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A NOTE ON THE GENERALIZED ORDER-k MODIFIED PELL AND PELL-LUCAS NUMBERS

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Abstract. This paper deals with the generalized order-k Pell, Modified Pell and Pell-Lucas numbers. Certain important properties and sum formulae of the generalized order-k Modified Pell and Pell-Lucas numbers are obtained. Moreover, their the generalized Binet formulae, combinatorial representations and the generating functions are developed via matrix approach.

1 Introduction

Over the years, several articles have been appeared in many journals relating to Fibonacci and Pell numbers. The Pell numbers and their generalization have marvelous properties and applications to nearly every fields in the modern science and art. Monograph [1] presents the well-known systematic investigations on the subject.

The usual Pell numbers $\{P_n\}$ is defined by the following recursive relation:

$$P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2,$$

where $P_0 = 0$ and $P_1 = 1$. In addition, the Pell-Lucas $\{Q_n\}$ and the Modified Pell $\{q_n\}$ numbers are defined by the same recurrence but with initial terms such that $Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$ respectively. It should be noted that due to the equation $Q_n = 2q_n$ given in [2], the known properties of the Pell-Lucas numbers can be written for the Modified Pell numbers. Hence, a study of the one involves inevitably familiarity with the other one.

Many investigations on the Pell, Pell-Lucas and Modified Pell numbers have been presented by a great number of researchers. Ercolano gave the generating matrices for the Pell numbers [3]. Horadam presented many properties of the Modified Pell numbers [4]. Melham introduced many sum formulae and properties of both Fibonacci and Pell numbers [5]. Kilic and Tasci derived the generalizations of the usual Pell numbers and obtain their generating matrices and certain sum formulae [6]. Daşdemir investigated some properties of the Pell, Pell-Lucas and Modified Pell numbers by employing the matrix approach [7]. Catarino obtained the Binet formulae, the generating functions and some properties of k-Pell numbers [8]. Vasco and Catarino developed some sums and certain products involving terms of k-Pell, k-Pell-Lucas and Modified k-Pell sequences [9]. Daşdemir gave the recurrence relations corresponding to a generalizations of the Pell-Lucas and Modified Pell numbers [10].

In this paper, the generalized order-k Pell, Modified Pell and Pell-Lucas numbers are considered, and certain properties, sum formulae, the generalized Binet formulae, combinatorial representations and the generating functions of the generalized order-k Modified Pell and Pell-Lucas numbers are presented by employing the matrix method.

2 Preliminaries

In this section, certain results given before are recalled.

In [6], Kilic and Tasci presented a generating matrix to obtain all the terms of the generalized order-k Pell numbers as follows:

$$E_n = R^n, \tag{2.1}$$

where

$$R = \begin{bmatrix} 2 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } E_n = \begin{bmatrix} P_n^1 & P_n^2 & \cdots & P_n^k \\ P_{n-1}^1 & P_{n-1}^2 & \cdots & P_{n-1}^1 \\ \vdots & \vdots & \vdots & \vdots \\ P_{n-k+1}^1 & P_{n-k+1}^1 & \cdots & P_{n-k+1}^1 \end{bmatrix}. \tag{2.2}$$

In addition, in the same paper, the generalized Binet formulae of the generalized order-k Pell numbers were investigated as follows: Let $f(\lambda)$ be the characteristic polynomial of the generating matrix R such as $f(\lambda) = \lambda^k - 2\lambda^{k-1} - \cdots - \lambda - 1$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$'s are the eigenvalues of the matrix R. In [6], it is shown that $\lambda_1, \lambda_2, \ldots, \lambda_k$'s are distinct, and a $k \times k$ Vandermonde matrix denoted by V and a $k \times 1$ column vector denoted by w_k^i are defined as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \lambda_3^{k-1} & \cdots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \lambda_3^{k-2} & \cdots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_k \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \text{ and } w_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}.$$

Hence, the following theorem was given.

Theorem 2.1. *([6]) Let P_n^i be the nth term of ith Pell sequence, for $1 \leq i \leq k$. Then,*

$$P_{n-i+1}^j = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}, \tag{2.3}$$

where $V_j^{(i)}$ denotes a $k \times k$ matrix obtained from V by replacing the jth column of V by w_k^i.

