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SPLIT DOMINATION OF CARTESIAN PRODUCT GRAPHS

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Abstract. A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with u adjacent to v. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$. A *dominating set* D of a graph $G = (V, E)$ is a *split dominating set* if the induced graph $\langle V - D \rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ is the minimum cardinality of a *split domination set*. The Cartesian graph product of $G_1 \times G_2$ called graph product of graphs with disjoint vertex sets and edge sets and is the graph with the vertex set $V_1 \times V_2$ and $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever $[u_1 = v_1$ and $u_2$adj $v_2]$ or $[u_2 = v_2$ and $u_1$adj $v_1]$. In this paper we have obtained the bounds for the cartesian product of paths, cycles and path with a cycle.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual $|V| = n$ and $|E| = q$ denote the number of vertices and edges of the graph G. Any undefined term will confirm to that in [1].

A subgraph H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced subgraph H of G is a subgraph with the added property that if $u, v \in V(H)$, then $uv \in E(H)$ if and only if $uv \in E(G)$ and it is denoted by $\langle H \rangle$.

The Cartesian graph product $G_1 \times G_2$ called graph product of graphs with disjoint vertex sets and edge sets and is the graph with the vertex set $V_1 \times V_2$ and $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever $[u_1 = v_1$ and $u_2$adj $v_2]$ or $[u_2 = v_2$ and $u_1$adj $v_1]$.

A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with u adjacent to v. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$. For a complete review on the topic of domination and its related parameters, see [6].

The concept of split domination has been studied by V. R. Kulli and B. Janikiram [2]. A *dominating set* D of a graph $G = (V, E)$ is a *split dominating set* if the induced graph $\langle V - D \rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ is the minimum cardinality of a *split domination set*. In this paper we have obtained the bounds for the cartesian product of paths, cycles and path with a cycle.

2 Main Results

Cartesian product of $P_m \times P_n$:

Theorem [4]: For $n \geq 2$, $\gamma(P_2 \times P_n) = \lfloor \frac{n+2}{2} \rfloor$.

Theorem 2.1. *For* $n \geq 2$

$$\gamma_s(P_2 \times P_n) = \begin{cases} \lfloor \frac{n+2}{2} \rfloor & n \text{ is even or } n = 3 \\ \lfloor \frac{n+2}{2} \rfloor + 1 & \text{Otherwise.} \end{cases}$$

Proof. Let $V(P_2 \times P_n) = \{(u_1, v_i), (u_2, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first and second row, respectively. We consider the following cases:
Case 1: $n = 2$.

The set $A = \{(u_2, v_1), (u_1, v_2)\}$ is the γ-set of the graph $P_2 \times P_n$ and $\langle (P_2 \times P_n) - A \rangle$ is disconnected. Hence $\gamma_s(P_2 \times P_n) = |A| = 2 = \lfloor \frac{n+2}{2} \rfloor$.
Case 2: $n = 3$.
The set $B = \{(u_1, v_2), (u_2, v_2)\}$ is the γ-set of the graph $P_2 \times P_n$ and $\langle (P_2 \times P_n) - B \rangle$ is disconnected. Hence $\gamma_s(P_2 \times P_n) = |B| = 2 = \lfloor \frac{n+2}{2} \rfloor$.
Case 3: n is even and $n \cong 0(mod4)$.
The set $C = \{(u_2, v_n), (u_1, v_i), i = 4p-1, 1 \leq p \leq \frac{n}{4}, (u_2, v_i), i = 4p-3, 1 \leq p \leq \frac{n}{4}\}$ is the γ-set of $P_2 \times P_n$ and the graph $\langle (P_2 \times P_n) - C \rangle$ is disconnected. Hence $\gamma_s(P_2 \times P_n) = |C| = \lfloor \frac{n+2}{2} \rfloor$.
Case 4: n is even and $n \not\cong 0(mod4)$.
The set $D = \{(u_1, v_n), (u_1, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil\}$ is the γ-set of $P_2 \times P_n$ and the graph $\langle (P_2 \times P_n) - D \rangle$ is disconnected. Hence $\gamma_s(P_2 \times P_n) = |D| = \lfloor \frac{n+2}{2} \rfloor$.
Case 5: n is odd and $n = 4k+1, k \geq 1$.
The set $E = \{(u_1, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n}{4} \rceil\}$ is the γ-set of $P_2 \times P_n$. No other γ-set with $|E|$ splits the graph and the graph $\langle (P_2 \times P_n) - (E \cup (u_1, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(P_2 \times P_n) = \lfloor \frac{n+2}{2} \rfloor + 1$.
Case 6: n is odd and $n \neq 4k+1, k \geq 1$.
The set $F = \{(u_1, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil\}$ is the γ-set of $P_2 \times P_n$. No other γ-set with $|F|$ splits the graph and the graph $\langle (P_2 \times P_n) - (F \cup (u_1, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(P_2 \times P_n) = \lfloor \frac{n+2}{2} \rfloor + 1$.□

Theorem [4]: For $n \geq 3$, $\gamma(P_3 \times P_n) = \lfloor \frac{3n+4}{4} \rfloor$.

