Palestine Journal of Mathematics

Vol. 8(2)(2019) , 355–364

Some bounds on sum Connectivity Banhatti Index of graphs

V. R. Kulli, †B. Chaluvaraju and †† H.S. Boregowda

Communicated by Ayman Badawi

MSC 2010 Classifications: 05C05, 05C012, 05C35.

Keywords: Graph; Molecular descriptor; Sum connectivity Banhatti index.

Abstract. Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The sum connectivity Banhatti index of a graph G is defined as $SB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u)+d_G(e)}}$, where ue means that the vertex u and edge e are incident in G. In this paper, we obtain lower and upper bounds of $SB(G)$ in terms order, size, minimum / maximum degrees and minimal non-pendant vertex degree by using some classical inequalities. Also, we obtain the relationship between $SB(G)$ in terms of some degree based topological indices such as sum connectivity, product connectivity, K Banhatti and Zagreb-type indices of G. Additionally, we give the Nordhaus-Gaddum-type result for $SB(G)$.

1 Introduction

All graphs considered in this paper are finite, connected, undirected without loops and multiple edges. For all further notation and terminology, we refer the reader to [5].

Let $G = (V, E)$ be a connected graph with n vertices and m edges. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The degree of an edge $e = uv$ in G is defined by $d_G(e) = d_G(u) + d_G(v) - 2$.

A molecular graph is a graph such that its vertices correspond to the atoms and the edges to the bonds. Chemical graph theory is a branch of Mathematical chemistry which has an important effect on the development of the chemical sciences. A single number that can be used to characterize some property of the graph of a molecular is called a topological index for that graph. There are numerous molecular descriptors, which are also referred to as topological indices, see [3] that have found some applications in theoretical chemistry, especially in QSPR/QSAR research.

One of the best known and widely used topological index is the product-connectivity index (or Randić index, connectivity index) by Randić [11], who has shown this index to reflect molecular branching. The product connectivity index of a graph G is defined as $P(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$. Motivated by Randić definition of the product connectivity index, the sum connectivity index was initiated by Zhou and Trinajstic [16] and [17], which is defined by $S(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)+d_G(v)}}$. For more details on these type of connectivity indices, we refer to [1], [2] and [9].

The first and second K Banhatti indices of G are defined as $B_1(G) = \sum_{ue}[d_G(u) + d_G(e)]$ and $B_2(G) = \sum_{ue}[d_G(u)d_G(e)]$, where ue means that the vertex u and edge e are incident in G. The K Banhatti indices were introduced by Kulli in [6]. The K Banhatti indices are closely related to Zagreb - types indices. For more details on these two types of indices refer to Gutman et al., [4].

In [7], Kulli et al., introduce the sum connectivity Banhatti index of G, which is defined as $SB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u)+d_G(e)}}$, where ue means that the vertex u and edge e are incident in G.

2 Existing Results

Here, we use the following existing results of the sum connectivity Banhatti index of some standard classes of graphs such as Cycle C_n, Complete graph K_n and Complete bipartite graph $K_{r,s,}$.

Theorem 2.1. [8]

(i) $SB(C_n) = n$ *, for* $n \geq 3$ *vertices,*

(ii) $SB(K_n) = \frac{n(n-1)}{\sqrt{3n-5}}$*, for* $n \geq 3$ *vertices,*

(iii) $SB(K_{r,s}) = rs\left[\frac{1}{\sqrt{r+2s-2}} + \frac{1}{\sqrt{2r+s-2}}\right]$*, for* $1 \leq r \leq s$ *and* $s \geq 2$ *vertices,*

(iv) $SB(G) = \frac{nr}{\sqrt{3r-2}}$*, where* G *is a* r*-regular graph with* $r \geq 1$.

In order to prove some bounds on the product connectivity Banhatti index $PB(G)$, we make use of the following results.

