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# SOLUTION TO PRODUCTION INVENTORY SYSTEMS WITH ORBIT, BUFFER AND DIFFERENT SERVICE RATES

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**Abstract**. This paper analyses two $(s, S)$ production inventory systems with different service rates. The time for producing and adding each item to the inventory is exponentially distributed. Arrivals of customers follow a Poisson process and service times are exponentially distributed. Upon arrival, the customers enter into a buffer of finite capacity. A potential customer who finds the buffer full, moves to an orbit. They can retry from there and inter-retrial times are exponentially distributed. When the inventory level depletes to $s$, service is given at a reduced rate. This is to minimise customers' loss from the system. Various system performance measures are computed using matrix analytic method. The effect of the underlying parameters on the performance measures are also analyzed. The minimum expected total cost by varying different parameters is computed numerically. The optimum values of $s$ and $S$ of the efficient model are obtained.

## 1 Introduction

The mixing of queuing systems with inventory has drawn much attention of many researchers over the recent years. Some earlier attempts in this direction can be found in the works by Berman et al. [4] and Berman and Sapna [5]. When customers arrive into a system and if the item is out of stock, such customers need not be backlogged or lost; otherwise they move to an orbit and may retry from there. This is known as retrial of unsatisfied customers. Artalejo et al. [1] were the first to study inventory policies with positive lead time along with retrial of unsatisfied customers and their approach turns out to be algorithmic. They formulated a bidimensional Markov process and numerically investigated the essential characteristics of the system. Manuel et al. [13] considered a perishable $(s, S)$ retrial inventory system where the arrival of customers was according to a Markovian arrival process and service times were assumed to have phase-type distribution. The life time of each item, the lead time of reorders and inter-retrial times were exponentially distributed. The joint probability distribution of the number of customers in the waiting room, number of customers in the orbit and the inventory level was obtained in the steady-state case and total expected cost rate was calculated. Sivakumar [21] studied an $(s, S)$ retrial inventory system with Server vacation. The author assumed that server vacation period and inter retrial times were exponentially distributed. The joint probability distribution of the inventory level and the number of customers in the orbit in the steady state and the total expected cost rate were calculated. Jeganathan et al. [8] described an $(s, S)$ retrial inventory system with priority customers. The service times and inter retrial times were exponentially distributed and retrial was permitted only to lower priority customers. Some important system performance measures in the steady state were derived and numerical examples were presented to illustrate the effect of the system parameters and costs on these measures. Rajkumar et al. [19] studied an $(s, S)$ inventory system with retrial and reneging. All underlying distributions were assumed to be exponential and the condition for ergodicity of the system was obtained. The measures of system performance in the steady state and total expected cost rate were derived. Vijaya Laxmi and Soujanya [23] described an $(s, S)$ perishable inventory system with service interruptions and retrial of negative customers. The service might be interrupted according to a Poisson process and inter retrial times were exponentially distributed. Various performance measures and cost analysis were shown with numerical results. Padmavathi et al. [17] analyzed a stochastic $(s, S)$ inventory system with retrial. They considered two models which differ in the way that server go for vacation. The joint probability distribution of the inventory level, the number of demands in the orbit and the server status were obtained in the steady state case.

Another significant area of literature related to this paper is that of inventory systems with production. Krishnamoorthy and Jose [11] compared three production inventory systems with positive service time and retrial of customers by assuming all the underlying distributions to be exponential. Various system performance measures were derived and a cost function was constructed. They obtained that the model with buffer size equal to the inventoried items was the best profitable model for practical purposes. Benjaafar et al. [3] considered a production inventory system and compared the performance of the optimal policy against several other policies and obtained that performance was poor for those models that ignored impatience of the customer. Chang and Lu [6] studied a serial production inventory system by providing a phase-type approximation and obtained good estimates for performance measures such as fill rate and mean queue-length distributions of each station. They also calculated the optimum base stock level by constructing a cost model. Pal et al. [18] analyzed a production-inventory model where all the members in the supply chain were considered. They obtained the optimum production rate and raw material order size for maximum expected average profit. Wang et al. [24] studied the characteristics of the optimal production policy and extended the well-known optimal service scheduling policy in the classical service system, by considering a flexible production service system. Karimi-Nasab and Sabri-Laghaie [10] constructed three randomized approximation algorithms to optimize an imperfect production problem that created defectives randomly. The algorithms in the model enabled to find the global optimum in polynomial time under certain condition. Krishnamoorthy et al. [12] analyzed an $(s, S)$ production inventory system and obtained joint distribution of the number of customers and the number of items in the inventory as the product of their marginals, under the assumption that customers did not join when inventory level was zero. Yu and Dong [26] considered a production lot size problem and obtained numerically the optimal solution to the problem. They also analyzed the effect of change of some parameters on the optimal solution of the problem. Wensing and Kuhn [25] analyzed periodic replenishment processes that exhibited order crossover. Anoop and Jacob [14] analyzed a multi-server Markovian queuing system where each server provided service only to one customer and after getting service, that server was also removed from the system. They numerically studied the dependence of system performance measures on the system parameters. Rashid et al. [20] considered a production inventory system and calculated transitional probabilities in steady state. They considered demand and production times as stochastic parameters to calculate long run inventory costs. They extended the problem to multi item by proposing a new heuristic algorithm. Baek and Moon [2] studied an $(s, S)$ production inventory system with an attached Markovian service queue. They derived an explicit stationary joint probability in product form. Jose and Salini [9] considered two production inventory systems with positive service time and retrial of customers.They assumed different rates of production depending on the inventory level. Various system performance measures were derived. A cost function was constructed and analyzed numerically and graphically.

This paper is organized as follows. Section 2 describes the mathematical formulation of model 1, which includes stability and performance measures of the system. Section 3 describes the mathematical formulation of model 2. Numerical results and interpretations are presented in section 4. Section 5 contains the cost analysis of the model. Conclusion and future study are included in section 6.

Notations used in this paper are:

i) $S$: Maximum inventory level

ii) $s$: Inventory level at which production starts.

iii) $I(t)$: Number of items in the inventory at time $t$.

iv) $N(t)$ : Number of customers in the orbit at time $t$.

v) $M(t)$ : Number of customers in the buffer at time $t$.

vi) $J(t) = \begin{cases} 0, & \text{if the production is in OFF mode} \\ 1, & \text{if the production is in ON mode} \end{cases}$

vii) $\mathbf{e} : (1, 1, ..., 1)'$ a column vector of 1's of appropriate order.

## 2 Mathematical formulation of Model 1

In model 1, a production inventory system is considered where item produces one unit at a time according to $(s, S)$ policy. When the inventory level falls to $s$, production starts and stops when the inventory level reaches back to $S$. The time for producing each item to the inventory is exponentially distributed with parameter $\beta$. The demand from customers is according to a Poisson process with rate $\lambda$. The customers on their arrival enter into a buffer having capacity equal to the maximum inventory level $S$. Orders are fulfilled if inventory is available. Service times are exponentially distributed with parameter $\mu$. When the inventory level depletes to $s$ due to service provided to the potential customers, service rate reduces to $\alpha\mu$ where $0 < \alpha < 1$ and this rate is maintained until inventory level reaches zero. When a customer enters into the system and finds the buffer full, he moves to an orbit of infinite capacity with probability $\gamma$ and is lost forever from the system with probability $(1-\gamma)$. If a customer retries from the orbit and finds the buffer full, he returns to the orbit with probability $\delta$ and is lost forever with probability $(1-\delta)$. The time between retrials of customers in the orbit is exponentially distributed with linear rate $i\theta$ when there are $i$ customers in the orbit.

Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), J(t), I(t), M(t))$ is a level dependent quasi birth-death process and its state space is $\{(i, 0, j, k); i \geq 0; j = s+1, ..., S; k = 0, 1, ..., S\} U \{(i, 1, j, k); i \geq 0; j = 0, ..., S-1; k = 0, 1, ..., S\}$. The infinitesimal generator $Q$, of the process is a block tri-diagonal matrix and it has the following form:

$$Q = \begin{bmatrix} A_{1,0} & A_0 & & & & \\ A_{2,1} & A_{1,1} & A_0 & & & \\ & A_{2,2} & A_{1,2} & A_0 & & \\ & & A_{2,3} & A_{1,3} & A_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.1}$$

where the blocks $A_0, A_{1,i} (i \geq 0)$ and $A_{2,i} (i \geq 1)$ are square matrices, each of order $(S+1)(2S-s)$ and they are given by

$$A_0 = \begin{array}{c} \underline{0, s+1} \\ \underline{0, s+2} \\ \vdots \\ \underline{0, S} \\ \underline{1, 0} \\ \underline{1, 1} \\ \vdots \\ \underline{1, S-1} \end{array} \begin{bmatrix} F & & & & & & & \\ & F & & & & & & \\ & & \ddots & & & & & \\ & & & F & & & & \\ & & & & F & & & \\ & & & & & F & & \\ & & & & & & \ddots & \\ & & & & & & & F \end{bmatrix}$$

$$A_{2,i} = \begin{array}{c} \underline{0, s+1} \\ \underline{0, s+2} \\ \vdots \\ \underline{0, S} \\ \underline{1, 0} \\ \underline{1, 1} \\ \vdots \\ \underline{1, S-1} \end{array} \begin{bmatrix} M & & & & & & & \\ & M & & & & & & \\ & & \ddots & & & & & \\ & & & M & & & & \\ & & & & M & & & \\ & & & & & M & & \\ & & & & & & \ddots & \\ & & & & & & & M \end{bmatrix}$$

$$A_{1,i} = \begin{array}{c} \underline{0,s+1} \\ \underline{0,s+2} \\ \vdots \\ \underline{0,S} \\ \underline{1,0} \\ \underline{1,1} \\ \vdots \\ \underline{1,s} \\ \underline{1,s+1} \\ \vdots \\ \underline{1,S-1} \end{array} \begin{bmatrix} K & & & & & & & J & & & \\ J & K & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & J & K & & & & & & & \\ & & & & L_0 & V & & & & & \\ & & & & J_1 & L_1 & V & & & & \\ & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & J_1 & L_1 & V & & \\ & & & & & & & J & L & \ddots & \\ & & & & & & & & \ddots & \ddots & V \\ & & & V & & & & & & J & L \end{bmatrix}$$

$(p,q)^{th}$ element of the matrices contained in $A_0$, $A_{1,i}$ and $A_{2,i}$ are given by

$$[F]_{pq} = \begin{cases} \lambda\gamma, & p = q = S+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[M]_{pq} = \begin{cases} i\theta, & 1 \le p \le S, q = p+1 \\ i\theta(1-\delta), & p = q = S+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[K]_{pq} = \begin{cases} -(\lambda + i\theta), & p = q = 1 \\ -(\lambda + \mu + i\theta), & 2 \le p \le S, q = p \\ -(\lambda\gamma + \mu + i\theta(1-\delta)), & p = q = S+1 \\ \lambda, & 1 \le p \le S, q = p+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[L_0]_{pq} = \begin{cases} -(\lambda + \beta + i\theta), & 1 \le p \le S, q = p \\ -(\lambda\gamma + \beta + i\theta(1-\delta)), & p = q = S+1 \\ \lambda, & 1 \le p \le S, q = p+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[L_1]_{pq} = \begin{cases} -(\lambda + \beta + i\theta), & p = q = 1 \\ -(\lambda + \beta + \alpha\mu + i\theta), & 2 \le p \le S, q = p \\ -(\lambda\gamma + \beta + \alpha\mu + i\theta(1-\delta)), & p = q = S+1 \\ \lambda, & 1 \le p \le S, q = p+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[L]_{pq} = \begin{cases} -(\lambda + \beta + i\theta), & p = q = 1 \\ -(\lambda + \beta + \mu + i\theta), & 2 \le p \le S, q = p \\ -(\lambda\gamma + \beta + \mu + i\theta(1-\delta)), & p = q = S+1 \\ \lambda, & 1 \le p \le S, q = p+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[J]_{pq} = \begin{cases} \mu, & 2 \le p \le S+1, q = p-1 \\ 0, & \text{otherwise} \end{cases}$$

$$[J_1]_{pq} = \begin{cases} \alpha\mu, & 2 \le p \le S+1, q = p-1 \\ 0, & \text{otherwise} \end{cases}$$

$$[V]_{pq} = \begin{cases} \beta, & 1 \le p \le S+1, q = p \\ 0, & \text{otherwise} \end{cases}$$

Neuts-Rao [16] truncation method is used to modify the infinitesimal generator $Q$ where $A_{1,i} = A_1$ and $A_{2,i} = A_2$ for $i \ge N$.

### 2.1 System Stability

In order to find system stability, we use Tweedi's [22] result. Using Lyapunov test function (Falin and Templeton [7]),define $\phi(s) = i$, if $s$ is a state in the level $i$. The mean drift $y_s$ for any $s$ belonging to the level $i \geq 1$ is given by,

$$\begin{aligned} y_s &= \sum_{p \neq s} q_{sp}(\phi(p) - \phi(s)) \\ &= \sum_{u} q_{su}(\phi(u) - \phi(s)) + \sum_{v} q_{sv}(\phi(v) - \phi(s)) + \sum_{w} q_{sw}(\phi(w) - \phi(s)) \end{aligned}$$

where $u, v$ and $w$ varies over the states belonging to the levels $(i-1), i$ and $(i+1)$ respectively. Then, by using the definition of $\phi$, we can define $\phi(u) = i - 1, \phi(v) = i$ and $\phi(w) = i + 1$ so that

$$\begin{aligned} y_s &= -\sum_{u} q_{su} + \sum_{w} q_{sw} \\ &= \begin{cases} -i\theta(1-\delta) + \lambda\gamma, & \text{if the bufffer is full} \\ -i\theta, & \text{otherwise} \end{cases} \end{aligned}$$

Since $(1 - \delta) > 0$, for any $\epsilon > 0$, we can find $N'$ large enough so that $y_s < -\epsilon$ for any $s$ belonging to the level $i \geq N'$. Therefore, the system under consideration is stable.

### 2.2 Rate Matrix $R$ and Truncation Level $N$

We use iterative method to find $R$. Denote the sequence of $R$ by $\{R_n(N)\}$ and is defined by $R_0(N) = 0$ and $R_{n+1}(N) = (-R_n^2(N)A_2(N) - A_0(N))A_1^{-1}(N)$. The value of $N$ must be chosen such that $|\eta(N) - \eta(N+1)| < \epsilon$, where $\epsilon$ is an arbitrarily small value and $\eta(R)$, the spectral radius of $R(N)$. For detailed discussion of selection of the value of $N$, one can refer to Neuts [15].

### 2.3 System Performance Measures

The $(i+1)^{th}$ component of the steady state probability vector $\mathbf{x} = (x_0, x_1, x_2, ..., x_{N-1}, x_N, ...)$ is given by $x_i = (y_{i,0,s+1,0}, ..., y_{i,0,s+1,S}, y_{i,0,s+2,0}, ..., y_{i,0,s+2,S}, ..., y_{i,0,S,0}, ..., y_{i,0,S,S}, y_{i,1,0,0}, ..., y_{i,1,0,S}, y_{i,1,1,0}, ..., y_{i,1,1,S}, ..., y_{i,1,S-1,0}, ..., y_{i,1,S-1,S})$. Then,

i) Expected Inventory level, $EI$, in the system is given by,

$$EI = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{S} j y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{S} j y_{i,1,j,k}$$

ii) Expected number of customers, $EO$, in the orbit is given by,

$$EO = \left[\sum_{i=1}^{\infty} i x_i\right] \boldsymbol{e} = \left[\left(\sum_{i=1}^{N-1} i x_i\right) + x_N \left(N(I-R)^{-1} + R(I-R)^{-2}\right)\right] \boldsymbol{e}$$

iii) Expected number of customers, $EB$, in the buffer is given by,

$$EB = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{S} k y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{S} k y_{i,1,j,k}$$

iv) Expected switching rate, $ESR$, is given by,

$$ESR = \mu \sum_{i=0}^{\infty} \sum_{k=1}^{S} y_{i,0,s+1,k}$$

v) Expected number of departures, $EDS$, after completing service is,

$$EDS = \mu \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=1}^{S} y_{i,0,j,k} + \alpha\mu \sum_{i=0}^{\infty} \sum_{j=1}^{s} \sum_{k=1}^{S} y_{i,1,j,k} + \mu \sum_{i=0}^{\infty} \sum_{j=s+1}^{S-1} \sum_{k=1}^{S} y_{i,1,j,k}$$