Theorem 2.2. *For* $n \geq 3$

$$\gamma_s(P_3 \times P_n) = \begin{cases} \lfloor \frac{3n+4}{4} \rfloor & n \cong 0(mod4) \text{ or } n = 3 \\ \lfloor \frac{3n+4}{4} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Proof. Let $V(P_3 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second and third row, respectively. We consider the following cases:
Case 1: $n = 3$.
The set $A = \{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ is the γ_s set of $P_3 \times P_n$ and the graph $\langle (P_3 \times P_n) - A \rangle$ is disconnected. Hence $\gamma_s(P_3 \times P_n) = |A| = 3 = \lfloor \frac{3n+4}{4} \rfloor$.
Case 2: $n \cong 0(mod4)$.
The set $B = \{(u_1, v_i), (u_3, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_2, v_n)\}$ is the γ-set of $P_3 \times P_n$ and the graph $\langle (P_2 \times P_n) - B \rangle$ is disconnected. Hence $\gamma_s(P_3 \times P_n) = |B| = \lfloor \frac{3n+4}{4} \rfloor$.
Case 3: $n = 4k+1, k \geq 1$.
The set $C = \{(u_1, v_i), (u_3, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n}{4} \rceil\}$ is the γ-set of $P_3 \times P_n$. No other γ-set with $|C|$ splits the graph and the graph $\langle (P_3 \times P_n) - (C \cup (u_3, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(P_3 \times P_n) = |C \cup (u_3, v_{n-1})| = \lfloor \frac{3n+4}{4} \rfloor + 1$.
Case 4: $n = 4k+2, k \geq 1$.
The set $D = \{(u_1, v_i), (u_3, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil, (u_2, v_n)\}$ is the γ-set of $P_3 \times P_n$. No other γ-set with $|D|$ splits the graph and the graph $\langle (P_3 \times P_n) - (D \cup (u_3, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(P_3 \times P_n) = |D \cup (u_3, v_{n-1})| = \lfloor \frac{3n+4}{4} \rfloor + 1$.
Case 5: $n = 4k+3, k \geq 1$.
The set $E = \{(u_1, v_i), (u_3, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-4}{4} \rceil, (u_2, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil, (u_2, v_n), (u_2, v_{n-1})\}$ is the γ-set of $P_3 \times P_n$. No other γ-set with $|E|$ splits the graph and the graph $\langle (P_3 \times P_n) - (E \cup (u_3, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(P_3 \times P_n) = |E \cup (u_3, v_{n-1})| = \lfloor \frac{3n+4}{4} \rfloor + 1$.□

Theorem [4]: For $n \geq 4$,

$$\gamma(P_4 \times P_n) = \begin{cases} n+1 & n = 5, 6, 9 \\ n & \text{otherwise.} \end{cases}$$

Theorem 2.3. *For* $n \geq 4$, $\gamma_s(P_4 \times P_n) = n + 1$.