Theorem 2.2. [7] *For any connected graph* G *with* $n \geq 3$ *vertices and no pendant vertices,*

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}} \leq PB(G) \leq n.$$

Further, equality holds in lower bound if and only if $G \cong C_3$ *and an equality holds in upper bound if and only if* $G \cong C_n; n \geq 3$.

3 Bounds on sum connectivity Banhatti index

First, we start with upper bound of $SB(G)$ in terms of the sum connectivity index $S(G)$ of a graph G.

Theorem 3.1. *For any* (n, m)*- connected graph* G *with* $\delta(G) \geq 2$ *and* $n \geq 3$ *vertices,*

$$SB(G) \leq 2\, S(G).$$

Further, equality is attained if and only if $G \cong C_n$.

Proof. Let G be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Consider the sum connectivity Banhatti index of G is

$$SB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}}$$

and the sum connectivity index of G is

$$S(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) + d_G(v)}}.$$

Since $\delta(G) \geq 2$, $d_G(uv) \geq d_G(u)$ and $d_G(uv) \geq d_G(v)$ for any edge $e = uv \in E(G)$.
Therefore $\sqrt{d_G(u) + d_G(uv)} \geq \sqrt{d_G(u) + d_G(v)}$.
Hence

$$\begin{aligned} SB(G) &= \sum_{uv \in E(G)} \left[\frac{1}{\sqrt{d_G(u) + d_G(uv)}} + \frac{1}{\sqrt{d_G(v) + d_G(uv)}}\right] \\ &\leq \sum_{uv \in E(G)} \frac{2}{\sqrt{d_G(u) + d_G(v)}}. \end{aligned}$$

Thus the upper bound of $SB(G)$ follows.
The equality case attains directly from (i) of Theorem 2.1. □

In order to prove the lower bound along with characterization of $SB(G)$ in terms of the size m and first K Banhatti index $B_1(G)$ of G, we recall the following facts.

If real valued function $f(x)$ defined on an interval has a second derivative $f^{''}(x)$ then a necessary and sufficient condition for it to be strictly convex on that interval is that $f^{''}(x) > 0$. For positive integer k, if $f(x)$ is strictly convex, then (by Jensen's inequality) we have $f\left(\sum_{i=1}^{k} \frac{x_i}{k}\right) \leq f(x_i)$ with equality if and only if $x_1 = x_2 = \cdots = x_k$, and if $-f(x)$ is strictly convex, then the inequality is reversed.

Theorem 3.2. *For any (n, m)-connected graph G with $n \geq 3$ vertices,*

$$SB(G) \geq \frac{(2m)^{\frac{3}{2}}}{\sqrt{B_1(G)}}.$$

Further, equality is attained if and only if G is a regular graph.

Proof. Let G be a connected graph with $n \geq 3$ vertices. Then

$$SB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}} = \sum_{ue} [d_G(u) + d_G(e)]^{-\frac{1}{2}}.$$

By Jensen's inequality, $\frac{1}{\sqrt{x}}$ is a convex function for $x > 0$, we have

$$\sum_{ue} \frac{[d_G(u) + d_G(e)]^{-\frac{1}{2}}}{2m} \geq \left[\sum_{ue} \frac{d_G(u) + d_G(e)}{2m}\right]^{-\frac{1}{2}}.$$

Therefore

$$\begin{aligned} SB(G) &\geq 2m\left[\sum_{ue} \frac{d_G(u) + d_G(e)}{2m}\right]^{-\frac{1}{2}} \\ &\geq \frac{2\sqrt{2}\ m\sqrt{m}}{\sqrt{\sum_{ue} [d_G(u) + d_G(e)]}}. \end{aligned}$$

Thus the result follows.
The equality case attains directly from (iv) of Theorem 2.1. □

Now, we obtain lower and upper bounds of $SB(G)$ in terms of the minimum and maximum degrees, the number of pendant vertices and minimal non-pendant vertices of G.

