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A NOTE ON CLOSED FORMS FOR THE HORADAM SEQUENCE GENERAL TERM

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Abstract. We present—as an alternative to the standard technique deployed historically—a matrix based method that delivers both of the characteristic root dependent closed forms for the general term of the celebrated Horadam sequence.

1 Introduction

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a,b;p,q)\}_0^{\infty}$, in standard format, the four-parameter Horadam sequence arising from the second order linear recursion

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{1.1}$$

for which $w_0 = a$ and $w_1 = b$ are initial values and whose associated characteristic equation is

$$\lambda^2 - p\lambda + q = 0. \tag{1.2}$$

In the non-degenerate characteristic roots case ($p^2 \neq 4q$) the distinct roots

$$\lambda_1(p,q) = (p + \sqrt{p^2 - 4q})/2, \qquad \lambda_2(p,q) = (p - \sqrt{p^2 - 4q})/2) \tag{1.3}$$

combine in a closed (traditionally referred to as a Binet) form

$$w_n(a,b;p,q) = w_n(\lambda_1(p,q), \lambda_2(p,q), a, b) = \frac{(b - a\lambda_2)\lambda_1^n - (b - a\lambda_1)\lambda_2^n}{\lambda_1 - \lambda_2}, \tag{1.4}$$

while in the degenerate characteristic roots case ($p^2 = 4q$)

$$w_n(a,b;p,p^2/4) = w_n(\lambda_r(p), a, b) = bn\lambda_r^{n-1} - a(n-1)\lambda_r^n, \tag{1.5}$$

the roots co-inciding as $\lambda_1 = \lambda_2 = p/2 = \lambda_r(p)$, say.

The standard method of deriving these closed forms uses the characteristic roots as building blocks for a general closed form (in either of the aforementioned cases), followed by application of initial conditions (that is, the sequence start values w_0, w_1) to evaluate unknown constants; the details are omitted here, as the exercise is a very familiar undergraduate level one. In this paper we present a different line of argument, reproducing the representations (1.4) and (1.5) using matrix methods—while routine diagonalisation drawn from linear algebra yields (1.4) easily enough, the route to (1.5) has added features which are not without interest and do not appear to have been documented in the literature.

2 Derivations

Noting that the recursion (1.1) readily delivers the matrix power relation (for $n \geq 1$)

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \mathbf{A}^{n-1}(p,q) \begin{pmatrix} b \\ a \end{pmatrix}, \tag{2.1}$$

where

$$\mathbf{A}(p,q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

our results are immediate from appropriate decompositions of $\mathbf{A}(p,q)$ to use in (2.1).

2.1 Non-Degenerate Roots Case

This is straightforward using diagonalisation of $\mathbf{A}(p,q)$ as

$$\mathbf{A}(p,q)=\mathbf{P}(\lambda_1,\lambda_2)\mathbf{D}(\lambda_1,\lambda_2)\mathbf{P}^{-1}(\lambda_1,\lambda_2), \tag{D.1}$$

with $\lambda_{1,2}=\lambda_{1,2}(p,q)$ according to (1.3). It is routine to find that the distinct eigenvalues (characteristic roots) $\lambda_{1,2}$ of $\mathbf{A}(p,q)$ have eigenvectors $(\lambda_{1,2},1)^T$, where T denotes transposition (reader exercise: as a check we see that $\mathbf{A}(p,q)(\lambda_{1,2},1)^T=(p\lambda_{1,2}-q,\lambda_{1,2})^T=(\lambda_{1,2}^2,\lambda_{1,2})^T$ (each of $\lambda_{1,2}$ satisfy the characteristic equation (1.2)) $=\lambda_{1,2}(\lambda_{1,2},1)^T$), so that

$$\mathbf{D}(\lambda_1,\lambda_2)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{D.2}$$

and

$$\mathbf{P}(\lambda_1,\lambda_2)=\begin{pmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{pmatrix}, \tag{D.3}$$

in conventional fashion. Equation (D.1) now gives

$$\begin{aligned}\mathbf{A}^{n-1}(p,q) &= \mathbf{P}(\lambda_1,\lambda_2)\mathbf{D}^{n-1}(\lambda_1,\lambda_2)\mathbf{P}^{-1}(\lambda_1,\lambda_2)\\ &= \begin{pmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}\lambda_1^{n-1} & 0\\ 0 & \lambda_2^{n-1}\end{pmatrix}\cdot\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{pmatrix}\\ &= \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}\lambda_1^n-\lambda_2^n & \lambda_1\lambda_2^n-\lambda_1^n\lambda_2\\ \lambda_1^{n-1}-\lambda_2^{n-1} & \lambda_1\lambda_2^{n-1}-\lambda_1^{n-1}\lambda_2\end{pmatrix}\end{aligned} \tag{D.4}$$

after a little algebra, whence, from (2.1),

$$\begin{aligned}w_n(\lambda_1(p,q),\lambda_2(p,q),a,b) &= \frac{(\lambda_1^n-\lambda_2^n)b+(\lambda_1\lambda_2^n-\lambda_1^n\lambda_2)a}{\lambda_1-\lambda_2}\\ &= \frac{(b-a\lambda_2)\lambda_1^n-(b-a\lambda_1)\lambda_2^n}{\lambda_1-\lambda_2},\end{aligned} \tag{D.5}$$

which is $w_n(a,b;p,q)$ of (1.4).