Proof. Let $V(P_4 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second, third and fourth row, respectively. We consider the following cases:
Case 1: $n = 4k, k \geq 1$.
The set $A = \{(u_1, v_i), i = 4p-2, 1 \leq p \leq \frac{n}{4}, (u_2, v_i), i = 4p, 1 \leq p \leq \frac{n}{4}, (u_3, v_i), i = 4p-3, 1 \leq p \leq \frac{n}{4}, (u_4, v_i), i = 4p-1, 1 \leq p \leq \frac{n}{4}\}$ is the γ-set of $P_4 \times P_n$. No other γ-set with $|A|$ splits the graph and the induced graph $\langle (P_4 \times P_n) - (A \cup (u_4, v_2)) \rangle$ is disconnected. Hence $\gamma_s(P_4 \times P_n) = |A \cup (u_4, v_2)| = n + 1$.
Case 2: $n = 5$ or 9.
The set $B = \{(u_1, v_i), i = 4p-2, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_1, v_n), (u_2, v_i), i = 4p, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil, (u_3, v_i), i = 4p-3, 1 \leq p \leq \lceil \frac{n}{4} \rceil, (u_4, v_i), i = 4p-1, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil\}$ is the γ-set of $P_4 \times P_n$ and the graph $\langle (P_4 \times P_n) - B \rangle$ is disconnected. Hence $\gamma_s(P_4 \times P_n) = |B| = n + 1$.
Case 3: $n = 6$.
The set $C = \{(u_1, v_i), i = 4p-2, 1 \leq p \leq \lceil \frac{n}{4} \rceil, (u_2, v_4), (u_3, v_1), (u_3, v_6), (u_4, v_3), (u_4, v_5)\}$ is the γ-set of $P_4 \times P_n$ and the graph $\langle (P_4 \times P_n) - C \rangle$ is disconnected. Hence $\gamma_s(P_4 \times P_n) = |C| = 7 = n + 1$.
Case 4: $n \neq 4k, 5, 6, 9, k \geq 1$.
We divide $P_4 \times P_n$ into m number of $P_4 \times P_4$ and $P_4 \times P_3$ blocks $B_i, i = 1, 2, 3....m$ such that m is minimum, $|V(B_i)| \geq |V(B_{i+1})|$ and $V(B_i) \cap V(B_{i+1}) = \phi$. Denote the vertices of $P_4 \times P_4$ as $(u_i, v_j), i = j = 1, 2, 3, 4$ and $P_4 \times P_3$ as $(p_i, q_j), i = 1, 2, 3, 4, j = 1, 2, 3$. Let $D = \{(u_3, v_1), (u_1, v_2), (u_2, v_4), (u_4, v_3)\}$ is the γ-set of each block of $P_4 \times P_4$. We consider the following sub-cases:
(i) When B_i contains only one copy of $P_4 \times P_3$.
Let the set $H = \{(p_1, q_2), (p_3, q_3), (p_4, q_1)\}$ are the vertices belongs to $P_4 \times P_3$ block. Then, the set $\{D \cup H\}$ is the γ-set of $(P_4 \times P_n)$ with $|D \cup H| = n$ and $\langle (P_4 \times P_n) - (D \cup H \cup (u_4, v_2)) \rangle, (u_4, v_2) \in B_1$ is disconnected. Hence $\gamma_s(P_4 \times P_n) = n + 1$.
(ii) When B_i contains two copies of $P_4 \times P_3$ say (B_i, B_{i+1}).
Let the set $F = \{(p_1, q_2), (p_3, q_3), (p_4, q_1)\}$ are the vertices belongs to B_i and $K = \{(p_1, q_1), (p_2, q_3), (p_4, q_2)\}$ are the vertices belongs to B_{i+1}. Then, the set $\{F \cup K \cup D\}$ is the γ-set of $(P_4 \times P_n)$ with $|F \cup K \cup D| = n$ and $\langle (P_4 \times P_n) - (D \cup F \cup K \cup (u_4, v_2)) \rangle, (u_4, v_2) \in B_1$ is disconnected. Hence $\gamma_s(P_4 \times P_n) = n + 1$.
(iii) When B_i contains three copies of $P_4 \times P_3$ say (B_i, B_{i+1}, B_{i+2}).
The set $M = \{(p_1, q_2), (p_3, q_3), (p_4, q_1)\}$ are the vertices belongs to each of B_i and B_{i+2} and $N = \{(p_1, q_1), (p_2, q_3), (p_4, q_2)\}$ are the vertices belongs to B_{i+1}.Then, the set $\{M \cup N \cup D\}$ is the γ-set of $(P_4 \times P_n)$ with $|M \cup N \cup D| = n$ and $\langle (P_4 \times P_n) - (D \cup M \cup N \cup (u_4, v_2)) \rangle, (u_4, v_2) \in B_1$ is disconnected. Hence $\gamma_s(P_4 \times P_n) = n + 1$.□

Theorem 2.4. *For* $m, n \geq 2$, $\gamma_s(P_m \times P_n) = n + 3(p-1) + 2(p-1)(q-1) + 1$, $m = 3p+1, n = 3q + 1, p \geq 1, q \geq 1$.