Theorem 3.3. *For any (n, m)- connected graph G with η pendant vertices and minimal non-pendant vertex degree $\delta_1(G)$,*

$$SB(G) \leq \eta\left[\frac{\sqrt{2\delta_1(G) - 1} + \sqrt{\delta_1(G)}}{\sqrt{\delta_1(G)(2\delta_1(G) - 1)}}\right] + \left[\frac{2(m - \eta)}{\sqrt{3\delta_1(G) - 2}}\right]$$

and

$$SB(G) \geq \eta\left[\frac{\sqrt{2\Delta(G) - 1} + \sqrt{\Delta(G)}}{\sqrt{\Delta(G)(2\Delta(G) - 1)}}\right] + \left[\frac{2(m - \eta)}{\sqrt{3\Delta(G) - 2}}\right].$$

Proof. We have

$$\begin{aligned}
SB(G) &= \sum_{e=uv\in E(G)}\left[\frac{1}{\sqrt{d_G(u)+d_G(e)}}+\frac{1}{\sqrt{d_G(v)+d_G(e)}}\right]\\
&= \sum_{e=uv\in E(G);d_G(u)=1}\left[\frac{1}{\sqrt{d_G(v)}}+\frac{1}{\sqrt{2d_G(v)-1}}\right]\\
&+ \sum_{e=uv\in E(G);d_G(u),d_G(v)\neq 1}\left[\frac{1}{\sqrt{d_G(u)+d_G(e)}}+\frac{1}{\sqrt{d_G(v)+d_G(e)}}\right]\\
&= \sum_{e=uv\in E(G);d_G(u)=1}\frac{\sqrt{2d_G(v)-1}+\sqrt{d_G(v)}}{\sqrt{d_G(v)}\sqrt{2d_G(v)-1}}\\
&+ \sum_{e=uv\in E(G);d_G(u),d_G(v)\neq 1}\left[\frac{1}{\sqrt{d_G(u)+d_G(e)}}+\frac{1}{\sqrt{d_G(v)+d_G(e)}}\right].
\end{aligned}$$

Since $3(\Delta(G)-2)\geq d_G(u)+d_G(e)\geq 3(\delta_1(G)-2)$

$$\Rightarrow \frac{1}{\sqrt{3\Delta(G)-2}}\leq\frac{1}{\sqrt{d_G(u)+d_G(e)}}\leq\frac{1}{\sqrt{3\delta_1(G)-1}}$$

and $\frac{1}{\sqrt{\Delta(G)}}\leq\frac{1}{\sqrt{d_G(u)}}\leq\frac{1}{\sqrt{\delta_1(G)}}$.
Thus the upper bound follows.
Similarly, the lower bound of

$$SB(G)\geq\eta\left[\frac{\sqrt{2\Delta(G)-1}+\sqrt{\Delta(G)}}{\sqrt{\Delta(G)(2\Delta(G)-1)}}\right]+\left[\frac{2(m-\eta)}{\sqrt{3\Delta(G)-2}}\right]$$

follows. □

Remark 3.4. Equality is attained on both sides if and only if $d_G(u)=d_G(v)=\Delta(G)=\delta_1(G)$ for each $uv\in E(G)$ with $d_G(u),d_G(v)\neq 1$ and $d_G(v)=\Delta(G)=\delta_1(G)$ for each $uv\in E(G)$ with $d_G(u)=1$.

To obtain the relation between sum and product connectivity Banhatti indices, we make use of the following definition:

The product connectivity Banhatti index of a graph G is defined as

$$PB(G)=\sum_{ue}\frac{1}{\sqrt{d_G(u)\,d_G(e)}},$$

where ue means that the vertex u and edge e are incident in G. This connectivity based index is put forward by Kulli et al., [7].

Theorem 3.5. *For any (n,m)- connected graph G with $\delta(G)\geq 2$ and $n\geq 3$ vertices,*

$$PB(G)\leq SB(G)\leq\sqrt{m\,PB(G)}.$$

Further, equality in both lower and upper bounds is attained if and only if $G\cong C_n$.