By the same token, Daşdemir gave the following generating matrix for the generalized order-k Pell-Lucas and Modified Pell numbers by aid of an auxiliary matrix [10]:

$$K_n = R^n.S \text{ or } K_n = E_n.S, \tag{2.4}$$

where

$$K_n = \begin{bmatrix} q_n^1 & q_n^2 & \cdots & q_n^k \\ q_{n-1}^1 & q_{n-1}^2 & \cdots & q_{n-1}^1 \\ \vdots & \vdots & \vdots & \vdots \\ q_{n-k+1}^1 & q_{n-k+1}^1 & \cdots & q_{n-k+1}^1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}. \tag{2.5}$$

In addition, the author showed that there exist the following relationships [10]: for all $n, m \in \mathbb{Z}^+$ and $1 \leqslant i \leqslant k$,

$$q_n^i = P_{n+1}^i - P_n^i \tag{2.6}$$

$$Q_n^i = 2q_n^i \tag{2.7}$$

Finally, the following theorem is recalled.

Theorem 2.2. *([11]) The (i,j) entry $a_{ij}^{n}(c_1,c_2,\ldots,c_k)$ in the matrix $A_k^n(c_1,c_2,\ldots,c_k)$ is given by the following formula:*

$$a_{ij}^{(n)}(c_1,c_2,\ldots,c_k) = \sum_{(t_1,\ldots,t_k)} \frac{t_j+t_{j+1}+\cdots+t_k}{t_1+t_2+\cdots+t_k} \times \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} c_1^{t_1}\ldots c_k^{t_k}, \quad (2.8)$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+kt_k = n-i+j$, and the coefficients are defined as 1 for $n = i-j$.

3 Main Results

In this section, certain relationships between the generalized order-k Pell, Pell-Lucas and Modified Pell numbers and their Binet formulae and combinatorial representations are presented by employing the matrix method. Beginning of this section, the following theorem is given.

Theorem 3.1. *Let P_n^k and q_n^k be the generalized order-k Pell and Modified Pell numbers, respectively. For $n \geqslant 0$,*

i. $q_n^k = P_{n-1}^k + q_n^{k-1}$

ii. $q_n^k - q_n^{k-1} = P_n^{k-1} + P_n^k$

iii. $q_n^k - q_{n-1}^k = P_n^k + P_{n-k}^k$

Proof. Only the proof of Theorem 3.1.i is given here, because the others are analogous. To prove Theorem 3.1.i, the induction method on n is used. Consider the recurrence relations of the generalized order-k Pell and Modified Pell numbers given in [6, 10]. It is seen that

$$q_1^k = P_0^k + q_1^{k-1} = 1.$$

Let it be true for $i = 1,2,\cdots,n-1$. It must be shown that this equation holds for $i = n$. Hence,

$$\begin{aligned} q_n^k &= 2q_{n-1}^k + q_{n-2}^k + \cdots + q_{n-k-1}^k \\ &= 2\left(P_{n-2}^k + q_{n-1}^{k-1}\right) + \left(P_{n-3}^k + q_{n-2}^{k-1}\right) + \cdots + \left(P_{n-k-2}^k + q_{n-k-1}^{k-1}\right) \\ &= \left(2P_{n-2}^k + P_{n-3}^k + \cdots + P_{n-k-2}^k\right) + \left(2q_{n-1}^{k-1} + q_{n-2}^{k-1} + \cdots + q_{n-k-1}^{k-1}\right) \\ &= P_{n-1}^k + q_n^{k-1} \end{aligned}$$

can be written. This completes the proof. □

Now certain sum formulae consisting of convolutions of the generalized order-k Pell and Modified Pell numbers are given.