Proof. Let $V(P_m \times P_n) = \{(u_1, v_i), (u_2, v_i), ...(u_m, v_i), i = 1, 2, 3, ...n\}$.
Where $(u_1, v_i), (u_2, v_i), ...(u_m, v_i)$ are the vertices of first column, second column, third column and so on, respectively.
1^{st} column: Let $H_1 = \{(u_1, v_i), i \equiv 0(mod 6) \cup (u_1, v_i), i = 6k-4, k \geq 1, i = 1$ to $n\}$.
2^{nd} column: Let $H_2 = \{(u_2, v_i), i = 6k-2, k \geq 1, i = 1$ to $n\}$.
3^{rd} column: Let $H_3 = \{(u_3, v_i), i = 6k-5, k \geq 1, i = 1$ to $n\}$.
4^{th} column: Let $H_4 = \{(u_4, v_i), i = 6k-3 \cup (u_4, v_i), i = 6k-1, k \geq 1, i = 1$ to $n\}$.
5^{th} column: Let $H_5 = \{(u_5, v_i), i = 6k-5, k \geq 1, i = 1$ to $n\}$.
6^{th} column: Let $H_6 = \{(u_3, v_i), i = 6k-2, k \geq 1, i = 1$ to $n\}$.
7^{th} column onwards: For each $n = 1, 2, 3, 4, 5......$,
$H_{j+6n} = H_j, j = 1, 2, 3, 4, 5, 6$, and $u_{j+6} = u_j$,for $j = 1, 2, 3, 4, 5, 6............$
Then $D = (H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5 \cup H_6 \cup H_7...............)$ is the γ-set of the graph $P_m \times P_n$ with $|H_1 \cup H_2 \cup H_3 \cup H_4| = n$ and $|H_5 \cup H_6 \cup H_7 \cup ....| = 3(p-1) + 2(p-1)(q-1)$ with $|D| = n + 3(p-1) + 2(p-1)(q-1)$ and the induced graph $\langle (P_m \times P_n) - D \rangle$ is connected and the induced graph $\langle (P_m \times P_n) - (D \cup (u_2, v_1)) \rangle$ is disconnected. Hence $\gamma_s(P_m \times P_n) = n + 3(p-1) + 2(p-1)(q-1) + 1$.□

Cartesian Product of $C_m \times P_n$:

Theorem [3]: For $n \geq 2$,

$$\gamma(C_3 \times P_n) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1 & n \cong 0(mod 4) \\ \lceil \frac{3n}{4} \rceil & \text{otherwise.} \end{cases}$$

Theorem 2.5. *For $n \geq 2$, $\gamma_s(C_3 \times P_n) = \lceil \frac{3n}{4} \rceil + 1$.*

Proof. Let $V(C_3 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second and third row, respectively. We consider the following cases:
Case 1: $n = 2$.
The set $A = \{(u_1, v_2), (u_3, v_1)\}$ is the γ-set of $C_3 \times P_n$. No other γ-set with $|A|$ splits the graph and the graph $\langle (C_3 \times P_n) - (A \cup (u_2, v_2)) \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |A \cup (u_2, v_2)| = 3 = \lceil \frac{3n}{4} \rceil + 1$.
Case 2: $n = 3$.
The set $B = \{(u_1, v_3), (u_2, v_2), (u_3, v_1)\}$ is the γ-set of $C_3 \times P_n$. No other γ-set with $|B|$ splits the graph and the graph $\langle (C_3 \times P_n) - (B \cup (u_3, v_3)) \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |B \cup (u_3, v_3)| = 4 = \lceil \frac{3n}{4} \rceil + 1$.
Case 3: $n = 4k, k \geq 1$.
The set $C = \{(u_1, v_i), i = 4p - 3, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_2, v_i), (u_3, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil, (u_1, v_n)\}$ is the γ-set of $C_3 \times P_n$ and the graph $\langle (C_3 \times P_n) - C \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |C| = \lceil \frac{3n}{4} \rceil + 1$.
Case 4: $n = 4k + 1.k \geq 1$.
The set $D = \{(u_1, v_i), i = 4p - 3, 1 \leq p \leq \lceil \frac{n}{4} \rceil, (u_2, v_i), (u_3, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil\}$ is the γ-set of $C_3 \times P_n$. No other γ-set with $|D|$ splits the graph and the graph $\langle (C_3 \times P_n) - (D - (u_1, v_n)) \cup ((u_1, v_{n-1}), (u_2, v_n)) \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |(D - (u_1, v_n)) \cup ((u_1, v_{n-1}), (u_2, v_n))| = \lceil \frac{3n}{4} \rceil + 1$.
Case 5: $n = 4k + 2, k \geq 1$.
The set $E = \{(u_1, v_i), i = 4p - 3, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil, (u_2, v_i), (u_3, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_2, v_n)\}$ is the γ-set of $C_3 \times P_n$. No other γ-set with $|E|$ splits the graph and the graph $\angle (C_3 \times P_n) - (E \cup (u_3, v_n)) \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |E \cup (u_3, v_n)| = \lceil \frac{3n}{4} \rceil + 1$.
Case 6: $n = 4k + 3, k \geq 1$.
The $F = \{(u_1, v_i), i = 4p - 3, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil, (u_2, v_i), (u_3, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n}{4} \rceil\}$ is the γ-set of $C_3 \times P_n$. No other γ-set with $|E|$ splits the graph and the graph $\langle C_3 \times P_n - (F \cup (u_1, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |F \cup (u_1, v_{n-1})| = \lceil \frac{3n}{4} \rceil + 1.\square$

Theorem [3]: For $n \geq 2$, $\gamma(C_4 \times P_n) = n$.

