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Optimal Production Model for Inventory Items with Verhulst's Demand and Time Dependent Amelioration Rate

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Abstract. In this paper, we develop an optimal production model for inventory items with Verhulst's demand and time-dependent amelioration rate. The model analyzes various time points of EOQ problems in respect of manufacturer's drive to maximize profit per unit time. This is achieved by determining optimal production run time and optimal cycle period under the assumptions that manufacturer's production rate is linearly proportional to the product demand rate. The rate of amelioration is considered a time-dependent function and the unit production cost is inversely proportional to the rate of demand. The model is employed in demonstrating three-time points of optimization; that is, point one; where the product is newly produced, point two; where the product is in its growing phase and point three; where the product is in the matured stage.

1 Introduction

It is observed in the marketplace that the value or utility or quantity of some commodities increases with time. It is a common knowledge in wine manufacturing industries, the utility or value of some wine increases with age. Also in farming yards, the amount or quantity of high breed fishes, fast growing animals like a broiler, pig and so on, increases with time in the growing phase. In our present research, we term such increase as amelioration. The first EOQ model of ameliorating items have established by Hwang [1, 2]. Later, Moon et al. [4] discussed analytically an EOQ model of ameliorating and deteriorating items. They developed models with zero-ending inventory for fixed order interval over a finite planning horizon, considering, the linear trend in demand, shortages, effects of inflation and the time value of money. Hwang et al. [3] studied an optimal issuing policy for fish-breeding supply center for items with Weibull amelioration in which they develop ameliorating inventory models to find optimal issuing policy in fish-breeding supply center. They considered two issuing policies, that is, FIFO first-in-first-out and LIFO last-in-last-out, to determine optimal issuing policy. Hwang [1] derived the equations of inventory levels for both the policies and developed two ameliorating inventory models to find the economic order quantity (EOQ) and the economic selling quantity (ESQ). They also developed a computer program which they in a fish-breeding supply center problem to show the effectiveness of the proposed models. There after, researchers expended much energy in studying different policies of ameliorating items with different parameters and demand patterns. For instance, an interesting multi-item EOQ (Economic Order Quantity) model was established by [6] when the time varying demand is influenced by enterprises' initiatives like advertising media and salesmen' effort. The model was developed for deteriorating and ameliorating items with the capacity constraint for storage facility. The effect of inflation and time value of money in the profit and cost parameters was also considered. Sana maximized the associated gain function using Euler's-Lagrange's method and illustrated it by various time varying demands like quadratic, linear and exponential demand functions. Recently, Kandpal and Tinani [4] developed a stochastic inventory model for ameliorating items under suppliers' trade credit policy to determine an optimal ordering policy for ameliorating items under permissible delay in payment and allowable shortage for future supply uncertainty for two suppliers. The models introduced the aspect of part payment where a part of the purchased cost is to be paid during the permissible delay period. They used spectral theory to derive an explicit expression for the transition probabilities of a four state continuous time Markov chain representing the status of the systems. They used these probabilities to compute the exact form of the average cost expression and also used concepts from renewal reward processes to develop average cost objective function. An inventory model for Weibull ameliorating / deteriorating items under the influence of inflation was also studied by Misra et al. [5] with the objective of discussing the development of an inventory model for ameliorating items in which the replenishment is instantaneous under cost minimization of the influence of inflation and the time value of money. A time varying the type of demand rate with infinite time horizon, constant deterioration and without shortage is considered. The effects of inflation and time discounting on an inventory model with general ramp type demand rate, time dependent Weibull deterioration rate and partial backlogging of unsatisfied demand was studied by Valliathal and Uthayakumar [7]. They considered the model under the replenishment policy, starting with shortages under two different types of backlogging rates, and provided their comparative study. Huang and Wang [8] presented an EOQ model to describe deteriorating inventory items with Verhulst's demand and time-dependent deterioration rate. They proposed the so-called Verhulst's population increase rate as a market demand rate for perishable products. The explore inventory management with profit optimality concern, through which they analyze a variety of time points of EOQ problem in respect of manufacturer's maximum profit per unit time by simultaneously determining its optimal production run time and optimal cycle time under assumptions that manufacturer's production rate is in linear proportion to product demand rate. Deterioration rate was considered time-dependent function and unit production cost a form of the reciprocal of demand. Finally, three distinct time points of optimization models for a new product, product in the middle of growth-stage and mature product are respectively demonstrated. Inspired by the Verhulst's population increase rate model, we aspire to formulate a profit maximizing inventory model to describe the behavior of items that pass through three stages. Industrialists that deal with those articles could employ our model to optimal profit per unit time by determining its production run time and cycle time under the proposed Verhult's demand model.

