-y_1) \\ &+ \frac{1}{2}y_1(m-x_2) + \frac{1}{4}y_1(n-x_1) + \frac{1}{2}y_2(n-x_1) + \frac{1}{4}y_2(m-x_2). \end{aligned}$$

Solving the system $f_{x_1}(x_1, x_2, y_1, y_2) = 0$, $f_{x_2}(x_1, x_2, y_1, y_2) = 0$ and $f_{y_1}(x_1, x_2, y_1, y_2) = 0$, $f_{y_2}(x_1, x_2, y_1, y_2) = 0$ gives $x_1 = y_1 = \frac{n}{2}$ and $x_2 = y_2 = \frac{m}{2}$. When we examine the maximum of the function depending on n and m being odd and even, we have

$$\begin{cases} \begin{cases} f(\frac{n}{2},\frac{m}{2},\frac{n}{2},\frac{m}{2}) = \frac{m^2}{4} + \frac{13mn}{16} + \frac{n^2}{4} & n \text{ even} \\ \left.\begin{array}{l} f(\lceil\frac{n}{2}\rceil,\frac{m}{2},\lfloor\frac{n}{2}\rfloor,\frac{m}{2}) \\ f(\lfloor\frac{n}{2}\rfloor,\frac{m}{2},\lceil\frac{n}{2}\rceil,\frac{m}{2}) \end{array}\right\} = \frac{m^2}{4} + \frac{13mn}{16} + \frac{n^2}{4} & n \text{ odd} \end{cases} & \text{if } m \text{ is even,} \\ \begin{cases} \left.\begin{array}{l} f(\frac{n}{2},\lceil\frac{m}{2}\rceil,\frac{n}{2},\lfloor\frac{m}{2}\rfloor) \\ f(\frac{n}{2},\lfloor\frac{m}{2}\rfloor,\frac{n}{2},\lceil\frac{m}{2}\rceil) \end{array}\right\} = \frac{m^2}{4} + \frac{13mn}{16} + \frac{n^2}{4} & n \text{ even} \\ \left.\begin{array}{l} f(\lceil\frac{n}{2}\rceil,\lfloor\frac{m}{2}\rfloor,\lceil\frac{n}{2}\rceil,\lfloor\frac{m}{2}\rfloor) \\ f(\lfloor\frac{n}{2}\rfloor,\lceil\frac{m}{2}\rceil,\lfloor\frac{n}{2}\rfloor,\lceil\frac{m}{2}\rceil) \end{array}\right\} = \frac{m^2}{4} + \frac{13mn}{16} + \frac{n^2}{4} + \frac{5}{16} & n \text{ odd} \end{cases} & \text{if } m \text{ is odd.} \end{cases}$$

But, we must look at the boundaries for x_1, x_2, y_1 and y_2 for the maximum of the function and compare results. Since complements sets are equivalent, we can ignore searching along $x_1 = n, x_2 = m, y_1 = n$ and $y_2 = m$. Therefore, we do our search along $x_1 = 0, x_2 = 0, y_1 = 0$ and $y_2 = 0$.

Case 1. For $x_1 = 0$, we maximize $f(0, x_2, y_1, y_2) = \frac{5my_1}{8} - \frac{y_1y_2}{4} + \frac{ny_1}{2} + \frac{my_2}{2} + \frac{5ny_2}{8} - x_2y_1 - \frac{x_2y_2}{2} + \frac{mx_2}{2} + nx_2 - \frac{y_1^2}{4} - \frac{y_2^2}{4} - \frac{x_2^2}{4}$ by setting $f_{x_2}(0, x_2, y_1, y_2) = 0$, $f_{y_1}(0, x_2, y_1, y_2) = 0$ and $f_{y_2}(0, x_2, y_1, y_2) = 0$ and solving for x_2, y_1, y_2. Then we find $x_2 = \frac{m}{2} \in [0, m]$, $y_1 = \frac{n}{2} \in [0, n]$ and $y_2 = n + \frac{m}{2}$. If $m \geqslant 2n$, then $n + \frac{m}{2} \in [0, m]$.