Proof. Let G be a connected graph with $\delta(G)\geq 2$ and $n\geq 3$ vertices. Then

$$\begin{aligned}
d_G(u)\,d_G(e) &\geq d_G(u)+d_G(e)\\
\sum_{ue}\frac{1}{\sqrt{d_G(u)\,d_G(e)}} &\leq \sum_{ue}\frac{1}{\sqrt{d_G(u)+d_G(e)}}.
\end{aligned}$$

Thus the lower bound follows.
To prove the upper bound of $SB(G)$, we consider

$$SB(G) = \sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}}.$$

By Cauchy-Schwartz inequality, we have

$$SB(G) \leq \sqrt{2m \sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}}}$$

and

$$\sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}} \leq \sum_{ue} \frac{1}{2\sqrt{d_G(u)\, d_G(e)}} = \frac{PB(G)}{2}.$$

Therefore

$$SB(G) \leq \sqrt{2m \times \frac{PB(G)}{2}}.$$

Thus the upper bound follows.
Clearly, equality in both lower and upper bounds is attained

$$\begin{aligned} &\Leftrightarrow \quad d_G(u)\, d_G(e) = d_G(u) + d_G(e) \\ &\Leftrightarrow \quad d_G(u) = d_G(v) = 2 \\ &\Leftrightarrow \quad G \cong C_n. \end{aligned}$$

□

In order to prove our next result (lower and upper bounds) of $SB(G)$ in terms of order n and size m, we recall the following facts.

If real valued function $f(x)$ defined on an interval has a second derivative $f^{''}(x)$ then a necessary and sufficient condition for it to be strictly convex on that interval is that $f^{''}(x) > 0$. For positive integer k, if $f(x)$ is strictly convex, then (by Jensen's inequality) we have $f\left(\sum_{i=1}^{k} \frac{x_i}{k}\right) \leq \frac{1}{k}\sum_{i=1}^{k} f(x_i)$ with equality if and only if $x_1 = x_2 = \cdots = x_k$, and if $-f(x)$ is strictly convex, then the inequality is reversed.

Theorem 3.6. *For any (n, m)- connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,*

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}} \leq SB(G) \leq \sqrt{mn}.$$

Further, equality holds in lower bound if and only if $G \cong C_3$ and an equality holds in upper bound if and only if $G \cong C_n; n \geq 3$.

Proof. From Theorems 2.2 and 3.4, the lower bound follows.
For positive integer k, if $f(x)$ is strictly convex, then by Jensen's inequality we have $f\left(\sum_{i=1}^{k} \frac{x_i}{k}\right) \leq f(x_i)$. Let $x_i = \frac{1}{\sqrt{d_G(u)+d_G(e)}}$ and $f(x) = x^2$. Clearly, $f(x)$ is convex. Therefore

$$\begin{aligned} f\left(\sum_{ue} \frac{1}{2m} \frac{1}{\sqrt{d_G(u) + d_G(e)}}\right) &\leq \frac{1}{2m}\sum_{ue} f\left(\frac{1}{\sqrt{d_G(u) + d_G(e)}}\right) \\ \frac{1}{4m^2}\left[\sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}}\right]^2 &\leq \frac{1}{2m}\sum_{ue} \frac{1}{\sqrt{d_G(u) + d_G(e)}}. \end{aligned}$$

We know that for all $a, b > 0$,

$$\frac{a+b}{ab} \geq \frac{4}{a+b} \quad \Rightarrow \quad \frac{1}{a+b} \leq \frac{1}{4}\left(\frac{a+b}{ab}\right).$$

$$\frac{1}{4m^2}[SB(G)]^2 \leq \frac{1}{8m}\sum_{ue}\frac{d_G(u)+d_G(e)}{d_G(u)\,d_G(e)}$$
$$[SB(G)]^2 \leq \frac{m}{2}\sum_{ue}\left(\frac{1}{d_G(e)}+\frac{1}{d_G(u)}\right).$$