2 Notation and Assumptions

The mathematical model have formulated under the following notation and assumptions Notation:

- C = setup cost per setup
- p = unit selling price
- c = unit holding cost per unit time
- T_1 = production run time (decision variable)
- T_2= inventory cycle time (decision variable)
- $Q_1(t,v)$stock level with the reference time point at time $t, i = 1, 2$

Assumptions

- The planning horizon is infinite.
- Lead time is zero.
- The initial and final inventory levels are both zero.
- Demand rate is assumed to be as in the Equation 4.
- Production rate is considered to be $\alpha R(t)$, α >1 a constant.
- The time-dependent amelioration rate is assumed to be λt, $0 < \lambda < 1$.
- The unit production cost is assumed to be $\lambda_1 \left(R(t)\right)^{-\gamma}$, $\lambda_1 \geq 0, \gamma \geq 0$

3 The Verhulst's demand

A Belgian mathematician Verhulst had in 1840 presented a mathematical model that determines the rate of increase in the population of species in a given ecological environment. Verhulst postulated that the rate of growth in population species is directly proportional to the number of the specie present at that time which cannot exceed a certain maximum capacity determined by surrounding environment. According to him the pattern of increase in the population of a species in any ecological environment is given by the following differential equations;

$$\frac{dP(t)}{dt} = \mu\rho(t)\left(\varphi - \rho(t)\right) \tag{3.1}$$

Where $\rho(t)$ is a number of specie at time t and φ represents a maximum capacity for the species number Intuitively, we can interpret the manufacturer's design and manufacture products to be purchased as "'*the more population, the more demand for the product*"'. Thus, we can adopt the Verhulst's model as a product demand to explore related inventory management problems. More details about Verhulst's model will be summarized as follows.

Let the variation of the demand with respect to time be assumed to be directly proportional to the amount of demand itself at that time. We also assumed that no product is demanded infinitely. This implies that there exists an upper limit for demand and the variation of demand will eventually grow moderately and at the long run vanishes. Hence, the demand and the difference between it and its upper bound constitutes two main factors that contribute to its behavior.

Again, let $R(t)$ be the demand function of time t; R_0 an initial demand; μ a positive constant; φ a constant representing an upper bound of the product. Hence, the variation of demand with respect to time is given by

$$\frac{dR(t)}{dt} = \mu R(t)\left(\varphi - R(t)\right), t \geq 0 \tag{3.2}$$

with the initial condition $R(0) = R_0$.

Solution of equation 3.2 is given below

$$R(c) = \frac{\varphi}{1 + ce^{-\mu\varphi t}}, t > 0 \tag{3.3}$$

Where $R(t) = \frac{\varphi}{R_0} - 1$

The figure below describes the behavior of the solution of equation 3.2

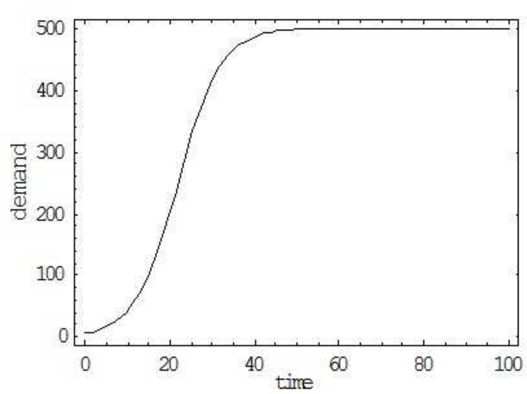

Figure 1. Figure. 1 Behavior of the Verhulst's demand

Let v be the reference time point that enables investigating economic order quantity model with particular time point of product being $v \in [0, \infty)$ with $v = 0$ being the time point that new products come to the market and $v \to \infty$ for the time point of mature products.