vi) Expected number of external customers lost, $EL_1$, before entering the orbit per unit time is,

$$EL_1 = (1-\gamma)\,\lambda \left[ \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} y_{i,0.j,S} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} y_{i,1,j,S} \right]$$

vii) Expected number of customers lost, $EL_2$, due to retrials per unit time is,

$$EL_2 = \theta(1-\delta) \left[ \sum_{i=1}^{\infty} \sum_{j=s+1}^{S} i y_{i,0,j,S} + \sum_{i=1}^{\infty} \sum_{j=0}^{S-1} i y_{i,1,j,S} \right]$$

viii) Overall rate of retrials, $ORR$, is given by,

$$ORR = \theta \left[ \sum_{i=1}^{\infty} i x_i \right] \mathbf{e}$$

ix) Successful rate of retrials, $SRR$, is given by,

$$SRR = \theta \sum_{i=1}^{\infty} i \left[ \sum_{j=s+1}^{S} \sum_{k=0}^{S-1} y_{i,0,j,k} + \sum_{j=0}^{S-1} \sum_{k=0}^{S-1} y_{i,1,j,k} \right]$$

## 3 Mathematical formulation of Model 2

In model 2, we consider a buffer of varying (finite) capacity, equal to the current inventory level instead of constant capacity. All other assumptions in model 1 hold for model 2.

Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), J(t), I(t), M(t))$ is a level dependent quasi birth-death process and its state space is $\{(i,0,j,k); i \geq 0; j = s+1, \ldots, S; k = 0, 1, \ldots, j\} U$ $\{(i,1,j,k); i \geq 0; j = 0, \ldots, S-1; k = 0, 1, \ldots, j\}$. Then the infinitesimal generator $Q$ has the form $(1)$ where the blocks $A_0, A_{1,i} (i \geq 0)$ and $A_{2,i} (i \geq 1)$ are square matrices of the same order $\frac{1}{2}[(S-s)(S+s+3)+S(S+1)]$ and they are given by

$$A_0 = \begin{array}{c} \underline{0,s+1} \\ \vdots \\ \underline{0,S} \\ \underline{1,0} \\ \vdots \\ \underline{1,S-1} \end{array} \begin{bmatrix} B_{s+1} & & & & & \\ & \ddots & & & & \\ & & B_S & & & \\ & & & B_0 & & \\ & & & & \ddots & \\ & & & & & B_{S-1} \end{bmatrix}$$

$$A_{2,i} = \begin{array}{c} \underline{0,s+1} \\ \vdots \\ \underline{0,S} \\ \underline{1,0} \\ \vdots \\ \underline{1,S-1} \end{array} \begin{bmatrix} C_{s+1} & & & & & \\ & \ddots & & & & \\ & & C_S & & & \\ & & & C_0 & & \\ & & & & \ddots & \\ & & & & & C_{S-1} \end{bmatrix}$$

$$A_{1,i} = \begin{array}{c} \underline{0, s+1} \\ \underline{0, s+2} \\ \vdots \\ \underline{0, S} \\ \underline{1, 0} \\ \underline{1, 1} \\ \vdots \\ \underline{1, s} \\ \underline{1, s+1} \\ \vdots \\ \underline{1, S-1} \end{array} \begin{bmatrix} G_{s+1} & & & & & & & H_{s+1} & & & \\ H_{s+2} & G_{s+2} & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & H_S & G_S & & & & & & & \\ & & & & E_0 & T_0 & & & & & \\ & & & & U_1 & E_1 & T_1 & & & & \\ & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & U_s & E_s & T_s & & \\ & & & & & & & H_{s+1} & D_{s+1} & \ddots & \\ & & & & & & & & \ddots & \ddots & T_{S-2} \\ & & & T_{S-1} & & & & & & H_{S-1} & D_{S-1} \end{bmatrix}$$

$(p,q)^{th}$ element of the matrices contained in $A_0,\ A_{1,i}$ and $A_{2,i}$ are given by,

$[B_0]_{pq} = \lambda\gamma,\ p = q = 1$

$$[B_n]_{pq} = \left.\begin{cases} \lambda\gamma, & p = q = n+1 \\ 0, & \text{otherwise} \end{cases}\right\} 1 \le n \le S$$

$[C_0]_{pq} = i\theta\,(1-\delta)\,,\ p = q = 1$

$$[C_n]_{pq} = \left.\begin{cases} i\theta, & 1 \le p \le n\ , q = p+1 \\ i\theta\,(1-\delta)\,, & p = q = n+1 \\ 0, & \text{otherwise} \end{cases}\right\} 1 \le n \le S$$

$[E_0]_{pq} = -(\lambda\gamma + \beta + i\theta(1-\delta)),\ p = q = 1$

$$[E_n]_{pq} = \left.\begin{cases} -(\lambda + \beta + i\theta), & p = q = 1 \\ -(\lambda + \beta + \alpha\mu + i\theta), & 2 \le p \le n\ , q = p \\ -(\lambda\gamma + \beta + \alpha\mu + i\theta(1-\delta)), & p = q = n+1 \\ \lambda, & 1 \le p \le n, q = p+1 \\ 0, & \text{otherwise} \end{cases}\right\} 1 \le n \le s$$

$$[D_n]_{pq} = \left.\begin{cases} -(\lambda + \beta + i\theta), & p = q = 1 \\ -(\lambda + \beta + \mu + i\theta), & 2 \le p \le n, q = p \\ -(\lambda\gamma + \beta + \mu + i\theta(1-\delta)), & p = q = n+1 \\ \lambda, & 1 \le p \le n, q = p+1 \\ 0, & \text{otherwise} \end{cases}\right\} s+1 \le n \le S-1$$

$$[G_n]_{pq} = \left.\begin{cases} -(\lambda + i\theta), & p = q = 1 \\ -(\lambda + \mu + i\theta), & 2 \le p \le n, q = p \\ -(\lambda\gamma + \mu + i\theta(1-\delta)), & p = q = n+1 \\ \lambda, & 1 \le p \le n, q = p+1 \\ 0, & \text{otherwise} \end{cases}\right\} s+1 \le n \le S$$

$$[H_n]_{pq} = \left.\begin{cases} \mu, & 2 \le p \le n+1, q = p-1 \\ 0, & \text{otherwise} \end{cases}\right\} s+1 \le n \le S$$

$$[U_n]_{pq} = \left.\begin{cases} \alpha\mu, & 2 \le p \le n+1, q = p-1 \\ 0, & \text{otherwise} \end{cases}\right\} 1 \le n \le s$$

$$[T_n]_{pq} = \left.\begin{cases} \beta, & 1 \le p \le n+1, q = p \\ 0, & \text{otherwise} \end{cases}\right\} 0 \le n \le S-1$$

### 3.1 System Performance Measures

We partition the steady state probability vector $\boldsymbol{x} = (x_0, x_1, ..., x_{N-1}, x_N, ...)$ such that its $(i\ +1)^{th}$ component is given by $x_i = (y_{i,0,s+1,0}, \dots, y_{i,0,s+1,s+1}, y_{i,0,s+2,0}, ..., y_{i,0,s+2,s+2}, \dots,$ $y_{i,0,S,0}, \dots, y_{i,0,S,S}, y_{i,1,0,0}, y_{i,1,1,0}, y_{i,1,1,1}, ..., y_{i,1,S-1,0}, \dots, y_{i,1,S-1,S-1})$
Then,

i) Expected Inventory level, $EI$, in the system is given by,

$$EI = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{j} j y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{j} j y_{i,1,j,k}$$

ii) Expected number of customers, $EO$, in the orbit is given by,

$$EO = \left[\sum_{i=1}^{\infty} i x_i\right] \boldsymbol{e} = \left[\left(\sum_{i=1}^{N-1} i x_i\right) + x_N \left((I-R)^{-1} + R(I-R)^{-2}\right)\right] \boldsymbol{e}$$

iii) Expected number of customers, $EB$, in the buffer is given by,

$$EB = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{j} k y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{j} k y_{i,1,j,k}$$