Since $\delta(G) \geq 2$, $d_G(e) \geq d_G(v)$ for all $v \in V(G)$ and $e \in E(G)$. Therefore

$$[SB(G)]^2 \leq \frac{m}{2}\sum_{uv\in E(G)} 2\left(\frac{1}{d_G(u)}+\frac{1}{d_G(v)}\right)$$
$$[SB(G)]^2 \leq mn,$$

since $n \geq \sum_{uv\in E(G)}\left(\frac{1}{d_G(u)}+\frac{1}{d_G(v)}\right)$.

Hence the upper bound follows.

The equality case attains directly from (i) of Theorem 2.1. □

In order to prove our next result (lower bound) of $SB(G)$ in terms of size, degrees and inverse edge degree of G, we make use of the following definition.

An inverse edge degree [12] of G with no isolated edges is defined as

$$IED(G) = \sum_{e=uv\in E(G)}\frac{1}{d_G(e)}.$$

In addition, we apply the Polya-Szego Inequality [10] as follows.

Theorem 3.7. *Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two sequences of positive numbers. If $0 < \alpha \leq a_i \leq A < \infty$ and $0 < \beta \leq b_i \leq B < \infty$ for each $i \in \{1, 2, \ldots, n\}$, then*

$$\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 \leq \frac{(\alpha\beta + AB)^2}{4\alpha\beta AB}\left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

The equality holds iff $p = n.\frac{A}{\alpha}/\left(\frac{A}{\alpha}+\frac{B}{\beta}\right)$ and $q = n.\frac{B}{\beta}/\left(\frac{A}{\alpha}+\frac{B}{\beta}\right)$ are integers and p of the numbers $a_1, a_2, \ldots, a_n$ are equal to α and q of these numbers are equal to A, and if the corresponding numbers b_i are equal to B and β, respectively.

Theorem 3.8. *For any (n, m)- connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,*

$$SB(G) \geq \frac{2\sqrt{2m \times IED(G)}\,[(3\Delta(G)-2)(3\delta(G)-2)]^{frac14}}{(\sqrt{3\Delta(G)-2}+\sqrt{3\delta(G)-2})}.$$

Proof. Let G be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Then

$$\frac{1}{\sqrt{3\Delta(G)-2}} \leq \frac{1}{\sqrt{d_G(u)+d_G(e)}}\frac{1}{\sqrt{3\delta(G)-2}}.$$

Let $a_i = \frac{1}{\sqrt{d_G(u)+d_G(e)}}$ and $b_i = 1$ in the Polya-Szego inequality. Clearly, $\alpha = \frac{1}{3\Delta(G)-2}$,

$A = \dfrac{1}{\sqrt{3\delta(G)-2}}$, $\beta = 1$ and $B = 1$. We have

$$\sum_{uv} a_i^2 \cdot \sum_{uv} b_i^2 \leq \frac{(\alpha\beta + AB)^2}{4\alpha\beta AB}\left(\sum_{uv} a_i b_i\right)^2$$

$$2m\frac{1}{d_G(u)+d_G(e)} \leq \frac{\left(\frac{1}{\sqrt{3\Delta(G)-2}} + \frac{1}{\sqrt{3\delta(G)-2}}\right)^2}{\frac{4}{\sqrt{(3\Delta(G)-2)(3\delta(G)-2)}}}[SB(G)]^2$$

$$\big[SB(G)\big]^2 \geq \frac{8m\sqrt{(3\Delta(G)-2)(3\delta(G)-2)}}{\left(\sqrt{3\Delta(G)-2}+\sqrt{3\delta(G)-2}\right)^2}\sum_{uv}\frac{1}{d_G(u)+d_G(e)}$$

Since $\delta(G) \geq 2$, $d_G(u) \leq d_G(e)$ for all $u \in V(G)$ and $e \in E(G)$, where $e = uv \in E(G)$. Therefore $d_G(u) + d_G(e) \leq 2d_G(e)$ implies $\dfrac{1}{d_G(u)+d_G(e)} \leq \dfrac{1}{d_G(e)}$.