Suppose σ is a maturity degree of product defined by the ratio of $R(v)$ over upper bound φ as follows;

$$\sigma = \frac{R(v)}{\varphi} = \frac{1}{1 - ce^{\mu\varphi v}} \tag{3.4}$$

with$\sigma \in \left[\frac{R_0}{\varphi}, 1\right)$

Solving equation 3.4, the reference time point v can be expressed regarding σ as

$$v = \frac{1}{\mu\varphi} ln \frac{c\sigma}{1 + \sigma} \tag{3.5}$$

where $\sigma = \frac{R_0}{\varphi}$ with $v = 0$ for a brand new products and $\sigma \to 1$ accompanied with $v \to \infty$ for matured products.

4 Model formulations

In the inventory system under formulation, production starts at $t = 0$ with zero stock level at the initial stage. The inventory starts accumulating and stops at time $t = T_1$. Due to the demand factor, the inventory level gradually diminishes over the time interval $[t_1, t_2]$ and finally drops to zero at time $t = t_2$.

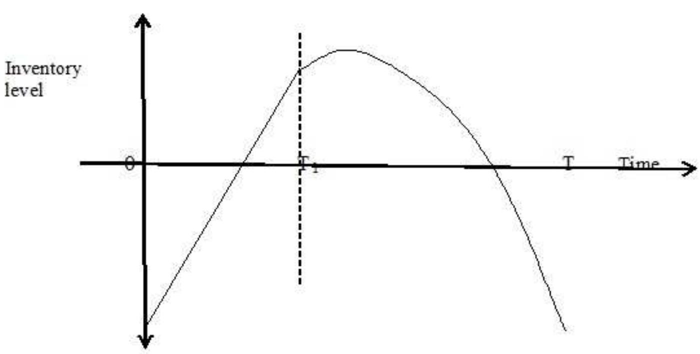

Figure 2. Figure2Inventory movement with an ameliorating inventory system with Verhulst's Demand

According to the assumptions, in the interval $[0, t_1]$, the inventory undergo amelioration at the rate; $\lambda t Q(t, v)$; the demand rate and production rate are respectively given by $R(t + v)$, and $\alpha R(t + v)$ the instantaneous rate of change of inventory level with respect to time in the interval, $0 \leq t \leq T_1$ is governed by

$$\frac{dQ_1(t, v)}{dt} - \lambda t Q_1(t, v) = (\alpha - 1)R(t + v), 0 \leq t \leq T_1 \tag{4.1}$$

with initial condition $Q_1\,(0, v) = 0$

In the same vein, during $[T_1, T_2]$ the inventory level undergo amelioration and is depleted by the effects of demand, so that the variation of inventory level with respect to time is governed by

$$\frac{dQ_2(t,v)}{dt} - \lambda Q_2(t,v) = -R(t+v), T_1 \le t \le T_2 \tag{4.2}$$

with boundary condition $Q_1\left(T_2, v\right) = 0$.

Solutions of equations 4.1 and 4.2 are given by

$$Q_1(t,v) = e^{\frac{\lambda t^2}{2}} \int_0^t \frac{\varphi\left(\alpha - 1\right) e{-}^{\frac{\lambda w^2}{2}}}{1 - ce^{\mu\varphi(w+v)}} dw, 0 \le t \le T_1 \tag{4.3}$$

$$Q_2(t,v) = e^{\frac{\lambda t^2}{2}} \int_t^{T_1} \frac{\varphi e^{\frac{\lambda w^2}{2}}}{1 - ce^{\mu\varphi(w+v)}} dw, 0 \le t \le T_1 \tag{4.4}$$

with continuous conditions at time $t = T_1$ that $Q_1(t,v) = Q_2(t,v)$ we have

$$\int_0^{T_1} \frac{\alpha e^{\frac{\lambda w^2}{2}}}{1 - ce^{\mu\varphi(w+v)}} dw = \int_0^{T_2} \frac{e^{\frac{\lambda w^2}{2}}}{1 - ce^{\mu\varphi(w+v)}} dw \tag{4.5}$$