Substituting these values into the function gives the maximum of the function as follows:

$$\begin{cases} \begin{cases} f(0,\frac{m}{2},\frac{n}{2},n+\frac{m}{2}) = \frac{m^2}{4} + \frac{13mn}{16} + \frac{7n^2}{16} & n \text{ even} \\ \left.\begin{array}{l} f(0,\frac{m}{2},\lceil\frac{n}{2}\rceil,n+\frac{m}{2}) \\ f(0,\frac{m}{2},\lfloor\frac{n}{2}\rfloor,n+\frac{m}{2}) \end{array}\right\} = \frac{m^2}{4} + \frac{13mn}{16} + \frac{7n^2}{16} - \frac{1}{16} & n \text{ odd} \end{cases} & \text{if } m \text{ is even,} \\ \begin{cases} \left.\begin{array}{l} f(0,\lceil\frac{m}{2}\rceil,\frac{n}{2},n+\lceil\frac{m}{2}\rceil) \\ f(0,\lfloor\frac{m}{2}\rfloor,\frac{n}{2},n+\lfloor\frac{m}{2}\rfloor) \end{array}\right\} = \frac{m^2}{4} + \frac{13mn}{16} + \frac{7n^2}{16} - \frac{1}{16} & n \text{ even} \\ \left.\begin{array}{l} f(0,\lceil\frac{m}{2}\rceil,\lfloor\frac{n}{2}\rfloor,n+\lfloor\frac{m}{2}\rfloor) \\ f(0,\lfloor\frac{m}{2}\rfloor,\lceil\frac{n}{2}\rceil,n+\lceil\frac{m}{2}\rceil) \end{array}\right\} = \frac{m^2}{4} + \frac{13mn}{16} + \frac{7n^2}{16} + \frac{1}{8} & n \text{ odd} \end{cases} & \text{if } m \text{ is odd.} \end{cases}$$

But, we must look at the boundaries for x_2, y_1, y_2 and then compare the obtained results.

Case 1.1. For $y_1 = 0$, we maximize $f(0, x_2, 0, y_2) = \frac{my_2}{2} + \frac{5ny_2}{8} - \frac{x_2y_2}{2} + \frac{mx_2}{2} + nx_2 - \frac{y_2^2}{4} - \frac{x_2^2}{4}$ by solving $f_{x_2}(0, x_2, 0, y_2) = 0$ and $f_{y_2}(0, x_2, 0, y_2) = 0$. But, this does not give a solution. Thus, we search the maximum of the function along $x_2 = 0, x_2 = m, y_2 = 0$ and $y_2 = m$.

For $x_2 = 0$ and $y_2 = 0$, we maximize $f(0, 0, 0, y_2) = \frac{my_2}{4} + \frac{5ny_2}{8} + \frac{y_2(m-y_2)}{4}$ and $f(0, x_2, 0, 0) = \frac{mx_2}{4} + nx_2 + \frac{x_2(m-x_2)}{4}$ and we have $y_2 = m + \frac{5n}{4} \notin [0, m]$, $x_2 = m + 2n \notin [0, m]$, respectively. After looking at the boundaries of x_2 and y_2, we find the maximum values of two function as $f(0, 0, 0, m) = \frac{m^2}{4} + \frac{5mn}{8}$, $f(0, m, 0, 0) = \frac{m^2}{4} + mn$, respectively. Notice that, $f(0, 0, 0, m) < f(0, m, 0, 0)$. Thus, $f(0, m, 0, 0)$ is the maximizing choice for these cases.

For $x_2 = m$, maximizing $f(0, m, 0, y_2) = \frac{5ny_2}{8} + mn + \frac{(m+y_2)(m-y_2)}{4}$ gives $y_2 = \frac{5n}{4}$. If $5n \leqslant 4m$, then $\frac{5n}{4} \in [0, m]$. Thus, we compute the function when $5n \leqslant 4m$ depending on n being odd and even. With the boundaries of y_2, we find the maximum value of the function as

$f(0,m,0,m) = \frac{13mn}{8}$.