$$\begin{aligned}\sum_{uv}\frac{1}{d_G(u)+d_G(e)} &= \sum_{e=uv\in E(G)}\frac{1}{d_G(u)+d_G(e)} + \sum_{e=uv\in E(G)}\frac{1}{d_G(v)+d_G(e)}\\ &\geq \sum_{e=uv\in E(G)}\frac{1}{2d_G(e)} + \sum_{e=uv\in E(G)}\frac{1}{2d_G(e)}\\ &\geq \sum_{e=uv\in E(G)}\frac{1}{d_G(e)} = IED(G).\end{aligned}$$

Hence the lower bound of $SB(G)$ follows. □

In order to prove our next results (upper bounds) of $SB(G)$ in terms of Randić (or, product connectivity) index $P(G)$, the first Zagreb index [3] is defined as $M_1(G) = \sum_{uv\in E(G)}\big[d_G(u) + d_G(v)\big]$ and the modified second Zagreb index [14] is defined as $M_2^*(G) = \sum_{uv\in E(G)}\frac{1}{d_G(u)\,d_G(v)}$ of a graph G. In addition, we make use of the well known Chebyschev's inequality [13] as follows.

Theorem 3.9. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be real numbers. Then*

$$n\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Theorem 3.10. *For any (n,m)- connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,*

(i) $SB(G) \leq \sqrt{m(m+1)P(G)}$,

(ii) $SB(G) \leq \sqrt{mM_1(G)}$,

(iii) $SB(G) \leq \sqrt{m(m+1)M_2^*(G)}$.

Proof. Let G be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. If $a_i = \dfrac{1}{\sqrt{d_G(u)+d_G(e)}}$ and $b_i = \dfrac{1}{\sqrt{d_G(u)+d_G(e)}}$. Then by Chebyschev's inequality, we have

$$\left(\sum_{ue}\frac{1}{\sqrt{d_G(u)+d_G(e)}}\right)^2 \leq 2m \times \sum_{ue}\frac{1}{d_G(u)+d_G(e)}.$$

Consider

$$\sum_{ue} \frac{1}{d_G(u)+d_G(e)} \leq \frac{1}{4}\sum_{ue} \frac{d_G(u)+d_G(e)}{d_G(u)d_G(e)}.$$

Since $\frac{a+b}{2} \geq \frac{2ab}{a+b}$ or $\frac{1}{a+b} \geq \frac{a+b}{4ab}$ for any two positive integers,

$$\begin{aligned}\sum_{ue} \frac{1}{d_G(u)+d_G(e)} &\leq \frac{1}{4}\sum_{ue}\left(\frac{1}{d_G(e)}+\frac{1}{d_G(u)}\right)\\ &\leq \frac{1}{4}\sum_{e=uv\in E(G)}\left[\left(\frac{1}{d_G(e)}+\frac{1}{d_G(u)}\right)+\left(\frac{1}{d_G(e)}+\frac{1}{d_G(v)}\right)\right].\end{aligned}$$

But as $\delta(G) \geq 2$, we have $d_G(u) \leq d_G(e)$ implies $\frac{1}{d_G(e)} \leq \frac{1}{d_G(u)}$.