Theorem 2.6. *For $n \geq 2$, $\gamma_s(C_4 \times P_n) = n + 2$.*

Proof. Let $V(C_4 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second, third and fourth row, respectively. We consider the following cases:
Case 1: n is even.
The set $A = \{(u_2, v_i), i = 2p - 1, 1 \leq p \leq \lceil \frac{n-1}{2} \rceil, (u_4, v_i), i = 2p, 1 \leq p \leq \frac{n}{2}\}$ is the γ-set of $C_4 \times P_n$. No other γ-set with $|A|$ splits the graph and the graph $\langle (C_4 \times P_n) - (A \cup (u_1, v_1), u_3, v_1)) \rangle$ is disconnected. Hence $\gamma_s(C_3 \times P_n) = |A \cup ((u_1, v_1), (u_3, v_1))| = n + 2$.
Case 2: n is odd.
The set $B = \{(u_2, v_i), i = 2p - 1, 1 \leq p \leq \lceil \frac{n}{2} \rceil, (u_4, v_i), i = 2p, 1 \leq p \leq \lceil \frac{n-1}{2} \rceil\}$ is the γ-set of $C_4 \times P_n$. No other-γ set with $|B|$ splits the graph and the graph $\langle (C_4 \times P_n) - (B \cup (u_1, v_1), u_3, v_1)) \rangle$ is disconnected. Hence $\gamma_s(C_4 \times P_n) = |B \cup ((u_1, v_1), (u_3, v_1))| = n + 2.\square$

Theorem [3]: For $n \geq 2$

$$\gamma(C_5 \times P_n) = \begin{cases} 3 & n = 2 \\ 4 & \text{n=3} \\ n + 2 & \text{otherwise} \end{cases}$$

Theorem 2.7. *For* $n \geq 2$

$$\gamma_s(C_5 \times P_n) = \begin{cases} n+2 & n = 2 \\ n+3 & \textit{otherwise} \end{cases}$$

Proof. Let $V(C_5 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i), (u_5, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second, third, fourth and fifth row, respectively. We consider the following cases:
Case 1: $n = 2$.
The set $A = \{(u_1, v_1), (u_3, v_2), (u_4, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|A|$ splits the graph and the graph $\langle (C_5 \times P_n) - (A - (u_3, v_2)) \cup ((u_2, v_2), (u_3, v_1)) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |(A - (u_3, v_2)) \cup ((u_2, v_2), (u_3, v_1))| = 4 = n + 2$.
Case 2: $n = 3$.
The set $B = \{(u_1, v_2), (u_3, v_1), (u_3, v_3), (u_5, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|B|$ splits the graph and the graph $\langle (C_5 \times P_n) - (B \cup ((u_1, v_3), (u_2, v_2)) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |B \cup (u_1, v_3), (u_2, v_2))| = 6 = n + 3$.
Case 3: $n = 5k - 2, k \geq 2$.
The set $C = \{(u_1, v_i), i = 5p - 3, 1 \leq p \leq \lceil \frac{n-1}{5} \rceil$, (u_1, v_n), $(u_2, v_i), i = 5p, 1 \leq p \leq \frac{n-3}{5}$, $(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n}{5} \rceil, (u_3, v_1), (u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-2}{5} \rfloor$, $(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n-4}{5} \rceil$,$(u_5, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|C|$ splits the graph and the graph $\langle (C_5 \times P_n) - (C \cup (u_2, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |C \cup (u_2, v_{n-1})| = n + 3$.
Case 4: $n = 5k - 1, k \geq 1$.
The set $D = \{(u_1, v_i), i = 5p - 3, p \geq 1, 1 \leq p \leq \lceil \frac{n-2}{5} \rceil$,$(u_2, v_i), i = 5p, 1 \leq p \leq \lceil \frac{n-4}{5} \rceil, n \geq 9$,$(u_2, v_n)$, $(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n-1}{5} \rceil$,$(u_3, v_1)$,$(u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-3}{5} \rfloor, n \geq 9$, $(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n}{5} \rceil$,$(u_5, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|D|$ splits the graph and $\langle (C_5 \times P_n) - (D \cup (u_4, v_n)) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |(D \cup ((u_4, v_n))| = n + 3$.
Case 5: $n = 5k, k \geq 1$.
The set $E = \{(u_1, v_i), i = 5p - 3, 1 \leq p \leq \lceil \frac{n-3}{5} \rceil$,$(u_2, v_i), i = 5p, 1 \leq p \leq \lceil \frac{n}{5} \rceil$, $(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n-2}{5} \rceil$,$(u_3, v_1)$,$(u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-4}{5} \rfloor, n \geq 10$,$(u_4, v_n)$, $(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n-1}{5} \rceil$,$(u_5, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|E|$ splits the graph and $\langle (C_5 \times P_n) - (E \cup (u_3, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |(E \cup ((u_3, v_{n-1}))| = n + 3$.
Case 6: $n = 5k + 1, k \geq 1$.
The set $F = \{(u_1, v_i), i = 5p - 3, 1 \leq p \leq \lceil \frac{n-4}{5} \rceil$,$(u_2, v_i), i = 5p, 1 \leq p \leq \lceil \frac{n-1}{5} \rceil$,$(u_2, v_n)$, $(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n-3}{5} \rceil$,$(u_3, v_1)$,$(u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-5}{6} \rfloor, n \geq 11$,$(u_4, v_n)$,$(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n-2}{5} \rceil$,$(u_5, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|F|$ splits the graph and $\langle (C_5 \times P_n) - (F \cup (u_3, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |(E \cup ((u_3, v_{n-1}))| = n + 3$.
Case 7: $n = 5k + 2, k \geq 1$.
The set $H = \{(u_1, v_i), i = 5p - 3, 1 \leq p \leq \lceil \frac{n}{5} \rceil$,$(u_2, v_i), i = 5p, 1 \leq p \leq \lceil \frac{n-2}{5} \rceil$,$(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n-3}{5} \rceil$,$(u_3, v_1)$,$(u_3, v_n)$,$(u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-1}{5} \rfloor$,$(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n-3}{5} \rceil$,$(u_5, v_2)\}$ is the γ-set of $C_5 \times P_n$. No other γ-set with $|H|$ splits the graph and $\langle (C_5 \times P_n) - (H \cup (u_2, v_{n-1})) \rangle$ is disconnected. Hence $\gamma_s(C_5 \times P_n) = |(H \cup ((u_2, v_{n-1}))| = n + 3$.□