From equation 10 we see that T_2 can be expressed in term of T_1 . Taking derivative of equation 10 with respect to T_1 we obtain

$$\frac{dT_2}{dT_1} = \frac{\alpha\left(1 - ce^{\mu\varphi(T_2+v)}\right)}{1 - ce^{\mu\varphi(T_1+v)}} e^{\frac{\left(T_2^2 - T_1^2\right)}{2}} \tag{4.6}$$

The manufacturer's total profit is made up of the following elements

Element 1: setup cost: C

Element 2: sales revenue

$$S_R = \int_0^{T_2} sR(t+v)dt \tag{4.7}$$

$$= \frac{s}{\mu} ln \frac{e^{\mu\varphi(T_2+v)} + c}{e^{\mu\varphi v} + c} \tag{4.8}$$

Element 3: Production cost

$$C_p = \int_0^{t_1} \lambda_1 \left(R(t+V)\right)^{-\gamma} \alpha R(t+v)dt + \tag{4.9}$$

$$\int_0^{T_1} \frac{\lambda_1 \alpha \varphi^{1-\gamma}}{\left[1 - ce^{-\mu\varphi t}\right]^{1-\gamma}} dt$$

Element 4: inventory holding cost

$$C_h = \int_0^{T_1} \eta Q_1(t,v)dt + \int_{T_1}^{T_2} \eta Q_2(t,v)dt \tag{4.10}$$

$$\int_0^{T_1} \eta e^{\frac{\lambda t^2}{2}} \int_0^{T_1} \frac{\varphi\left(\alpha - 1\right) e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dwdt - \int_{T_1}^{T_2} \eta e^{\frac{\lambda t^2}{2}} \int_t^{T_2} \frac{\varphi e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dwdt$$

Thus the manufacturer's total profit is given over the interval $[0, T_2]$ is given by

$$P_T(T_1, T_2, V) = S_R - C - C_p - C_h \tag{4.11}$$

$$P_T(T_1, T_2, V) = \frac{s}{\mu} ln \frac{e^{\mu\varphi T_2} + c}{1 + c} - C - \int_0^{T_1} \frac{\lambda_1 \alpha \varphi^{1-\gamma}}{\left[1 - ce^{-\mu\varphi t}\right]^{1-\gamma}} dt - \tag{4.12}$$

$$\int_0^{T_1} \eta e^{\frac{\lambda t^2}{2}} \int_0^{T_1} \frac{\varphi(\alpha - 1) e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dw dt - \int_{T_1}^{T_2} \eta e^{\frac{\lambda t^2}{2}} {}_t^{T_2} \frac{\varphi e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dw dt$$

The total profit per unit time over the interval $[0, T_2]$ is given by

$$P_T T(T_1, T_2, v) = \frac{P_t(T_1, T_2, v)}{T_2} \tag{4.13}$$

As T_2 is a function of T_1 for convenience $P_T(T_1, T_2, v)$ and $P_T(T_1, T_2, v)$ are respectively reduced to a single variable T_1 denoted by $P_T(T_1, v)$ and $P_T(T_1, v)$. To maximize the total profit per unit time, we differentiate equation 4.13 on T_1 and set the result to zero to get

$$\frac{dP_T T(T_1, v)}{dT_1} = \frac{1}{T_1} \left[\frac{P_T(T_1, v)}{dT_1} - P_T T(T_1, v) \frac{dT_2}{dT_1}\right] \tag{4.14}$$