iv) Expected switching rate, $ESR$, is given by,

$$ESR\ = \mu \sum_{i=0}^{\infty} \sum_{k=1}^{s+1} y_{i,0,s+1,k}$$

v) Expected number of departures, $EDS$, after completing service is,

$$EDS = \mu \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=1}^{j} y_{i,0,j,k} + \alpha\mu \sum_{i=0}^{\infty} \sum_{j=1}^{s} \sum_{k=1}^{j} y_{i,1,j,k} + \mu \sum_{i=0}^{\infty} \sum_{j=s+1}^{S-1} \sum_{k=1}^{j} y_{i,1,j,k}$$

vi) Expected number of external customers lost, $EL_1$, before entering the orbit per unit time is,

$$EL_1 = (1-\gamma)\,\lambda \left[\sum_{i=0}^{\infty} \sum_{j=s+1}^{S} y_{i,0.j,j} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} y_{i,1,j,j}\right]$$

vii) Expected number of customers lost, $EL_2$, due to retrials per unit time is,

$$EL_2 = \theta(1-\delta) \left[\sum_{i=1}^{\infty} \sum_{j=s+1}^{S} i y_{i,0,j,j} + \sum_{i=1}^{\infty} \sum_{j=0}^{S-1} i y_{i,1,j,j}\right]$$

viii) Overall rate of retrials, $ORR$, is given by,

$$ORR = \theta \left[\sum_{i=1}^{\infty} i x_i\right] \mathbf{e}$$

ix) Successful rate of retrials, $SRR$, is given by,

$$SRR = \theta \sum_{i=1}^{\infty} i \left[\sum_{j=s+1}^{S} \sum_{k=0}^{j-1} y_{i,0,j,k} + \sum_{j=0}^{S-1} \sum_{k=0}^{j-1} y_{i,1,j,k}\right]$$

## 4 Numerical Results and Interpretations

This section provides values of system performance measures with variation in values of underlying parameters.

**Model 1**

**Effect of the service reduction value $\alpha$ on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \delta = 0.6; \mu = 3.$

| $\alpha$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0534 | 9.7872 | 3.1231 | 19.4044 | 0.1979 | 1.3830 | 1.9472 | 4.6846 | 1.2270 |
| 0.2 | 0.0535 | 9.5760 | 3.0904 | 19.3912 | 0.1958 | 1.3522 | 1.9894 | 4.6356 | 1.2552 |
| 0.3 | 0.0536 | 9.3238 | 3.0532 | 19.3724 | 0.1929 | 1.3161 | 2.0447 | 4.5798 | 1.2896 |
| 0.4 | 0.0537 | 9.0070 | 3.0127 | 19.3479 | 0.1895 | 1.2766 | 2.1111 | 4.5191 | 1.3275 |
| 0.5 | 0.0538 | 8.5870 | 2.9717 | 19.3189 | 0.1859 | 1.2365 | 2.1839 | 4.4576 | 1.3664 |
| 0.6 | 0.0539 | 8.0358 | 2.9337 | 19.2885 | 0.1824 | 1.1987 | 2.2560 | 4.4005 | 1.4037 |
| 0.7 | 0.0541 | 7.4036 | 2.9003 | 19.2589 | 0.1793 | 1.1650 | 2.3227 | 4.3504 | 1.4380 |
| 0.8 | 0.0543 | 6.8073 | 2.8706 | 19.2299 | 0.1766 | 1.1348 | 2.3841 | 4.3060 | 1.4689 |
| 0.9 | 0.0544 | 6.3193 | 2.8439 | 19.2010 | 0.1741 | 1.1078 | 2.4410 | 4.2658 | 1.4963 |

**Table 1.** Effect of the service reduction value $\alpha$

Table 1 indicates that as $\alpha$ increases, measures like expected inventory level, expected number of customers in the orbit, expected number of customers in the buffer, expected number of primary customers lost, expected number of retrial customers lost, overall rate of retrials decrease. But, expected switching rate, expected number of departures after completing service and successful rate of retrials increase.

**Effect of the production rate $\beta$ on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \mu = 3; \delta = 0.7; \alpha = 0.5.$

| $\beta$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.0537 | 6.2177 | 3.5535 | 19.5411 | 0.2138 | 1.2224 | 1.7552 | 5.3303 | 1.2558 |
| 1.2 | 0.0538 | 6.4393 | 3.5247 | 19.5246 | 0.2113 | 1.2032 | 1.7997 | 5.287 | 1.2763 |
| 1.3 | 0.0538 | 6.6730 | 3.4961 | 19.5075 | 0.2089 | 1.1845 | 1.8441 | 5.2441 | 1.2959 |
| 1.4 | 0.0538 | 6.9175 | 3.4675 | 19.4897 | 0.2064 | 1.1660 | 1.8887 | 5.2012 | 1.3145 |
| 1.5 | 0.0538 | 7.1707 | 3.4389 | 19.4711 | 0.2039 | 1.1478 | 1.9338 | 5.1583 | 1.3323 |
| 1.6 | 0.0538 | 7.4302 | 3.4102 | 19.4516 | 0.2014 | 1.1298 | 1.9796 | 5.1153 | 1.3493 |
| 1.7 | 0.0537 | 7.6941 | 3.3813 | 19.4311 | 0.1989 | 1.1119 | 2.0264 | 5.0719 | 1.3657 |
| 1.8 | 0.0537 | 7.9614 | 3.3521 | 19.4095 | 0.1964 | 1.0940 | 2.0742 | 5.0282 | 1.3814 |
| 1.9 | 0.0536 | 8.2329 | 3.3228 | 19.3868 | 0.1939 | 1.0763 | 2.1233 | 4.9842 | 1.3965 |

**Table 2.** Effect of the production rate $\beta$

Table 2 shows that expected number of customers in the orbit, expected number of customers in the buffer, expected number of primary customers lost, expected number of retrial customers lost and overall rate of retrials decrease with the increase in $\beta$. But, expected number of departures after completing service and successful rate of retrials increase. Expected switching rate increases first and then decreases.

**Effect of the probability $\gamma$ of primary arrivals joining the orbit on various performance measures**

$S = 20; s = 8; \lambda = 1.5; N = 25; \theta = 1.5; \beta = 2; \delta = 0.6; \mu = 3; \alpha = 0.5.$

| $\gamma$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0153 | 7.9226 | 1.2608 | 19.1151 | 0.7213 | 0.4231 | 2.0657 | 1.8912 | 0.8334 |
| 0.2 | 0.0164 | 7.9372 | 1.3331 | 19.1240 | 0.6454 | 0.4558 | 2.0684 | 1.9996 | 0.8601 |
| 0.3 | 0.0175 | 7.9522 | 1.4087 | 19.1332 | 0.5686 | 0.4904 | 2.0713 | 2.1131 | 0.8871 |
| 0.4 | 0.0187 | 7.9678 | 1.4878 | 19.1429 | 0.4909 | 0.5270 | 2.0742 | 2.2317 | 0.9143 |
| 0.5 | 0.0198 | 7.9838 | 1.5703 | 19.1529 | 0.4122 | 0.5655 | 2.0772 | 2.3555 | 0.9417 |
| 0.6 | 0.0210 | 8.0002 | 1.6563 | 19.1633 | 0.3324 | 0.6061 | 2.0803 | 2.4844 | 0.9691 |
| 0.7 | 0.0221 | 8.0171 | 1.7456 | 19.1739 | 0.2513 | 0.6487 | 2.0834 | 2.6184 | 0.9967 |
| 0.8 | 0.0233 | 8.0344 | 1.8382 | 19.1849 | 0.1689 | 0.6932 | 2.0866 | 2.7573 | 1.0242 |
| 0.9 | 0.0244 | 8.0520 | 1.9341 | 19.1960 | 0.0852 | 0.7398 | 2.0899 | 2.9012 | 1.0517 |

**Table 3.** Effect of the probability $\gamma$

Table 3 shows that all the performance measures except expected number of primary customers lost increase with the increase in $\gamma$.