Theorem 3.2. *Let P_n^k and q_n^k be the generalized order-k Pell and Modified Pell numbers, respectively. For $n,p,m \in \mathbb{Z}^+$,*

i. $q_n^k = \sum\limits_{i=1}^{k} P_{n-i+1}^k$

ii. $q_{n+m+p}^i = \sum\limits_{j=1}^{k} P_n^j q_{m+p+1-j}^i = \sum\limits_{j=1}^{k} P_{n+m}^j q_{p+1-j}^i$

iii. $q_{n+m}^i = \sum\limits_{j=1}^{k} P_{n+p}^j q_{m+1-p-j}^i = \sum\limits_{j=1}^{k} P_{n-p}^j q_{m+p+1-j}^i$

iv. $\sum\limits_{j=1}^{n} {q_j}^i = {P_{n+1}}^i - 1$

Proof. Each case is separately investigated.

$i.$ Consider the induction method on n. By the definition of the generalized order-k Pell and Modified Pell numbers,

$$q_1^k = \sum_{i=1}^{k} P_{n-i+1}^k = P_1^k + P_0^k + \cdots + P_{2-k}^k = 1$$

is obtained for the case where $n = 1$. Assume that the considered equation holds for first n term. From the assumption and Eq. (2.6),

$$\sum_{i=1}^{k} P_{n-i+2}^k = P_{n+1}^k + P_n^k + \cdots + P_{n-k+2}^k = P_{n+2}^k - P_{n+1}^k = q_{n+1}^k$$

is written for $n + 1$. Thus the proof is completed.

$ii.$ Consider Eq. (2.4). Hence, for all $n, p, m \in \mathbb{Z}^+$,

$$K_{n+m+p} = E_n K_{m+p} = E_{n+m} K_p$$

can be written. Consequently, by the matrix multiplication, the proof can directly be completed.

$iii.$ As in the proof of ii, from Eq. (2.4),

$$K_{n+m} = E_{n-p} K_{m+p} = E_{n+p} K_{m-p}$$

can be obtained. Hence, the proof is completed by employing the matrix multiplication.

$iv.$ Summing all the equations after writing equation (2.6) from 1 to n, the result follows.

So, the proof is completed. □

Now the generalized Binet formalae of the generalized order-k Modified Pell numbers are investigated. To do this, the following theorem is given.

Theorem 3.3. *Let q_n^i be the nth term of ith generalized Modified Pell sequence. Then for $1 \leq i \leq k$,*

$$q_n^i = \frac{\det\left(V_1^{(1)}\right) - \det\left(V_i^{(1)}\right)}{\det(V)}. \tag{3.1}$$

Proof. Taking the cases separately where $i = j = 1$ and $i = 1$ in Theorem 2.1 and Eq. (2.6) into account, the proof is completed. □

To present the combinatorial representations of the generalized order-k Modified Pell numbers, the following corollary is given as a result of Theorem 2.2 without the proof.

Corollary 3.4. *Let q_n^i be the generalized order-k Modified Pell number, for $1 \leq i \leq k$. Then,*

$$\begin{aligned} q_n^i = & \sum_{(d_1, d_2, \cdots, d_k)} \begin{pmatrix} d_1 + d_2 + \cdots + d_k \\ d_1, d_2, \cdots, d_k \end{pmatrix} 2^{d_1} \\ & - \sum_{r_1, r_2, \cdots, r_k} \frac{r_k}{r_1 + r_2 + \cdots + r_k} \times \begin{pmatrix} r_1 + r_2 + \cdots + r_k \\ r_1, r_2, \cdots, r_k \end{pmatrix} 2^{r_1} \end{aligned} \tag{3.2}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + \cdots + kr_k = n - i + k$ and $d_1 + 2d_2 + \cdots + kd_k = n$.

It should be noted that all the results presented above can be expressed in terms of the generalized order-k Pell-Lucas numbers. It is enough to consider Eq. (2.7) to do this. Consequently, the following statements can be given.