We then substitute the result of equation 4.6 into above equation to obtain

$$\frac{dP_T T(T_1, v)}{dT_1} - \frac{P_T T(T_1, v)}{T_1} \alpha \left[\frac{1 + ce^{-\mu\varphi T_2 + v}}{1 - ce^{-\mu\varphi T_1 + v}}\right] e^{\frac{\left(T_1^2 - T_1^2\right)}{2}} = 0 \tag{4.15}$$

5 Numerical Examples

Equations 4.1 to 4.5 are valid for any specific time point throughout product's lifespan. In our model, however, we illustrate three objective functions to describe the following three stages of the inventory as follows

Stage 1: For an infant inventory

The maturity degree is $\sigma = \frac{R_0}{\varphi}$ associated with $v = 0$. Substituting $v = 0$ into equation 4.10 we get

$$P_T(T_1, 0) = \frac{s}{\mu} ln \frac{e^{\mu\varphi T_2} + c}{1 + c} - C - \int_0^{T_1} \frac{\lambda_1 \alpha \varphi^{1-\gamma}}{\left[1 - ce^{-\mu\varphi t}\right]^{1-\gamma}} dt \tag{5.1}$$

$$- \int_0^{T_1} \eta e^{\frac{\lambda t^2}{2}} \int_0^{T_1} \frac{\varphi(\alpha - 1) e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dw dt - \int_{T_1}^{T_2} \eta e^{\frac{\lambda t^2}{2}} {}_t^{T_2} \frac{\varphi e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dw dt$$

$$P_T T(T_1, 0) = \frac{P_t(T_1, 0)}{T_1} \tag{5.2}$$

Stage 2: For a youthful inventory

The maturity degree is assumed as $v = \frac{1}{2}$ associated with $v = \frac{1}{\mu\varphi} lnc$ and from equation 4.10, then

$$P_T T\left(T_1, \frac{1}{\mu\varphi} lnc\right) = \frac{s}{\mu} ln \frac{e^{\mu\varphi T_2} + 1}{2} - C - \int_0^{T_1} \frac{\lambda_1 \alpha \varphi^{1-\gamma}}{\left[1 - ce^{-}\mu\varphi t\right]^{1-\gamma}} dt \tag{5.3}$$

$$- \int_0^{T_1} \eta e^{\frac{\lambda t^2}{2}} \int_0^{T_1} \frac{\varphi(\alpha - 1) e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dw dt - \int_{T_1}^{T_2} \eta e^{\frac{\lambda t^2}{2}} \int_t^{T_2} \frac{\varphi e^{\frac{\lambda w^2}{2}}}{1 - ce^{-\mu\varphi w}} dw dt$$

$$P_T T\left(T_1, \frac{1}{\mu\varphi} lnc\right) = \frac{P_T\left(T_1, \frac{1}{\mu\varphi} lnc\right)}{T_1} \tag{5.4}$$

Stage 3: For a matured inventory
The maturity level is $\sigma \to 1$ associated with $v \to \infty$, so from equation 4.10 we have

$$TP\left(T_1, \infty\right) = s\varphi T_2 - C - \lambda_1 \alpha \varphi^{1-\gamma} T_1 - \int_0^{T_1} \eta e - \frac{\lambda T^2}{2} \int_o^t \varphi\left(\alpha - 1\right) e^{\frac{\lambda w^2}{2}} dwdt - \int \eta e^{\frac{\lambda T^2}{2}} \int_t^{T_2} \varphi e^{\frac{\lambda w^2}{2}} dwdt \tag{5.5}$$

$$P_T T\left(T_1, \infty\right) = \frac{P_T\left(T_1, \infty\right)}{T_1} \tag{5.6}$$

6 Conclusions

Huang and Wang [9] presented an EOQ model to describe deteriorating inventory items with Verhulst's demand and time-dependent deterioration rate. The model studied an integrated demand pattern from the birth of new product to its maturity stage with a single mathematical expression which can be used to simultaneously analyze a variety of different time points of related economic order quantity problems during entire product lifespan. In our present research, we employ the concept used in Huang and Wang [9] to ameliorating inventory, and we obtained similar but opposite results. As a result, it helps decision makers predetermine any specific time point of optimal management policies before its scheduled time point is due.

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