For $y_2 = m$, we maximize $f(0,x_2,0,m) = \frac{5mn}{8} + nx_2 + \frac{1}{4}(m+x_2)(m-x_2)$ by setting $f_{x_2}(0,x_2,0,m) = 0$ and solving for x_2. We find $x_2 = 2n$. If $m \geqslant 2n$, then $2n \in [0,m]$. Thus, $f(0,2n,0,m) = \frac{m^2}{4} + n^2 + \frac{5mn}{8}$. For the boundary of x_2, we have $f(0,m,0,m) = \frac{13mn}{8}$. Since $f(0,2n,0,m) < f(0,m,0,m)$, the function $f(0,x_2,0,m)$ is maximized at $x_2 = m$.

Case 1.2. For $y_1 = n$, we maximize $f(0,x_2,n,y_2) = -\frac{y_2^2}{4} + \frac{3ny_2}{8} - \frac{x_2y_2}{2} + \frac{my_2}{2} + \frac{n^2}{4} + \frac{5mn}{8} - \frac{x_2^2}{4} + \frac{mx_2}{2}$. Solving $f_{x_2}(0,x_2,n,y_2) = 0$, $f_{y_2}(0,x_2,n,y_2) = 0$ does not give a solution. And so, we examine the maximum of the function at the boundaries for x_2 and y_2 as follows:

For $x_2 = 0$, maximizing $f(0,0,0,y_2) = -\frac{y_2^2}{4} + \frac{3ny_2}{8} + \frac{my_2}{2} + \frac{n^2}{4} + \frac{5mn}{8}$ gives $y_2 = m + \frac{3n}{4} \notin [0,m]$. Thus, the function is maximized at $y_2 = m$ and $f(0,0,0,m) = \frac{m^2}{4} + mn + \frac{n^2}{4}$.

For $x_2 = m$, maximizing $f(0,m,0,y_2) = \frac{1}{8}(m - y_2 + 2n)(2y_2 + 2m + n)$ gives $y_2 = \frac{3n}{4} \in [0,m]$. Boundaries of y_2 are $y_2 = 0$ and $y_2 = m$. Then, we substitute these values into the function and by comparing results for $k \in \mathbb{Z}^+$ gives

$$\begin{cases} \begin{cases} f(0,m,n,\lceil\frac{3n}{4}\rceil) = \frac{m^2}{4} + \frac{25n^2}{64} + \frac{5mn}{8} - \frac{1}{64} & n = 4k+1 \\ f(0,m,n,\lfloor\frac{3n}{4}\rfloor) = \frac{m^2}{4} + \frac{25n^2}{64} + \frac{5mn}{8} - \frac{1}{64} & n \neq 4k+1 \end{cases} & \text{if } n \text{ is odd,} \\ \begin{cases} f(0,m,n,\frac{3n}{4}) = \frac{m^2}{4} + \frac{25n^2}{64} + \frac{5mn}{8} & n = 4k \\ \left.\begin{array}{l} f(0,m,n,\lfloor\frac{3n}{4}\rfloor) \\ f(0,m,n,\lceil\frac{3n}{4}\rceil) \end{array}\right\} = \frac{m^2}{4} + \frac{25n^2}{64} + \frac{5mn}{8} - \frac{1}{16} & n \neq 4k \end{cases} & \text{if } n \text{ is even.} \end{cases}$$

For $y_2 = 0$ and $y_2 = m$, we maximize $f(0,x_2,0,0) = \frac{n^2}{4} + \frac{5mn}{8} - \frac{x_2^2}{4} + \frac{mx_2}{2}$ and $f(0,x_2,0,m) = \frac{m^2}{4} + mn + \frac{n^2}{4} - \frac{x_2^2}{4}$ by solving $f_{x_2}(0,x_2,0,0) = 0$ and $f_{x_2}(0,x_2,0,m) = 0$. Then, we have $x_2 = m \in [0,m]$ and $x_2 = 0 \in [0,m]$, respectively. Summarizing the maximum values for two functions gives $f(0,0,n,m) = \frac{m^2}{4} + mn + \frac{n^2}{4}$.