**Effect of the return probability $\delta$ of retrial customers on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \mu = 3; \alpha = 0.6.$

| $\delta$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0244 | 7.2060 | 1.5276 | 19.1055 | 0.1609 | 1.1960 | 2.1366 | 2.2913 | 0.9624 |
| 0.2 | 0.0243 | 7.1905 | 1.5629 | 19.1134 | 0.1617 | 1.0978 | 2.1350 | 2.3444 | 0.9722 |
| 0.3 | 0.0241 | 7.1732 | 1.6060 | 19.1226 | 0.1626 | 0.9976 | 2.1332 | 2.4091 | 0.9839 |
| 0.4 | 0.0239 | 7.1536 | 1.6599 | 19.1336 | 0.1637 | 0.8950 | 2.1311 | 2.4899 | 0.9982 |
| 0.5 | 0.0236 | 7.1310 | 1.7293 | 19.1469 | 0.1651 | 0.7890 | 2.1286 | 2.5940 | 1.0161 |
| 0.6 | 0.0233 | 7.1045 | 1.8226 | 19.1637 | 0.1669 | 0.6780 | 2.1257 | 2.7339 | 1.0389 |
| 0.7 | 0.0227 | 7.0722 | 1.9557 | 19.1859 | 0.1694 | 0.5593 | 2.1220 | 2.9335 | 1.0691 |
| 0.8 | 0.0215 | 7.0313 | 2.1648 | 19.2173 | 0.1731 | 0.4272 | 2.1169 | 3.2472 | 1.1112 |
| 0.9 | 0.0188 | 6.9752 | 2.5576 | 19.2666 | 0.1792 | 0.2661 | 2.1092 | 3.8364 | 1.1753 |

**Table 4.** Effect of the return probability $\delta$

Table 4 indicates that as $\delta$ increases, measures like expected switching rate, expected inventory level, expected number of retrial customers lost and expected number of departures after completing service decrease. However, expected number of customers in the orbit, expected number of customers in the buffer, expected number of primary customers lost, successful rate of retrials and overall rate of retrials increase.

**Effect of the arrival rate $\lambda$ on various performance measures**

$S = 20; s = 8; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6.$

| $\lambda$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.0217 | 7.4966 | 1.6823 | 19.0087 | 0.1130 | 0.4386 | 2.1794 | 2.5235 | 1.0581 |
| 1.2 | 0.0214 | 7.2938 | 1.7465 | 19.0626 | 0.1268 | 0.4679 | 2.1519 | 2.6198 | 1.0601 |
| 1.3 | 0.0215 | 7.1726 | 1.8134 | 19.1086 | 0.1407 | 0.4976 | 2.1354 | 2.7202 | 1.0614 |
| 1.4 | 0.0220 | 7.1047 | 1.8831 | 19.1493 | 0.1549 | 0.5280 | 2.1262 | 2.8247 | 1.0646 |
| 1.5 | 0.0227 | 7.0722 | 1.9557 | 19.1859 | 0.1694 | 0.5593 | 2.122 | 2.9335 | 1.0691 |
| 1.6 | 0.0234 | 7.0640 | 2.0311 | 19.2196 | 0.1841 | 0.5916 | 2.121 | 3.0467 | 1.0747 |
| 1.7 | 0.0242 | 7.0726 | 2.1095 | 19.2510 | 0.1992 | 0.6250 | 2.1225 | 3.1642 | 1.0811 |
| 1.8 | 0.0250 | 7.0929 | 2.1908 | 19.2805 | 0.2145 | 0.6594 | 2.1256 | 3.2862 | 1.0881 |
| 1.9 | 0.0259 | 7.1215 | 2.2750 | 19.3083 | 0.2301 | 0.6950 | 2.1298 | 3.4125 | 1.0957 |

**Table 5.** Effect of the arrival rate $\lambda$

Table 5 shows that expected number of customers in the orbit, expected number of customers in the buffer, expected number of primary customers lost, expected number of retrial customers lost, overall rate of retrials and successful rate of retrials increase with the increase in $\lambda$. However expected switching rate, expected inventory level and expected number of departures after completing service decrease.

**Effect of the retrial rate $\theta$ on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6.$

| $\theta$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.0189 | 6.9508 | 2.0697 | 18.9506 | 0.1511 | 0.3987 | 2.1059 | 2.2767 | 0.9476 |
| 1.2 | 0.0201 | 6.9850 | 2.0380 | 19.0230 | 0.1563 | 0.4392 | 2.1105 | 2.4456 | 0.9816 |
| 1.3 | 0.0211 | 7.0165 | 2.0085 | 19.0851 | 0.1610 | 0.4794 | 2.1147 | 2.6111 | 1.0130 |
| 1.4 | 0.0219 | 7.0454 | 1.9812 | 19.1388 | 0.1654 | 0.5195 | 2.1185 | 2.7736 | 1.0421 |
| 1.5 | 0.0227 | 7.0722 | 1.9557 | 19.1859 | 0.1694 | 0.5593 | 2.1220 | 2.9335 | 1.0691 |
| 1.6 | 0.0232 | 7.0971 | 1.9319 | 19.2276 | 0.1731 | 0.5990 | 2.1251 | 3.0910 | 1.0944 |
| 1.7 | 0.0238 | 7.1201 | 1.9096 | 19.2648 | 0.1766 | 0.6385 | 2.1279 | 3.2463 | 1.1181 |
| 1.8 | 0.0242 | 7.1416 | 1.8887 | 19.2981 | 0.1798 | 0.6778 | 2.1305 | 3.3997 | 1.1405 |
| 1.9 | 0.0245 | 7.1617 | 1.8691 | 19.3283 | 0.1829 | 0.7169 | 2.1329 | 3.5513 | 1.1616 |

**Table 6.** Effect of the retrial rate $\theta$

Table 6 shows that all the performance measures except expected number of customers in the orbit increase with the increase in $\theta$. Increase in retrial rate $\theta$ decreases the number of customers in the orbit.

**Model 2**

**Effect of the service reduction value $\alpha$ on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \delta = 0.6; \mu = 3.$

| $\alpha$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0198 | 8.8354 | 1.584 | 8.0642 | 0.1726 | 0.5993 | 1.5467 | 2.3761 | 0.8777 |
| 0.2 | 0.0197 | 8.4558 | 1.584 | 7.6844 | 0.1726 | 0.5993 | 1.5496 | 2.3759 | 0.8778 |
| 0.3 | 0.0196 | 7.8920 | 1.5839 | 7.1205 | 0.1725 | 0.5992 | 1.5803 | 2.3758 | 0.8779 |
| 0.4 | 0.0195 | 7.1284 | 1.5838 | 6.3567 | 0.1725 | 0.5991 | 1.6436 | 2.3757 | 0.8780 |
| 0.5 | 0.0194 | 6.2445 | 1.5837 | 5.4728 | 0.1725 | 0.5990 | 1.7241 | 2.3756 | 0.8781 |
| 0.6 | 0.0193 | 5.3807 | 1.5837 | 4.6089 | 0.1725 | 0.5989 | 1.7959 | 2.3755 | 0.8782 |
| 0.7 | 0.0193 | 4.6401 | 1.5836 | 3.8682 | 0.1725 | 0.5989 | 1.8433 | 2.3754 | 0.8782 |
| 0.8 | 0.0192 | 4.0523 | 1.5836 | 3.2803 | 0.1725 | 0.5989 | 1.8654 | 2.3754 | 0.8783 |
| 0.9 | 0.0192 | 3.6006 | 1.5836 | 2.8286 | 0.1725 | 0.5988 | 1.8686 | 2.3754 | 0.8783 |

**Table 7.** Effect of the service reduction value $\alpha$

Table 7 indicates that as $\alpha$ increases measures like expected switching rate, expected inventory level, expected number of customers in the orbit, expected number of customers in the buffer, expected number of retrial customers lost and overall rate of retrials decrease. However, both expected number of departures after completing service and successful rate of retrials increase. Expected number of primary customers lost decreases first and then remains constant.