Theorem 2.8. *For any graph* $m, n \geq 2$, $\gamma_s(C_m \times P_n) = \frac{mn}{4} + 2, m = 4p, n = 3q, p, q \geq 1$.

Proof. Let $V(C_m \times P_n) = \{(u_1, v_i), (u_2, v_i), ...(u_m, v_i)\}$. Where $(u_1, v_i), (u_2, v_i)$ denotes the vertices of first row, second row and so on, respectively. Divide $V(C_m \times P_n)$ into $m \times 3$ blocks and let B_i, $i = 1, 2, 3, ...(\frac{n}{3})$ be such blocks. For each block B_i denote the vertex set as $(u_i, v_j), i = 1, 2, 3, ...m, j = 1, 2, 3$. The set $A = \{(u_{4r}, v_1), (u_{4r}, v_3), (u_{4r-2}, v_2), r \geq 1, r = 1$ to $\frac{m}{4}\}$ with $|A| = \frac{3m}{4}$ is the γ-set of each block B_i. Let D be the γ set of $C_m \times P_n$ with $|D| = \frac{3m}{4} * \frac{n}{3}$. Therefore $\gamma(C_m \times P_n) = (\frac{mn}{4})$. No γ-set with $|D|$ or $|D + 1|$ splits the graph and the induced graph $\langle (C_m \times P_n) - (D \cup (u_{m-1}, v_2) \cup (u_{m-2}, v_1)) \rangle$ is disconnected. Hence $\gamma_s(C_m \times P_n) = \frac{mn}{4} + 2$.□

Cartesian product of $C_m \times C_n$:

Theorem[3]: For $n \geq 3$, $\gamma_s(C_3 \times C_n) = \lceil \frac{3n}{4} \rceil$.

Theorem 2.9. *For $n \geq 4$*

$$\gamma_s(C_3 \times C_n) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1 & n = 4k+2, k \geq 1, n = 3 \\ \lceil \frac{3n}{4} \rceil + 2 & n \neq 4k+2,\ k \geq 1. \end{cases}$$