Therefore

$$\begin{aligned}\sum_{ue} \frac{1}{d_G(u)+d_G(e)} &\leq \frac{1}{4}\sum_{uv\in E(G)} 2\left[\frac{1}{d_G(u)}+\frac{1}{d_G(v)}\right]\\ &\leq \frac{1}{2}\sum_{uv\in E(G)}\left[\frac{d_G(u)+d_G(v)}{d_G(u)d_G(v)}\right].\end{aligned}$$

By above inequality, we have
(i) Since for any $e = uv \in E(G)$, $d_G(u) + d_G(v) \leq m + 1$, implies

$$\begin{aligned}\sum_{ue} \frac{1}{d_G(u)+d_G(e)} &\leq \frac{m+1}{2}\sum_{uv\in E(G)} \frac{1}{d_G(u)d_G(v)}\\ &\leq \frac{m+1}{2}P(G).\end{aligned}$$

Since

$$\sum_{uv\in E(G)} \frac{1}{d_G(u)d_G(v)} \leq \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}},$$

$$\begin{aligned}[SB(G)]^2 &\leq 2m \times \frac{m+1}{2}P(G)\\ SB(G) &\leq \sqrt{m(m+1)P(G)}.\end{aligned}$$

(ii) Since G is connected with $n \geq 3$, we have $d_G(u)d_G(v) \geq 1$ implies

$$\frac{d_G(u)+d_G(v)}{2d_G(u)d_G(v)} \leq \frac{d_G(u)+d_G(v)}{2}.$$

Since

$$\sum_{uv\in E(G)} \frac{1}{d_G(u)+d_G(v)} \leq \frac{1}{2}\sum_{uv\in E(G)}\big(d_G(u)+d_G(v)\big) = \frac{1}{2}M_1(G).$$

Therefore

$$\begin{aligned}[SB(G)]^2 &\leq 2m \times \frac{1}{2}M_1(G)\\ SB(G) &\leq \sqrt{mM_1(G)}.\end{aligned}$$

(iii) Since for any $e = uv \in E(G)$, $d_G(u) + d_G(v) \leq m + 1$, implies

$$\begin{aligned} \sum_{ue} \frac{1}{d_G(u) + d_G(e)} &\leq \frac{m+1}{2} \sum_{uv \in E(G)} \frac{1}{d_G(u) d_G(v)} \\ &\leq \frac{m+1}{2} M_2^*(G). \end{aligned}$$

Therefore

$$\begin{aligned} [SB(G)]^2 &\leq m(m+1) M_2^*(G) \\ SB(G) &\leq \sqrt{m(m+1) M_2^*(G)}. \end{aligned}$$

□

4 Nordhaus- Gaddum Type Inequality

In [15], E. A. Nordhaus and J. W. Gaddum gave tight bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then, such type of results have been derived for several other graph invariants. Here, we derive such kind of relation for $SB(G)$.

Theorem 4.1. *For any (n, m)- connected graph G on $\delta(G) \geq 2$ and $n \geq 5$ vertices with a connected $\overline{G}$,*

(i) $\frac{n(n-1)}{\sqrt{3n-5}} \leq SB(G) + SB(\overline{G}) \leq n\sqrt{n-1}$,

(ii) $\frac{2n^2}{(n-1)(n-2)} \leq SB(G) \times SB(\overline{G}) \leq \frac{n^2(n-1)}{4}$.

Proof. (i) Since $m + \overline{m} = \frac{n(n-1)}{2}$, $d_G(u) + d_{\overline{G}}(u) = n - 1$ and $d_G(v) + d_{\overline{G}}(v) = n - 1$. Hence, we have

$$\begin{aligned} SB(G) + SB(\overline{G}) &= \sum_{uv \in E(G)} \big[(d_G(u) + d_G(u) + d_G(v) - 2)^{-\frac{1}{2}} \\ &+ (d_G(v) + d_G(u) + d_G(v) - 2)^{-\frac{1}{2}}\big] \\ &+ \sum_{uv \in E(G)} \big[(n - 1 - d_G(u) + n - 1 - d_G(u) \\ &+ n - 1 - d_G(v) - 2)^{-\frac{1}{2}} + (n - 1 - d_G(v) \\ &+ n - 1 - d_G(u) + n - 1 - d_G(v) - 2)^{-\frac{1}{2}}\big] \\ &\geq 2(3n-5)^{-\frac{1}{2}}\, m + 2(3n-5)^{-\frac{1}{2}}\, \overline{m} \\ &\geq \frac{n(n-1)}{\sqrt{3n-5}}. \end{aligned}$$