This method does not accommodate the degenerate roots case (for with $\lambda_1=\lambda_2$ the matrix $\mathbf{P}$ would be singular), the closed form for which can be found as $\lim_{\lambda_2\to\lambda_1=\lambda_r}\{w_n(\lambda_1,\lambda_2,a,b)\}=w_n(\lambda_r,a,b)$ directly from (D.5).[1] We can, however, deploy quasi-diagonalisation as a modified version of the above eigendecomposition which has its own mathematical nuances in our context.

2.2 Degenerate Roots Case

As stated, the diagonalisation seen in the above case breaks down here, for the matrix $\mathbf{P}(\lambda_r)$ would become singular (having identical columns). Noting that $q=q(p)=p^2/4$, we instead appeal to a matrix

$$\mathbf{J}(\lambda_r)=\begin{pmatrix}\lambda_r & 1\\ 0 & \lambda_r\end{pmatrix} \tag{D.6}$$

which drives a decomposition

$$\mathbf{A}(p,q(p))=\mathbf{S}(\lambda_r)\mathbf{J}(\lambda_r)\mathbf{S}^{-1}(\lambda_r), \tag{D.7}$$

and takes the so called Jordan normal (or canonical) form with $\lambda_r(p)=p/2$ being an eigenvalue of algebraic multiplicity two. The matrix $\mathbf{S}(\lambda_r)$ is such that

$$\mathbf{A}(p,q(p))\mathbf{S}(\lambda_r)=\mathbf{S}(\lambda_r)\mathbf{J}(\lambda_r) \tag{D.8}$$

[1] The limiting procedure is set out in the Appendix of [1] after a novel, and little known, construction of the Horadam sequence term closed form in the non-degenerate characteristic roots case; as a point of interest concerning the latter, this most unusual variant of method—found in a 1960 textbook authored by Niven and Zuckerman and seemingly applicable to linear recurrence equations of degree two only—is explained fully therein.

and, upon writing

$$\mathbf{S}(\lambda_r) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \tag{D.9}$$

(D.8) yields the equation set

$$\begin{aligned} ps_1 - qs_3 &= \lambda_r s_1, \\ ps_2 - qs_4 &= s_1 + \lambda_r s_2, \\ s_1 &= \lambda_r s_3, \\ s_2 &= s_3 + \lambda_r s_4, \end{aligned} \tag{D.10}$$

for the unknowns $s_1(\lambda_r), \ldots, s_4(\lambda_r)$ of $\mathbf{S}(\lambda_r)$. The third of these, when substituted into the first, gives $p(\lambda_r s_3) - qs_3 = \lambda_r(\lambda_r s_3) \Rightarrow 0 = (\lambda_r^2 - p\lambda_r + q)s_3 = (0)s_3 = 0$ which, being identically true, means that s_3 becomes an arbitrary parameter $s_3 = \Omega$, say, and in turn $s_1 = \lambda_r s_3 = \lambda_r\Omega$. The second and fourth equations now read, respectively,

$$(p - \lambda_r)s_2 - qs_4 = \lambda_r\Omega \tag{D.11}$$

and

$$s_2 - \lambda_r s_4 = \Omega, \tag{D.12}$$

elimination of s_4 delivering the equation $(\lambda_r^2 - p\lambda_r + q)s_2 = (q - \lambda_r^2)\Omega$ so that, both sides being identically zero ($q = q(p) = p^2/4 = \lambda_r^2$), $s_2 = \theta$ is also arbitrary. Equations (D.12),(D.11) separately give s_4 as (resp.)

$$s_4 = (\theta - \Omega)/\lambda_r \quad \text{and} \quad s_4 = [p\theta - (\theta + \Omega)\lambda_r]/q, \tag{D.13}$$

from reconciliation of which

$$0 = (\theta + \Omega)\lambda_r^2 - p\theta\lambda_r + (\theta - \Omega)q. \tag{D.14}$$