Proof. Let $V(C_3 \times C_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second and third row, respectively. We consider the following cases:
Case 1: $n = 3$
In $C_3 \times C_n$, the subset $H = \{(u_1, v_1), (u_2, v_1), (u_3, v_3)\}$ is the γ-set of $C_3 \times C_n$ with $|H| = 3 = \lceil \frac{3n}{4} \rceil$. No other γ-set with $|H|$ splits the graph . The set $H \cup \{(u_3, v_2)\}$ is the γ_s set of $C_3 \times C_n$. Hence $\gamma_s(C_3 \times C_n) = \lceil \frac{3n}{4} \rceil + 1$.
Case 2: $n = 4k + 2, k \geq 1$.
In $C_3 \times C_n$, the subset $D = \{A \cup B \cup C\}$ where, $A = \{(u_1, v_i), i = 4p - 3, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil\}$, $B = \{(u_2, v_i), (u_3, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil\}$ and $C = \{(u_2, v_n)\}$ is the γ-set of $C_3 \times C_n$ with $|D| = \lceil \frac{3n}{4} \rceil$. No other γ-set with $|D|$ splits the graph . The set $D \cup \{(u_3, v_n)\}$ is the γ_s set of $C_3 \times C_n$. Hence $\gamma_s(C_3 \times C_n) = \lceil \frac{3n}{4} \rceil + 1$.
Case 3:$n \neq 4k + 2, k \geq 1$.
In $C_3 \times C_n$, the subset $F = \{A \cup B\}$ where, $A = \{(u_1, v_i), i = 4p - 3, 1 \leq p \leq \lceil \frac{n}{4} \rceil\}$, $B = \{(u_2, v_i), (u_3, v_i), i = 4p - 1, p \geq 1, i = 1$ to $n\}$ is the γ-set of $C_3 \times C_n$ with $|F| = \lceil \frac{3n}{4} \rceil$. No other γ-set with $|F|$ splits the graph . The set $F \cup \{(u_1, v_2) \cup (u_1, v_4)\}$ is the γ_s set of $C_3 \times C_n$. Hence $\gamma_s(C_3 \times C_n) = \lceil \frac{3n}{4} \rceil + 2$.□

Theorem[3]: For $n \geq 4$, $\gamma(C_4 \times C_n) = n$.

Theorem 2.10. *For $n \geq 4$, $\gamma_s(C_4 \times C_n) = n + 2$.*

Proof. Let $V(C_4 \times C_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second, third and fourth row, respectively. We consider the following cases:
Case 1: $n = 4k, k \geq 1$.
Let $A = \{(u_1, v_i), i = 4p + 1, 1 \leq p \leq \lfloor \frac{n-3}{4} \rfloor, n \geq 8,\ (u_2, v_i), i = 4p - 2, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil, (u_2, v_n), (u_3, v_i), i = 4p, 1 \leq p \leq \lceil \frac{n-4}{4} \rceil, n \geq 8,\ (u_4, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil,$ $(u_4, v_1))\}$ is the γ-set of $C_4 \times C_n$ with $|A| = n$. No other γ-set with $|D|$ splits the graph . The set $A \cup \{(u_1, v_2), (u_3, v_2)\}$ is the γ_s set of $C_4 \times C_n$. Hence $\gamma_s(C_4 \times C_n) = n + 2$.
Case 2: $n = 4k + 1, k \geq 1$.
Let $B = \{(u_1, v_i), i = 4p + 1, 1 \leq p \leq \lfloor \frac{n}{4} \rfloor,\ (u_2, v_i), i = 4p - 2, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil, (u_3, v_i), i = 4p, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil,\ (u_4, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil,\ (u_4, v_1)\}$ is the γ-set of $C_4 \times C_n$ with $|B| = n$. No other γ-set with $|B|$ splits the graph . The set $B \cup \{(u_1, v_2), (u_3, v_2)\}$ is the γ_s set of $C_4 \times C_n$. Hence $\gamma_s(C_4 \times C_n) = n + 2$.
Case 3: $n = 4k + 2, k \geq 1$.
Let $C = \{(u_1, v_i), i = 4p + 1, 1 \leq p \leq \lfloor \frac{n-1}{4} \rfloor,\ (u_2, v_i), i = 4p - 2, 1 \leq p \leq \lceil \frac{n}{4} \rceil, (u_3, v_i), i = 4p, 1 \leq p \leq \lceil \frac{n-2}{4} \rceil,\ (u_4, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil,\ (u_4, v_1)\}$ is the γ-set of $C_4 \times C_n$ with $|C| = n$. No other γ-set with $|C|$ splits the graph . The set $C \cup \{(u_1, v_2), (u_3, v_2)\}$ is the γ_s set of $C_4 \times C_n$. Hence $\gamma_s(C_4 \times C_n) = n + 2$.
Case 4: $n = 4k + 3, k \geq 1$.
Let $D = \{(u_1, v_i), i = 4p + 1, 1 \leq p \leq \lfloor \frac{n-2}{4} \rfloor,\ (u_2, v_i), i = 4p - 2, 1 \leq p \leq \lceil \frac{n-1}{4} \rceil, (u_3, v_i), i = 4p, 1 \leq p \leq \lceil \frac{n-3}{4} \rceil,\ (u_4, v_i), i = 4p - 1, 1 \leq p \leq \lceil \frac{n}{4} \rceil,\ (u_4, v_1)\}$ is the γ-set of $C_4 \times C_n$ with $|D| = n$. No other γ-set with $|D|$ splits the graph . The set $D \cup \{(u_1, v_2), (u_3, v_2)\}$ is the γ_s set of $C_4 \times C_n$. Hence $\gamma_s(C_4 \times C_n) = n + 2$.□

Theorem[3]: For $n \geq 5$,

$$\gamma(C_5 \times C_n) = \begin{cases} n & n \cong 0 (mod 5) \\ n + 2 & \text{n} \cong 3 \text{ (mod 5)} \\ n + 1 & \text{otherwise} \end{cases}$$

Theorem 2.11. *For* $n \geq 5$, $\gamma_s(C_5 \times C_n) = n + 3$.