**Effect of the production rate $\beta$ on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \mu = 3; \delta = 0.7; \alpha = 0.5.$

| $\beta$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.0076 | 3.7724 | 2.1352 | 3.4408 | 0.2272 | 0.7719 | 1.2822 | 3.2028 | 0.6297 |
| 1.2 | 0.0078 | 4.0640 | 2.0749 | 3.6905 | 0.2205 | 0.7323 | 1.3405 | 3.1123 | 0.6713 |
| 1.3 | 0.0081 | 4.3564 | 2.0188 | 3.9394 | 0.2140 | 0.6953 | 1.3961 | 3.0281 | 0.7105 |
| 1.4 | 0.0085 | 4.6469 | 1.9666 | 4.1848 | 0.2076 | 0.6608 | 1.4492 | 2.9500 | 0.7473 |
| 1.5 | 0.0089 | 4.9330 | 1.9183 | 4.4244 | 0.2015 | 0.6287 | 1.4998 | 2.8774 | 0.7818 |
| 1.6 | 0.0094 | 5.2127 | 1.8735 | 4.6563 | 0.1955 | 0.5988 | 1.5482 | 2.8102 | 0.8142 |
| 1.7 | 0.0101 | 5.4848 | 1.8320 | 4.8795 | 0.1898 | 0.5711 | 1.5949 | 2.7480 | 0.8444 |
| 1.8 | 0.0111 | 5.7487 | 1.7938 | 5.0936 | 0.1843 | 0.5454 | 1.6400 | 2.6906 | 0.8726 |
| 1.9 | 0.0125 | 6.0046 | 1.7586 | 5.2990 | 0.1791 | 0.5217 | 1.6841 | 2.6379 | 0.8987 |

**Table 8.** Effect of the production rate $\beta$

Table 8 shows that expected number of customers in the orbit, expected number of primary customers lost, expected number of retrial customers lost and overall rate of retrials decrease with the increase in $\beta$. However expected switching rate, expected inventory level, expected number of customers in the buffer, expected number of departures after completing service and successful rate of retrials increase.

**Effect of the probability $\gamma$ of primary arrivals joining the orbit on various performance measures**

$S = 20; s = 8; \lambda = 1.5; N = 25; \theta = 1.5; \beta = 2; \delta = 0.6; \mu = 3; \alpha = 0.5.$

| $\gamma$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0072 | 5.2631 | 1.0469 | 4.4044 | 0.7283 | 0.3426 | 1.5268 | 1.5703 | 0.7137 |
| 0.2 | 0.0085 | 5.3776 | 1.0998 | 4.5261 | 0.6509 | 0.3661 | 1.5506 | 1.6497 | 0.7344 |
| 0.3 | 0.0102 | 5.5130 | 1.1603 | 4.6709 | 0.5735 | 0.3937 | 1.5782 | 1.7405 | 0.7563 |
| 0.4 | 0.0120 | 5.6569 | 1.2288 | 4.8260 | 0.4957 | 0.4256 | 1.6073 | 1.8433 | 0.7793 |
| 0.5 | 0.0138 | 5.8039 | 1.3055 | 4.9858 | 0.4170 | 0.4621 | 1.6368 | 1.9582 | 0.8031 |
| 0.6 | 0.0157 | 5.9515 | 1.3902 | 5.1477 | 0.3371 | 0.5031 | 1.6662 | 2.0853 | 0.8276 |
| 0.7 | 0.0175 | 6.0986 | 1.4830 | 5.3103 | 0.2557 | 0.5487 | 1.6954 | 2.2245 | 0.8527 |
| 0.8 | 0.0194 | 6.2445 | 1.5837 | 5.4728 | 0.1725 | 0.5990 | 1.7241 | 2.3756 | 0.8781 |
| 0.9 | 0.0213 | 6.3887 | 1.6924 | 5.6344 | 0.0873 | 0.6539 | 1.7525 | 2.5386 | 0.9037 |

**Table 9.** Effect of the probability $\gamma$

Table 9 shows that all the performance measures except expected number of primary customers lost increase with the increase in $\gamma$.

**Effect of the return probability $\delta$ of retrial customers on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \mu = 3; \alpha = 0.6.$

| $\delta$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0300 | 5.4258 | 1.2978 | 4.6318 | 0.1707 | 1.0557 | 1.7981 | 1.9467 | 0.7737 |
| 0.2 | 0.0291 | 5.4203 | 1.3299 | 4.6284 | 0.1708 | 0.9671 | 1.7978 | 1.9949 | 0.7860 |
| 0.3 | 0.0279 | 5.4128 | 1.3699 | 4.6235 | 0.1710 | 0.8776 | 1.7973 | 2.0549 | 0.8012 |
| 0.4 | 0.0259 | 5.4031 | 1.4211 | 4.6174 | 0.1713 | 0.7867 | 1.7967 | 2.1317 | 0.8205 |
| 0.5 | 0.0232 | 5.3918 | 1.4890 | 4.6114 | 0.1717 | 0.6941 | 1.7960 | 2.2335 | 0.8453 |
| 0.6 | 0.0193 | 5.3807 | 1.5837 | 4.6089 | 0.1725 | 0.5989 | 1.7959 | 2.3755 | 0.8782 |
| 0.7 | 0.0143 | 5.3743 | 1.7262 | 4.6177 | 0.1741 | 0.4999 | 1.7976 | 2.5894 | 0.9230 |
| 0.8 | 0.0089 | 5.3832 | 1.9722 | 4.6556 | 0.1775 | 0.3942 | 1.8039 | 2.9584 | 0.9873 |
| 0.9 | 0.0047 | 5.4274 | 2.5264 | 4.7612 | 0.1854 | 0.2698 | 1.8204 | 3.7896 | 1.0917 |

**Table 10.** Effect of the return probability $\delta$

Table 10 indicates that as $\delta$ increases, measures like expected switching rate, expected inventory level, expected number of customers in the buffer, expected number of retrial customers lost and expected number of departures after completing service decrease. However, expected number of customers in the orbit, expected number of primary customers lost, successful rate of retrials and overall rate of retrials increase.

**Effect of the arrival rate $\lambda$ on various performance measures**

$S = 20; s = 8; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6.$

| $\lambda$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.0123 | 4.7074 | 1.4302 | 3.8072 | 0.1175 | 0.375 | 1.6513 | 2.1453 | 0.8952 |
| 1.2 | 0.0128 | 4.8659 | 1.4965 | 4.0029 | 0.1309 | 0.4029 | 1.6886 | 2.2447 | 0.9017 |
| 1.3 | 0.0133 | 5.0304 | 1.5679 | 4.2038 | 0.1448 | 0.433 | 1.7254 | 2.3518 | 0.9085 |
| 1.4 | 0.0138 | 5.2003 | 1.6444 | 4.4091 | 0.1592 | 0.4653 | 1.7618 | 2.4666 | 0.9156 |
| 1.5 | 0.0143 | 5.3743 | 1.7262 | 4.6177 | 0.1741 | 0.4999 | 1.7976 | 2.5894 | 0.923 |
| 1.6 | 0.0149 | 5.5513 | 1.8134 | 4.8282 | 0.1895 | 0.5368 | 1.8328 | 2.72 | 0.9306 |
| 1.7 | 0.0155 | 5.7297 | 1.9058 | 5.0391 | 0.2053 | 0.5761 | 1.8671 | 2.8587 | 0.9386 |
| 1.8 | 0.0162 | 5.9076 | 2.0037 | 5.2484 | 0.2216 | 0.6176 | 1.9005 | 3.0055 | 0.9467 |
| 1.9 | 0.017 | 6.0832 | 2.1068 | 5.4544 | 0.2384 | 0.6616 | 1.9326 | 3.1603 | 0.955 |

**Table 11.** Effect of the arrival rate $\lambda$

Table 11 shows that expected number of customers in the orbit, expected number of customers in the buffer, expected number of primary customers lost, expected number of retrial customers lost, overall rate of retrials, successful rate of retrials, expected switching rate, expected inventory level and expected number of departures after completing service increase with the increase in $\lambda$.