Case 1.3. For $x_2 = 0$, we maximize $f(0,0,y_1,y_2) = \frac{5my_1}{8} - \frac{y_1y_2}{4} + \frac{ny_1}{2} + \frac{my_2}{2} + \frac{5ny_2}{8} - \frac{y_2^2}{4} - \frac{y_1^2}{4}$ and find $y_1 = m + \frac{n}{2} \notin [0,n]$ and for $m \geqslant 2n$, $y_2 = n + \frac{m}{2} \in [0,m]$. Hence, we must seek the maximum of $f(0,0,y_1,y_2)$ at the boundaries for y_1 and y_2.

Examining the maximum of the function along $y_1 = 0$ and $y_2 = 0$ gives the same value of Case 1.1 ($x_2 = 0$) and Case 1.2 ($x_2 = 0$), respectively.

For $y_2 = 0$ and $y_2 = m$, maximizing $f(0,0,y_1,0) = \frac{5my_1}{8} + \frac{ny_1}{4} + \frac{1}{4}y_1(n - y_1)$ and $f(0,0,y_1,m) = -\frac{y_1^2}{4} + \frac{3my_1}{8} + \frac{ny_1}{2} + \frac{m^2}{4} + \frac{5mn}{8}$ gives $y_1 = n + \frac{5m}{4}, y_1 = n + \frac{3m}{4}$, respectively. But, they are not in the range $[0,n]$. Thus, these functions are maximized at the boundaries and we compute $f(0,0,n,0) = \frac{n^2}{4} + \frac{5mn}{8}$, $f(0,0,n,m) = \frac{m^2}{4} + mn + \frac{n^2}{4}$. Notice that, $f(0,0,n,m) > f(0,0,n,0)$. Therefore, $f(0,0,n,m)$ is the maximizing choice.

Case 1.4. For $x_2 = m$, we maximize $f(0,m,y_1,y_2) = -\frac{y_1^2}{2} - \frac{y_1y_2}{4} - \frac{3my_1}{8} + \frac{ny_1}{2} - \frac{y_2^2}{4} + \frac{5ny_2}{8} + \frac{m^2}{4} + mn$ and find $y_1 = \frac{n}{2} - m$, $y_2 = n + \frac{m}{2}$. Since $\frac{n}{2} - m$ is outside the range $[0,n]$, we look at the boundaries for y_1 and y_2.

Examining the maximum of the function along $y_1 = 0$ and $y_1 = n$ is equivalent to Case 1.1 ($x_2 = m$) and Case 1.2 ($x_2 = m$), respectively.

For $y_2 = 0$ and $y_2 = m$, we maximize $f(0,m,y_1,0) = \frac{1}{8}(y_1 + 2m)(m - 2y_1 + 4n)$ and $f(0,m,y_1,m) = \frac{ny_1}{2} - \frac{5my_1}{8} + \frac{13mn}{8} - \frac{y_1^2}{4}$ by solving $f_{y_1}(0,m,y_1,0) = 0$, $f_{y_1}(0,m,y_1,m) = 0$ for y_1. We find $y_1 = n - \frac{3m}{4}$ and $y_1 = n - \frac{5m}{4}$, respectively. But, these values are not in the range $[0,n]$. Therefore, these functions are maximized at $y_1 = 0$ and comparing the maximum of the functions gives $f(0,m,0,m) = \frac{13mn}{8}$.

Case 1.5. For $y_2 = 0$, we maximize $f(0,x_2,y_1,0) = -\frac{5my_1}{8} + \frac{ny_1}{2} - x_2y_1 + \frac{mx_2}{2} + nx_2 - \frac{y_1^2}{4} - \frac{x_2^2}{4}$. Solving the system $f_{x_2}(0,x_2,y_1,0) = 0$ and $f_{y_1}(0,x_2,y_1,0) = 0$ gives $x_2 = \frac{m}{2}$ and $y_1 = n + \frac{m}{4}$. Since $n + \frac{m}{4}$ is outside the range $[0,n]$, we search the maximum of the function at boundaries.