Thus the lower bound follows.
From the Theorem 3.5, we have $SB(G) \leq \sqrt{mn}$ and $SB(\overline{G}) \leq \sqrt{\overline{m}n}$. Therefore

$$\begin{aligned} SB(G) + SB(\overline{G}) &\leq \sqrt{mn} + \sqrt{\overline{m}n} \\ &\leq \sqrt{n}\,(\sqrt{m} + \sqrt{\overline{m}}) \\ &\leq \sqrt{n}\,(\sqrt{2(m + \overline{m}})) \\ &\leq n\sqrt{n-1}. \end{aligned}$$

Thus the upper bound follows.
(ii) From the right hand side of Theorem 3.5 with due to the fact of $m\overline{m} \geq m + \overline{m}$, we have the upper bound of $SB(G) \times SB(\overline{G})$. □

References

[1] K. C. Das, S. Das and B. Zhou, Sum-connectivity index of a graph, Front. Math., China, **11(1)** (2016), 47–54.

[2] I. Gutman and B. Furtula (eds.), Recent Results in the Theory of Randić Index, Univ. Kragujevac, Kragujevac, 2008.

[3] I. Gutman and N. Trinajstic, Graph Theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett. **17** (1972), 535–538.

[4] I. Gutman, V. R. Kulli, B. Chaluvaraju and H. S. Boregowda, On Banhatti and Zagreb Indices. J. Int. Math. Virtual Inst. **7** (2017), 53–67.

[5] V. R. Kulli, College Graph Theory, Vishwa International Publications, Gulbarga, India (2012).

[6] V. R. Kulli, On K Banhatti indices of graphs, Journal of Computer and Mathematical Sciences, **7(4)** (2016), 213–218.

[7] V. R. Kulli, B. Chaluvaraju and H.S. Boregowda, The product connectivity Banhatti index of a graph, Discussion Mathematicae-Graph theory, to appear (2018).

[8] V. R. Kulli, B. Chaluvaraju and H.S. Boregowda, On sum-connectivity Banhatti index of some nanostructures, Submitted.

[9] X. Li and I. Gutman, Mathematical aspects of Randić-type molecular structure descriptors, Kragujevac (2006).

[10] G. Polya and G. Szego, Inequalities for the capacity of a condenser, Amer. J. Math. **67** (1945), 1–32.

[11] M. Randić, On characterization of molecular branching, J. Am. Chem. Soc., **97** (1975), 6609 – 6615.

[12] T. Mansour, M. A. Rostami, E. Suresh and G. B. A. Xavier, On the Bounds of the First Reformulated Zagreb Index, Turkish Journal of Analysis and Number Theory,**4 (1)** (2016), 8–15.

[13] D. S. Miltrinović and P. M. Vasić, Analytic Inequality, Springer-Verlag, Berlin, 1970.

[14] S. Nikolić, G. Kovačević, A. Miličević, and N. Trinajstić, The Zagreb indices 30 years after, Croat Chem Acta. **76 (2)** (2003), 113–124.

[15] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, Amer. Math. Monthly **63** (1956), 175 – 177.

[16] B. Zhou and N. Trinajstić, On a novel connectivity index. J. Math. Chem., **46** (2009), 1252 – 1270.

[17] B. Zhou and N. Trinajstić, On general sum connectivity index. J. Math. Chem., **47 (1)** (2010), 210 – 218.

Author information

V. R. Kulli, [†]B. Chaluvaraju and [††] H.S. Boregowda, Department of Mathematics, Gulbarga University, Gulbarga - 585 106, [†]Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bangalore-560 056, [††] Department of Studies and Research in Mathematics, University Constituent College Campus, Tumkur University, Tumkur - 571 103, INDIA.
E-mail: `vrkulli@gmail.com; bchaluvaraju@gmail.com; bgsamarasa@gmail.com`