**Effect of the retrial rate $\theta$ on various performance measures**

$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6.$

| $\theta$ | $ESR$ | $EI$ | $EO$ | $EB$ | $EL_1$ | $EL_2$ | $EDS$ | $ORR$ | $SRR$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.0082 | 5.4236 | 1.8798 | 4.4554 | 0.1561 | 0.3690 | 1.7906 | 2.0678 | 0.8377 |
| 1.2 | 0.0096 | 5.4062 | 1.8351 | 4.5024 | 0.1612 | 0.4021 | 1.7925 | 2.2022 | 0.8619 |
| 1.3 | 0.0111 | 5.3927 | 1.7951 | 4.5446 | 0.1658 | 0.4349 | 1.7943 | 2.3336 | 0.8840 |
| 1.4 | 0.0127 | 5.3823 | 1.7590 | 4.5827 | 0.1701 | 0.4675 | 1.7960 | 2.4626 | 0.9043 |
| 1.5 | 0.0143 | 5.3743 | 1.7262 | 4.6177 | 0.1741 | 0.4999 | 1.7976 | 2.5894 | 0.9230 |
| 1.6 | 0.0160 | 5.3685 | 1.6964 | 4.6500 | 0.1778 | 0.5322 | 1.7993 | 2.7142 | 0.9403 |
| 1.7 | 0.0176 | 5.3643 | 1.6690 | 4.6801 | 0.1813 | 0.5643 | 1.8009 | 2.8374 | 0.9565 |
| 1.8 | 0.0193 | 5.3615 | 1.6439 | 4.7082 | 0.1846 | 0.5962 | 1.8024 | 2.9590 | 0.9716 |
| 1.9 | 0.0209 | 5.3599 | 1.6206 | 4.7347 | 0.1876 | 0.6280 | 1.8040 | 3.0792 | 0.9857 |

**Table 12.** Effect of the retrial rate $\theta$

Table 12 shows that all the performance measures except expected number of customers in the orbit and expected inventory level increase with the increase in $\theta$. Increase in retrial rate $\theta$ decreases the number of customers in the orbit.

## 5 Cost Analysis

We define the expected total cost function as

$$ETC = (C + (S - s)c_1)ESR + c_2EI + c_3EO + c_4EB + c_5EL_1 + c_6EL_2 + (c_7 - c_8)EDS$$

where $C$ is the fixed cost, $c_1$ is the procurement cost /unit, $c_2$ is the holding cost of inventory /unit /unit time, $c_3$ is the holding cost of customers in the orbit /unit /unit time, $c_4$ is the holding cost of customers in the buffer /unit /unit time, $c_5$ is the cost due to loss of primary customers /unit /unit time, $c_6$ is the cost due to loss of retrial customers /unit /unit time, $c_7$ is the cost due to service /unit /unit time and $c_8$ is the revenue from service /unit /unit time.

This section provides graphical illustrations of variation of ETC with variation in values of underlying parameters.

Here, we compare the two models by calculating the expected total cost ($ETC$) per unit time by varying the parameters one at a time keeping others fixed. Fig. 1 compares the values of the cost function by varying the value of $\alpha$. For given parameter values, the cost function has minimum values 88.1788 at $\alpha = 0.1$ for model 1 and 60.6634 at $\alpha = 0.3$ for model 2. From fig.2, as $\beta$ varies $ETC$ has minimum values 44.1646 at $\beta = 1.4$ for model 1 and 21.7841 at $\beta = 1.5$ for model 2. From fig.3, as $\gamma$ varies $ETC$ has minimum values 28.7893 at $\gamma = 0.5$ for model 1 and 11.6076 at $\gamma = 0.2$ for model 2. The minimum values of $ETC$ at $\delta = 0.6$ for model 1 and at $\delta = 0.7$ for model 2, are 35.8605 and 18.1728 respectively as seen in fig. 4. In fig. 5 and fig. 6 one can also observe that $ETC$ is minimum for model 2. Hence in all cases model 2 is more efficient than model 1 in the given range of parameter values.

$$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \delta = 0.6; \mu = 3;$$
$$C = 20; c_1 = 1; c_2 = 1; c_3 = 1; c_4 = 1; c_5 = 1; c_6 = 1; c_7 = 28; c_8 = 1.$$

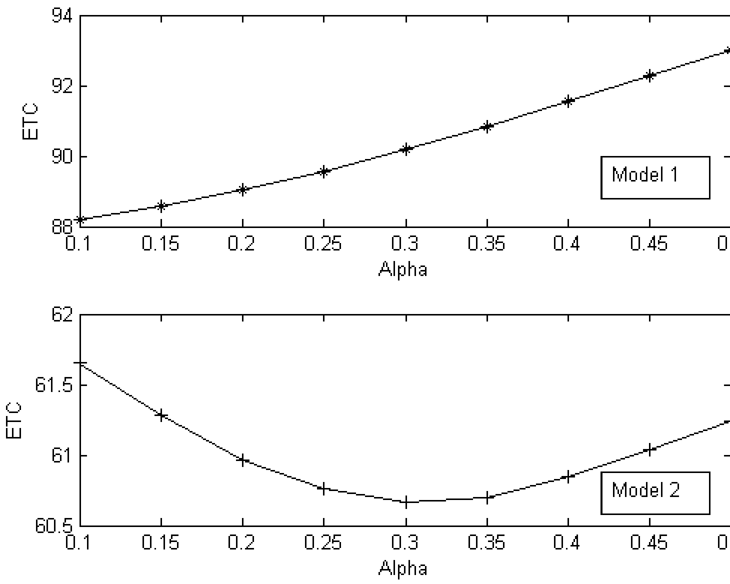


**Figure 1.** $ETC$ versus $\alpha$

$$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \delta = 0.7; \mu = 3; \alpha = 0.5;$$
$$C = 20; c_1 = 1; c_2 = 1; c_3 = 0.7; c_4 = 0.7; c_5 = 12.8; c_6 = 12.8; c_7 = 2; c_8 = 1.$$

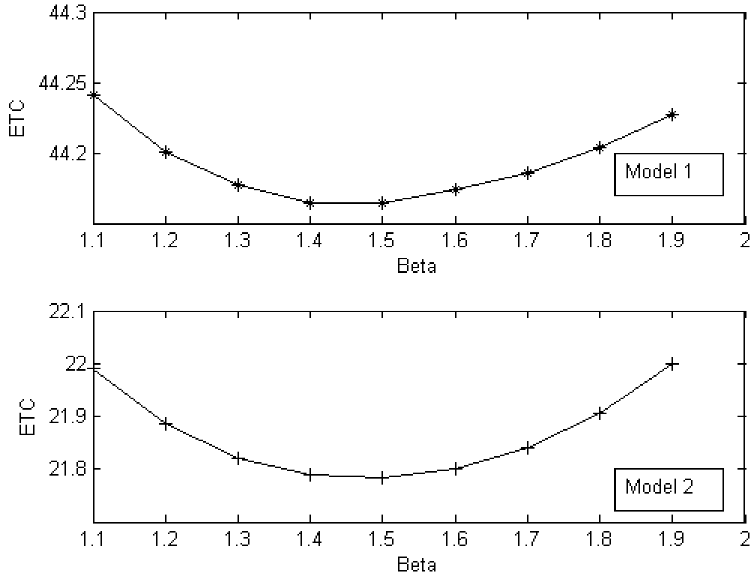


**Figure 2.** $ETC$ versus $\beta$

$$S = 20; s = 8; \lambda = 1.5; N = 25; \theta = 1.5; \beta = 2; \delta = 0.6; \mu = 3; \alpha = 0.5;$$
$$C = 20; c_1 = 1; c_2 = 1; c_3 = 1; c_4 = 1; c_5 = 2.33; c_6 = 1; c_7 = 2; c_8 = 1.$$

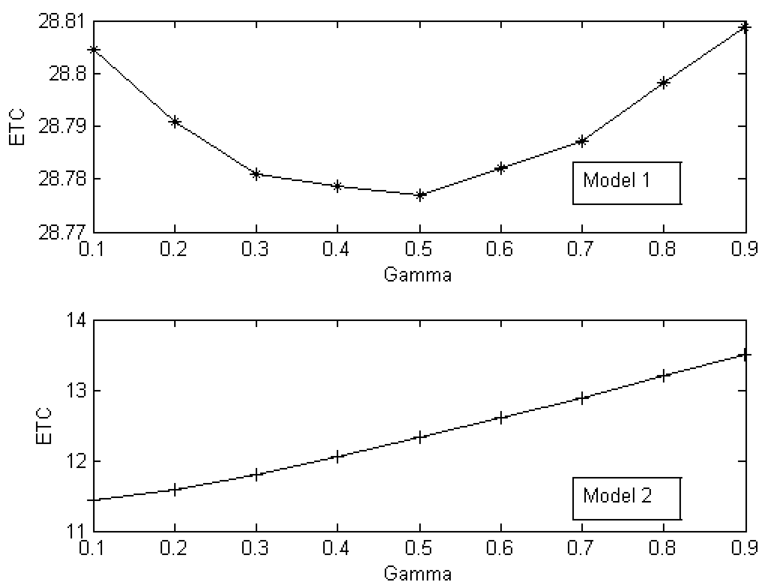


**Figure 3.** $ETC$ versus $\gamma$

$$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \mu = 3; \alpha = 0.6;$$
$$C = 20; c_1 = 1; c_2 = 1; c_3 = 1.3; c_4 = 1; c_5 = 2.5; c_6 = 1.1; c_7 = 3.5; c_8 = 1.$$