Proof. Let $V(C_5 \times C_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i), (u_5, v_i), i = 1, 2, 3, ...n\}$ be the vertices of first, second, third, fourth and fifth row, respectively.
Let $A = \{(u_4, v_1), (u_1, v_2), (u_2, v_n), (u_5, v_{n-1})\}$. We consider the following cases:
Case 1: $n \cong 0(mod5)$.
Let $B = \{(u_1, v_i), i = 5p + 2, 1 \leq p \leq \lfloor \frac{n-1}{5} \rfloor, n \geq 10$, $(u_2, v_i), i = 5p, 1 \leq p \leq \lceil \frac{n-5}{5} \rceil, n \geq 10$,$(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n-2}{5} \rceil$, $(u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-4}{5} \rfloor, n \geq 10$, $(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n-6}{5} \rceil, n \geq 10\} \cup A$ is the γ-set of $C_5 \times C_n$ with $|A \cup B| = n$. No other γ-set with $|(B \cup A)|$ splits the graph. The set $\{(A \cup B) \cup (u_3, v_1), (u_5, v_1), (u_4, v_n)\}$ is the γ_s set of $C_5 \times C_n$. Hence $\gamma_s(C_5 \times C_n) = n + 3$.
Case 2: $n \cong 3(mod5)$.
Let $C = \{(u_1, v_i), i = 5p{+}2, 1 \leq p \leq \lfloor \frac{n-6}{5} \rfloor, n \geq 13$, $(u_2, v_i), i = 5p, 1 \leq p \leq \lceil \frac{n-3}{5} \rceil$,$(u_3, v_i), i = 5p - 2, 1 \leq p \leq \lceil \frac{n-5}{5} \rceil$, $(u_4, v_i), i = 5p + 1, 1 \leq p \leq \lfloor \frac{n-2}{5} \rfloor$, $(u_5, v_i), i = 5p - 1, 1 \leq p \leq \lceil \frac{n-4}{5} \rceil$, $(u_3, v_{n-2}), (u_1, v_{n-2})\} \cup A$ is the γ-set of $C_5 \times C_n$ with $|C \cup B| = n + 2$. No other γ-set with $|(C \cup A)|$ splits the graph . The set $\{((A \cup C) \cup (u_2, v_{n-1}))\}$ is the γ_s set of $C_5 \times C_n$. Hence $\gamma_s(C_5 \times C_n) = n + 3$.
Case 3: $n \ncong 0(mod5)$, $n \ncong 3(mod5)$.
Let $D = \{(u_1, v_i), i = 5p + 2, p \geq 1, i = 9$ to n, $(u_2, v_i), i = 5p, p \geq 1, i = 5$ to n,$(u_3, v_i), i = 5p - 2, p \geq 1, i = 3$ to n, $(u_4, v_i), i = 5p + 1, p \geq 1, i = 6$ to n, $(u_5, v_i), i = 5p - 1, p \geq 1, i = 4$ to $n - 2\} \cup A$ is the γ-set of $C_5 \times C_n$ with $|A \cup D| = n + 1$. No other γ-set with $|(D \cup A)|$ splits the graph. We consider the following sub-cases:
(a) $n = 5k + 1, k \geq 1$.
The set $\{(D \cup A) \cup (u_5, v_1), (u_1, v_n))\}$ is the γ_s set of $C_5 \times C_n$. Hence $\gamma_s(C_5 \times C_n) = n + 3$.
(b) $n = 5k + 2, k \geq 1$.
The set $\{(D \cup A) \cup (u_1, v_{n-2}), (u_4, v_{n-2}))\}$ is the γ_s set of $C_5 \times C_n$. Hence $\gamma_s(C_5 \times C_n) = n + 3$.
(c) $n = 5k + 4, k \geq 1$.
The set $\{(D \cup A) \cup (u_2, v_{n-2}), (u_1, v_{n-1}))\}$ is the γ_s set of $C_5 \times C_n$. Hence $\gamma_s(C_5 \times C_n) = n + 3$.□

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