Maximizing the function along $x_2 = 0, x_2 = m$ and $y_1 = 0, y_1 = n$ is equivalent to Case 1.3, Case 1.4 and Case 1.1, Case 1.2, respectively.

Case 1.6. For $y_2 = m$, we maximize $f(0,x_2,y_1,m) = -\frac{y_1^2}{4} + \frac{3my_1}{8} - x_2y_1 + \frac{ny_1}{2} + nx_2 + \frac{m^2}{4} + \frac{5nm}{8} - \frac{x_2^2}{4}$ and find $x_2 = \frac{m}{2} \in [0,m]$, $y_1 = n - \frac{m}{4}$. If $4n \geqslant m$, then $n - \frac{m}{4} \in [0,n]$. We look at the lower and upper bounds for y_1 and x_2 depending on m being odd and even. Then, by checking the values at the boundaries of x_2 and y_1, we find the maximum of the function as follows:

$$
\begin{cases}
\begin{cases}
\begin{cases}
f(0,\lceil \frac{m}{2}\rceil, n-\lceil \frac{m}{4}\rceil, m) = \frac{13m^2}{64}+\frac{n^2}{4}+mn+\frac{11}{64} & m=4k+1\\
f(0,\lfloor \frac{m}{2}\rfloor, n-\lfloor \frac{m}{4}\rfloor, m) = \frac{13m^2}{64}+\frac{n^2}{4}+mn+\frac{11}{64} & m\neq 4k+1
\end{cases} & m \text{ odd}\\
\begin{cases}
f(0,\frac{m}{2}, n-\frac{m}{4}, m) = \frac{13m^2}{64}+\frac{n^2}{4}+mn & m=4k\\
\left.\begin{array}{l} f(0,\frac{m}{2},n-\lceil\frac{m}{4}\rceil,m)\\ f(0,\frac{m}{2},n-\lfloor\frac{m}{4}\rfloor,m)\end{array}\right\} = \frac{13m^2}{64}+\frac{n^2}{4}+mn-\frac{1}{16} & m\neq 4k
\end{cases} & m\text{ even}
\end{cases} & \text{if } \frac{m}{4}\leqslant n,\\
f(0,2n,0,m)=\frac{m^2}{4}+n^2+\frac{5mn}{8} & \text{if } n\leqslant \frac{m}{2},\\
f(0,m,0,m)=\frac{13mn}{8} & \text{otherwise.}
\end{cases}
$$

We prove for $x_2 = 0$, $y_1 = 0$ and $y_2 = 0$ similar to Case 1. Then we find the maximum value of the function for each of them and have following cases:

Case 2. For $x_2 = 0$, $k \in \mathbb{Z}^+$

$$
\begin{cases}
\begin{cases}
\begin{cases}
f(n,0,\lceil \frac{3m}{4}\rceil, m) = \frac{25m^2}{64}+\frac{n^2}{4}+\frac{5mn}{8}-\frac{1}{64} & m=4k+1\\
n,0,\lfloor \frac{3m}{4}\rfloor, m) = \frac{25m^2}{64}+\frac{n^2}{4}+\frac{5mn}{8}-\frac{1}{64} & m\neq 4k+1
\end{cases} & m \text{ odd}\\
\begin{cases}
f(n,0,\frac{3m}{4}, m) = \frac{25m^2}{64}+\frac{n^2}{4}+\frac{5mn}{8} & m=4k\\
\left.\begin{array}{l} f(n,0,\lceil\frac{3m}{4}\rceil,m)\\ f(n,0,\lfloor\frac{3m}{4}\rfloor,m)\end{array}\right\} = \frac{25m^2}{64}+\frac{n^2}{4}+\frac{5mn}{8}-\frac{1}{16} & m\neq 4k
\end{cases} & m\text{ even}
\end{cases} & \text{if } \frac{3m}{4}\leqslant n,\\
\begin{cases}
\begin{cases}
f(n,0,n,m-\lfloor \frac{5n}{4}\rfloor) = \frac{m^2}{4}+\frac{25n^2}{64}+mn-\frac{1}{64} & n=4k+1\\
f(n,0,n,m-\lceil \frac{5n}{4}\rceil) = \frac{m^2}{4}+\frac{25n^2}{64}+mn-\frac{1}{64} & n\neq 4k+1
\end{cases} & n \text{ odd}\\
\begin{cases}
f(n,0,n,m-\frac{5n}{4}) = \frac{m^2}{4}+\frac{25n^2}{64}+mn & n=4k\\
\left.\begin{array}{l} f(n,0,n,m-\lceil\frac{5n}{4}\rceil)\\ f(n,0,n,m-\lfloor\frac{5n}{4}\rfloor)\end{array}\right\} = \frac{m^2}{4}+\frac{25n^2}{64}+mn-\frac{1}{16} & n\neq 4k
\end{cases} & n\text{ even}
\end{cases} & \text{if } \frac{5n}{4}\leqslant m,\\
f(n,0,n,0)=\frac{13mn}{8} & \text{otherwise.}
\end{cases}
$$