Theorem 3.5. *Let P_n^k and Q_n^k be the generalized order-k Pell and Pell-Lucas numbers, respectively. For $n \geqslant 0$,* *i.* $Q_n^k = 2P_{n-1}^k + Q_n^{k-1}$

ii. $Q_n^k - Q_n^{k-1} = 2\left(P_n^{k-1} + P_n^k\right)$

iii. $Q_n^k - Q_{n-1}^k = 2\left(P_n^k + P_{n-k}^k\right)$

Theorem 3.6. *Let P_n^k and Q_n^k be the generalized order-k Pell and Pell-Lucas numbers, respectively. For $n, p, m \in \mathbb{Z}^+$,*

i. $\frac{1}{2}Q_n^k = \sum_{i=1}^{k} P_{n-i+1}^k$

ii. $Q_{n+m+p}^i = \sum_{j=1}^{k} P_n^j Q_{m+p+1-j}^i = \sum_{j=1}^{k} P_{n+m}^j Q_{p+1-j}^i$

iii. $Q_{n+m}^i = \sum_{j=1}^{k} P_{n+p}^j Q_{m+1-p-j}^i = \sum_{j=1}^{k} P_{n-p}^j Q_{m+p+1-j}^i$

iv. $\sum_{j=1}^{n} {Q_j}^i = 2({P_{n+1}}^i - 1)$

Theorem 3.7. *Let Q_n^i be the nth term of ith Pell-Lucas sequences, for $1 \le i \le k$. Then,*

$$\frac{1}{2}Q_n^i = \frac{\det\left(V_1^{(1)}\right) - \det\left(V_i^{(1)}\right)}{\det(V)}. \tag{3.3}$$

Corollary 3.8. *Let Q_n^i be the generalized order-k Pell-Lucas numbers, for $1 \le i \le k$. Then,*

$$\begin{aligned}\tfrac{1}{2}Q_n^i = &\sum_{(d_1,d_2,\cdots,d_k)} \begin{pmatrix} d_1 + d_2 + \cdots + d_k \\ d_1, d_2, \cdots, d_k \end{pmatrix} 2^{d_1} \\ &- \sum_{r_1,r_2,\cdots,r_k} \tfrac{r_k}{r_1+r_2+\cdots+r_k} \times \begin{pmatrix} r_1 + r_2 + \cdots + r_k \\ r_1, r_2, \cdots, r_k \end{pmatrix} 2^{r_1}\end{aligned} \tag{3.4}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + \cdots + kr_k = n - i + k$ and $d_1 + 2d_2 + \cdots + kd_k = n$.

Now, the generating functions of the generalized order-k Modified Pell numbers are derived. To do this, the limit of the adjacent generalized order-k Modified Pell numbers $\frac{q_n^k}{q_{n-1}^k}$ is considered under the case where $n \to \infty$. First of all, the following definition is introduced:

$$\lim_{n\to\infty} \frac{q_n^k}{q_{n-1}^k} = x. \tag{3.5}$$

The ratio of the adjacent generalized order-k Modified Pell numbers can be written as follows:

$$\frac{q_n^k}{q_{n-1}^k} = \frac{2q_{n-1}^k + q_{n-2}^k + \cdots + q_{n-k}^k}{q_{n-1}^k} = \frac{3q_{n-1}^k - q_{n-k-2}^k - q_{n-2}^k}{q_{n-1}^k}$$

or more smoothly

$$\frac{q_n^k}{q_{n-1}^k} = 3 - \frac{1}{\frac{q_{n-1}^k}{q_{n-2}^k}} - \frac{1}{\frac{q_{n-1}^k q_{n-2}^k q_{n-3}^k \cdots q_{n-k}^k}{q_{n-2}^k q_{n-3}^k \cdots q_{n-k}^k q_{n-k-1}^k}}. \tag{3.6}$$

Substituting Eq. (3.6) into Eq. (3.5), the following algebraic equation for the generalized order-k Modified Pell numbers is obtained:

$$x^{k+1} - 3x^k + x^{k-1} + 1 = 0 \tag{3.7}$$

or equally

$$(x-1)\left(x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1\right) = 0. \tag{3.8}$$

According to the famous Fundamental Theorem of Algebra, Eq. (3.7) possesses $(k+1)$ roots such as $x_1, x_2, \ldots, x_{k+1}$ and from [6], each is different from the other. It is clear that since $x-1 \neq 0$ for $k > 1$,

$$x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1 = 0 \tag{3.9}$$

is obtained. When $k = 2$, Eq. (3.9) is reduced to well-known form for the usual Pell numbers (or Modified Pell numbers).