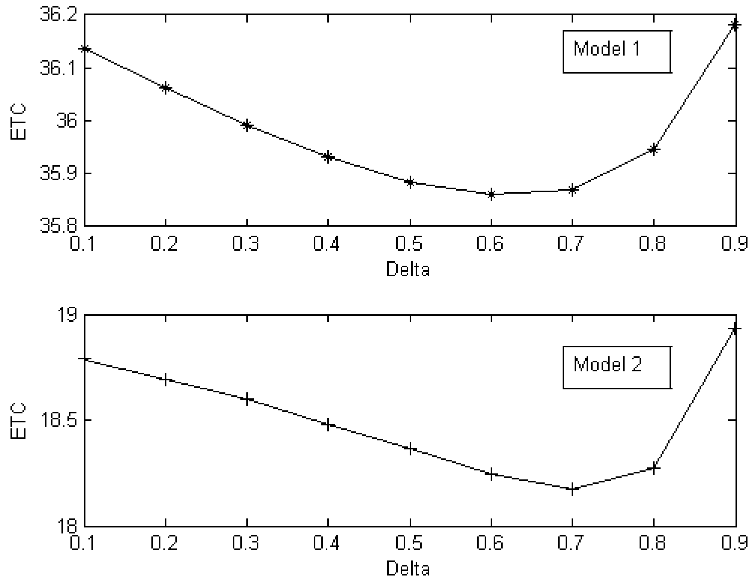


**Figure 4.** ETC versus $\delta$

$$S = 20; s = 8; \gamma = 0.8; N = 25; \theta = 1.5; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6;$$
$$C = 20; c_1 = 1; c_2 = 4.3; c_3 = 1; c_4 = 1; c_5 = 1; c_6 = 1; c_7 = 2; c_8 = 1.$$

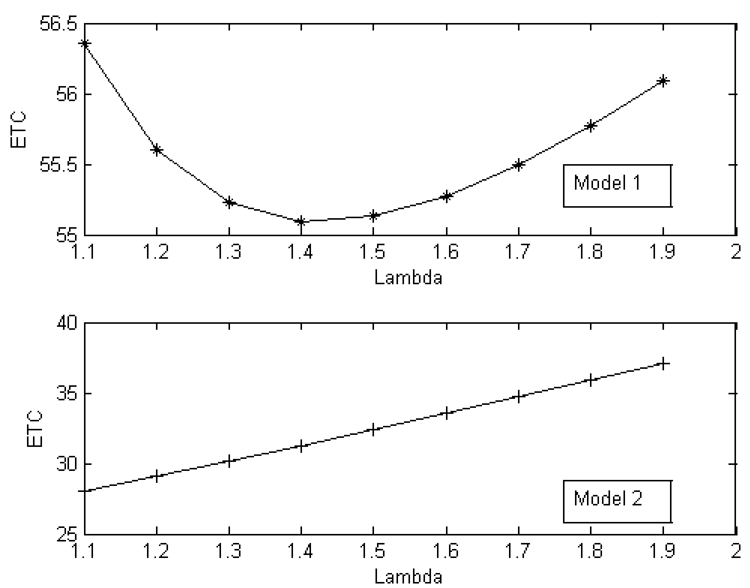


**Figure 5.** $ETC$ versus $\lambda$

$$S = 20; s = 8; \lambda = 1.5; \gamma = 0.8; N = 25; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6;$$
$$C = 20; c_1 = 1; c_2 = 2; c_3 = 3.5; c_4 = 1; c_5 = 1; c_6 = 1; c_7 = 2; c_8 = 1.$$

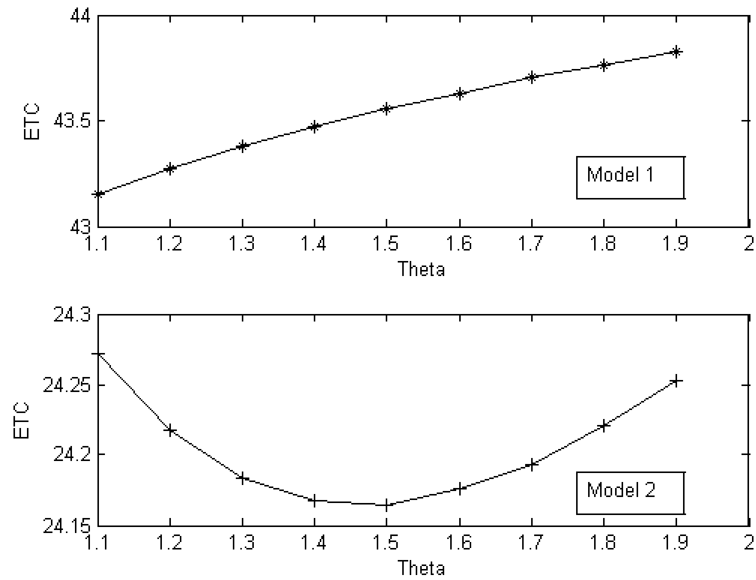


**Figure 6.** $ETC$ versus $\theta$

### 5.1 Optimum $(s, S)$ pair

Another important observation in this article is to find the optimum values of $s$ and $S$ for the efficient model. Optimum $(s, S)$ pairs of the efficient model are obtained by considering suitable parameter values and cost values. They are given in the following table.

$$\lambda = 1.5; \gamma = 0.8; N = 25; \beta = 2; \delta = 0.7; \mu = 3; \alpha = 0.6; \theta = 1.5; C = 20; c_1 = 1;$$
$$c_2 = 2; c_3 = 3.5; c_4 = 1; c_5 = 1; c_6 = 1; c_7 = 2; c_8 = 1.$$

| s \ S | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|
| 1 | 10.4043 | 10.7399 | 11.0903 | 11.4523 | 11.8232 |
| 2 | **10.2052** | **10.5163** | **10.8448** | **11.1872** | **11.5404** |
| 3 | 10.3915 | 10.678 | 10.9844 | 11.3072 | 11.6431 |
| 4 | 10.8278 | 11.0886 | 11.3718 | 11.6739 | 11.9915 |
| 5 | 11.4379 | 11.6712 | 11.9294 | 12.2091 | 12.5067 |
| 6 | 12.1745 | 12.3782 | 12.6095 | 12.8648 | 13.1405 |
| 7 | 13.0073 | 13.1788 | 13.3809 | 13.6097 | 13.8615 |

**Table 13.** Optimum $(s, S)$ pair

The minimum values of the cost function are 10.2052, 10.5163, 10.8448, 11.1872, 11.5404 and they are obtained at the pair values $(2, 16), (2, 17), (2, 18), (2, 19), (2, 20)$ respectively in the specified parameter values.

## 6 Conclusion and Future Study

This paper analyzed two production inventory systems with different service rates and retrials. The formulae for some important performance measures of the system and a suitable cost function were obtained. The optimum value of $\alpha$ corresponding to the minimum expected total cost is found. The minimum value of expected total cost by varying different parameters of the system was calculated and it was found that the model with buffer of varying capacity was efficient for practical purposes in the given range of parameter values. The optimum $(s, S)$ pair for the efficient model was calculated.

The models have many applications in industries like automobiles, drugs, textiles etc. For example, consider a manufacturing company of a particular brand of car. Here, each new car can be considered as an item in the inventory. The company receives many orders for purchasing a new car. When the stock of cars reduces to a particular level, then company reduces its rate of sale. This situation will reduce the loss of demands and increase the profit of the company.

The proposed models can be extended in several ways. For instance, it could be of interest to extend the exponential service to some other suitable probability distributions. The model may also be generalized by considering Markovian Arrival Process instead of Poisson arrivals.

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