Thus, for consistency with the characteristic equation (1.2), it follows that $\theta = 1$ and $\theta + \Omega = \theta - \Omega = 1 \Rightarrow \Omega = 0$, and so now $s_1 = \lambda_r\Omega = 0$ and $s_2 = \theta = 1$, together with $s_3 = \Omega = 0$ and (from (D.13)) $s_4 = 1/\lambda_r$. However, while

$$\mathbf{S}(\lambda_r) = \begin{pmatrix} 0 & 1 \\ 0 & 1/\lambda_r \end{pmatrix} \tag{D.15}$$

indeed satisfies (D.8), its singularity rules out the decomposition (D.7) sought and our efforts are wasted. Fortunately, we are at liberty (with $s_1 = \lambda_r\Omega$, $s_3 = \Omega$ settled as earlier) to choose $\theta = \Omega$ as a way forward,[2] whence $s_2 = \theta = \Omega$ and $s_4 = 0$ (by (D.13)), offering

$$\mathbf{S}(\lambda_r; \Omega) = \begin{pmatrix} \lambda_r\Omega & \Omega \\ \Omega & 0 \end{pmatrix} \tag{D.16}$$

for use in (D.7); setting $\Omega = 1$ for simplicity, we take

$$\mathbf{S}(\lambda_r) = \begin{pmatrix} \lambda_r & 1 \\ 1 & 0 \end{pmatrix}. \tag{D.17}$$

We are now in a position to apply (D.7) which, noting that for integer $m \geq 1$,

$$\mathbf{J}^m(\lambda_r) = \begin{pmatrix} \lambda_r^m & m\lambda_r^{m-1} \\ 0 & \lambda_r^m \end{pmatrix} \tag{D.18}$$

[2] The justification for this is that the quadratic (D.14) collapses to read $0 = \Omega\lambda_r(2\lambda_r - p)$, which holds since (dismissing the trivial solution $\lambda_r = 0$) it confirms $\lambda_r = p/2$.

(easily proven by induction, for instance), gives us

$$\begin{aligned}\mathbf{A}^{n-1}(p,q(p)) &= \mathbf{S}(\lambda_r)\mathbf{J}^{n-1}(\lambda_r)\mathbf{S}^{-1}(\lambda_r)\\ &= \begin{pmatrix}\lambda_r & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\lambda_r^{n-1} & (n-1)\lambda_r^{n-2}\\ 0 & \lambda_r^{n-1}\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & -\lambda_r\end{pmatrix}\\ &= \begin{pmatrix}n\lambda_r^{n-1} & -(n-1)\lambda_r^{n}\\ (n-1)\lambda_r^{n-2} & -(n-2)\lambda_r^{n-1}\end{pmatrix},\end{aligned} \tag{D.19}$$

the closed form (1.5) being immediate via (2.1).

As in the earlier non-degenerate roots case diagonalisation (D.1), in (D.7) $\mathbf{A}(p,q(p))$ is representing a similarity transformation of a matrix which this time is $\mathbf{J}(\lambda_r)$ (D.6) in normal Jordan form (the theory for which began, by all accounts, to be pulled together in the 1930s [2]) that comprises a single block where (a) the repeated eigenvalue features on the diagonal, (b) unity is the superdiagonal element, and (c) zero is the remaining entry. In the previous instance the algebraic multiplicity of each of $\lambda_{1,2}$ is one—so that each geometric multiplicity (that is, the dimension of the associated eigenspace) is also automatically one and the matrix is diagonalisable—whereas here the geometric multiplicity of λ_r, being also one, is less than its algebraic multiplicity and so standard diagonalisation is not possible. Any reader familiar with the theory of eigendecomposition will recognise the structure of $\mathbf{S}(\lambda_r)$ (D.17) as being formed by two columns—one being the components of the λ_r-eigenvector $(\lambda_r,1)^T$, and the other being those of the *generalised* eigenvector $\mathbf{g}=(1,0)^T$ which is a solution of the equation $(\mathbf{A}-\lambda_r\mathbf{I}_2)\mathbf{g}=(\lambda_r,1)^T$ (denoting by $\mathbf{I}_2$ the 2-square identity matrix).[3] Note that $\rho=2$ is the smallest integer for which $(\mathbf{A}-\lambda_r\mathbf{I}_2)^\rho=\mathbf{0}_2$ (where $\mathbf{0}_2$ is the 2-square zero matrix)—left as a trivial reader exercise to check, this index of nilpotency reflects the fact that the largest (and in this case the only) Jordan block is of size 2.

3 Summary

This short note presents a matrix based approach to the formulation of (characteristic root dependent) closed forms for the general term of the Horadam sequence. What looks to be an undemanding and slightly uninspiring piece of analysis becomes more intricate in the degenerate roots instance, and the work provides an interesting departure from the classic formulations that lie behind those closed forms which have—over many decades—proven to be so integral to research on this sequence.

References

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[3]The separated equations of which are, writing $\mathbf{g}=(g_1,g_2)^T$, $(p-\lambda_r)g_1-qg_2=\lambda_r$ and $g_1-\lambda_r g_2=1$, delivering the solution vector $\mathbf{g}=(\phi,(\phi-1)/\lambda_r)^T$ which we use with arbitrary ϕ chosen to be 1 for convenience; not surprisingly, these equations (in g_1,g_2) are the $\Omega=1$ versions of (D.11) and (D.12) for the column two variables s_2,s_4 of $\mathbf{S}(\lambda_r)$ (D.9).