4 Algorithms

The recurrence relations and the initial conditions of the considered number sequences vary depending the selection of i, k and n. Hence, it is troublesome and not economical to compute the terms of the generalized order-k Pell, Modified Pell and Pell-Lucas numbers and their sums. Consequently, certain algorithms for PC are given to address the issue. Fig. 1 displays an algorithm to obtain the terms of the generalized order-k Pell numbers and their sums. Furthermore, Figs. 2 and 3 show the algorithms for the generalized order-k Modified Pell and Pell-Lucas numbers, respectively. Note that the algorithms given in this paper are composed for Mathematica 8.0 and later versions. They may give errors for older versions. It should be recalled to get $1 \leqslant i \leqslant k$ while running the algorithms. In addition, introduce the notations

$$P\left(i,k,n\right) = P_n{}^i \; q\left(i,k,n\right) = q_n{}^i \text{ and } Q\left(i,k,n\right) = Q_n{}^i, \tag{4.1}$$

$$S\left(n\right) = \sum_{j=0}^{n} P\left(i,k,n\right) \; H\left(n\right) = \sum_{j=0}^{n} q\left(i,k,n\right) \text{ and } T\left(n\right) = \sum_{j=0}^{n} Q\left(i,k,n.\right) \tag{4.2}$$

These notations will appear in the output of the program.

```
(* Algorithm for computing terms of the generalized order-k Pell number and their sums *)

k = Input["Please enter the order of the sequence."];

i = Input["Please enter the value of i."];

n = Input["Please enter the number of terms of the sequence."];

GPN = Table[0, {n + k}];

Do[GPN[[k - i + 1]] = 1; GPN[[tt]] = 2 * GPN[[tt - 1]] + Sum[GPN[[tt - i]], {i, 2, k}], {tt, k + 1, n + k}];

P = Drop[GPN, k - 1];

S = Sum[P[[j + 1]], {j, 0, n}];

Print["P(", i ",", k ",", n, ")", "=", P[[n + 1]]]

Print["S(", n, ")", "=", S]
```

Figure 1. Algorithm for computing the terms of the generalized order-k Pell numbers and their sums.

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(* Algorithm for computing terms of the generalized order-k  Modified Pell number and their sums *)

k = Input["Please enter the order of the sequence."];

i = Input["Please enter the value of i."];

n = Input["Please enter the number of terms of the sequence."];

GMPN = Table[0, {n + k}];

Do[GMPN[[k - i + 1]] = -1;
  GMPN[[k]] = 1;
  GMPN[[tt]] = 2 * GMPN[[tt - 1]] + Sum[GMPN[[tt - i]], {i, 2, k}], {tt, k + 1, n + k}];

MP = Drop[GMPN, k - 1];

H = Sum[MP[[j + 1]], {j, 0, n}];

Print["q(", i ",", k ",", n, ")", "=", MP[[n + 1]]]

Print["H(", n, ")", "=", H]
```

Figure 2. Algorithm for computing the terms of the generalized order-k Modified Pell numbers and their sums.

```
(* Algorithm for computing terms of the generalized order-k Pell-Lucas number and their sums *)

k = Input["Please enter the order of the sequence."];

i = Input["Please enter the value of i."];

n = Input["Please enter the number of terms of the sequence."];

GPL = Table[0, {n + k}];

Do[GPL[[k - i + 1]] = -2;
  GPL[[k]] = 2;
  GPL[[tt]] = 2 * GPL[[tt - 1]] + Sum[GPL[[tt - i]], {i, 2, k}], {tt, k + 1, n + k}];

PL = Drop[GPL, k - 1];

T = Sum[PL[[j + 1]], {j, 0, n}];

Print["Q(", i ",", k ",", n, ")", "=", PL[[n + 1]]]

Print["T(", n, ")", "=", T]
```

Figure 3. Algorithm for computing the terms of the generalized order-k Pell-Lucas numbers and their sums.

5 Conclusions

In this study, the generalized order-k Pell, Modified Pell and Pell-Lucas numbers are investigated, and their certain important properties, interrelationships and sum formulae are given. In addition, their the generalized Binet formulae, combinatorial representations and the generating functions are developed via the matrix approach.

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