Case 3. For $y_1 = 0$, $k \in \mathbb{Z}^+$

$$
\begin{cases}
\begin{cases}
\begin{cases}
f(0,m,0,\lfloor \frac{5n}{4}\rfloor) = \frac{m^2}{4}+\frac{25n^2}{64}+mn-\frac{1}{64} & n=4k+1\\
f(0,m,0,\lceil \frac{5n}{4}\rceil) = \frac{m^2}{4}+\frac{25n^2}{64}+mn-\frac{1}{64} & n\neq 4k+1
\end{cases} & n \text{ odd}\\
\begin{cases}
f(0,m,0,\frac{5n}{4}) = \frac{m^2}{4}+\frac{25n^2}{64}+mn & n=4k\\
\left.\begin{array}{l} f(0,m,0,\lceil\frac{5n}{4}\rceil)\\ f(0,m,0,\lfloor\frac{5n}{4}\rfloor)\end{array}\right\} = \frac{m^2}{4}+\frac{25n^2}{64}+mn-\frac{1}{16} & n\neq 4k
\end{cases} & n\text{ even}
\end{cases} & \text{if } \frac{5n}{4}\leqslant m,\\
f(0,2n,0,m)=\frac{m^2}{4}+n^2+\frac{5mn}{8} & \text{if } n\leqslant \frac{m}{2},\\
f(n,0,n,0)=\frac{13mn}{8} & \text{otherwise.}
\end{cases}
$$

Case 4. For $y_2 = 0$, $k \in \mathbb{Z}^+$

$$
\begin{cases}
\begin{cases}
\begin{cases}
f(n,\lfloor \frac{m}{2}\rfloor, \lceil \frac{m}{4}\rceil, 0) = \frac{13m^2}{64}+\frac{n^2}{4}+mn+\frac{11}{64} & m=4k+1\\
f(n,\lceil \frac{m}{2}\rceil, \lfloor \frac{m}{4}\rfloor, 0) = \frac{13m^2}{64}+\frac{n^2}{4}+mn+\frac{11}{64} & m\neq 4k+1
\end{cases} & m \text{ odd}\\
\begin{cases}
f(n,\frac{m}{2}, \frac{m}{4}, 0) = \frac{13m^2}{64}+\frac{n^2}{4}+mn & m=4k\\
\left.\begin{array}{l} f(n,\frac{m}{2},\lceil\frac{m}{4}\rceil,0)\\ f(n,\frac{m}{2},\lfloor\frac{m}{4}\rfloor,0)\end{array}\right\} = \frac{13m^2}{64}+\frac{n^2}{4}+mn-\frac{1}{16} & m\neq 4k
\end{cases} & m\text{ even}
\end{cases} & \text{if } \frac{m}{4}\leqslant n,\\
f(n,m-2n,n,0)=\frac{m^2}{4}+n^2+\frac{5mn}{8} & \text{if } n\leqslant \frac{m}{2},\\
f(0,m,0,m)=\frac{13mn}{8} & \text{otherwise.}
\end{cases}
$$

Summarizing these results gives the theorem as stated. □

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