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THE GROUP OF HOMEOMORPHISMS AND THE CYCLIC GROUPS OF PERMUTATIONS

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Abstract In this paper we study the group of homeomorphisms of a topological space. A subgroup H of the group $S(X)$ of all permutations of a set X is called t-representable on X if there exists a topology τ on X such that the group of homeomorphisms of $(X,\ \tau) = K$. It is proved that the group generated by a permutation which is an arbitrary product of infinite cycles is a t-representable subgroup of $S(X)$. It is also proved that the group generated by a permutation which is a product of two disjoint finite cycles is not a t-representable subgroup of $S(X)$ when the order of the group is greater than two.

1 Introduction

Consider the topological space (X, τ), the set of all homeomorphisms on (X, T) onto itself form a group under composition which is a subgroup of the symmetric group $S(X)$. Many authors studied the concept of group of homeomorphisms. In 1959, J. De Groot proved that for any group G, there is a topological space (X, τ) such that the group of homeomorphisms of (X, τ) is isomorphic to G [4]. The problem of representing a subgroup of S(X) as the group of homeomorphisms of some topology on X was considered by P. T. Ramachandran. In [6, 7], P. T. Ramachandran showed that nontrivial proper normal subgroups of the group of all permutations of a set X can not be represented as the group of homeomorphisms of $(X,\ \tau)$ for any topology τ on X. If $X = \{a_1,\ a_2,\ \dots, a_n\}$, n $\geq$ 3, the group of permutations of X generated by the cycle $(a_1,\ a_2,\ \dots, a_n)$ cannot be represented as the group of homeomorphisms of (X, τ) for any topology τ on X whereas if X is an infinite set, then the cyclic group generated by an infinite cycle can be represented as the group of homeomorphisms of $(X,\ \tau)$ for a topology τ on X [6, 8].

A subgroup H of the group $S(X)$ of all permutations of a set X is called t-representable on X if there exists a topology τ on X such that the group of homeomorphisms of $(X,\ \tau) = H$. In [9] it is proved that the direct sum of finite t-representable permutation groups is t-representable, every permutation group of order two is t-representable and also determined the t-representability of finite transitive permutation groups. In [10], we determined the t-representability of group generated by a permutation which is a product of disjoint cycles having equal lengths.

The aim of this paper is to continue the study in [10]. In the second section we determine the t-representability of groups generated by an arbitrary product of infinite cycles. In the third section we prove that the group generated by a permutation which is a product of two disjoint finite cycles is not a t-representable subgroup of $S(X)$ provided the order of the group is greater than two.

We use an order theoretic method to determine the t-representability of permutation groups. Susan J. Andima and W. J Thron [1] associated each topology τ on a set X with a preorder relation '$\leq$' on X defined by $a \leq b$ if and only if every open set containing b contains a. Then any homeomorphisms of $(X,\ \tau)$ onto itself is also an order isomorphisms of $(X,\ \leq)$. Also we have the group of homeomorphisms of $(X,\ \tau)$ which is denoted by $H(X,\ \tau)$ is equal to the group of order isomorphisms of $(X,\ \leq)$ if X is finite [9].

A topological space $(X,\ \tau)$ is said to be a T_0 space if given any two distinct points in X, there exist an open set which contains one of them but not the other [11]. So $(X,\ \tau)$ is a T_0 space if and only if the corresponding preordered set $(X,\ \leq)$ is a partially ordered set. If X is a finite nonempty set, then the partially ordered set $(X,\ \leq)$ has both maximal and minimal elements. Also an order isomorphism of $(X,\ \leq)$ maps maximal elements to maximal elements and minimal elements to minimal elements.

The basic concepts to be used in our proofs will be introduced as needed and reference for each concept will be mentioned along with. In particular for the basic notions of topological spaces and groups we refer to [3] and [11].

2 t-representability of the groups generated by a product of disjoint infinite cycles

In this section we investigate the t-representability of infinite cyclic subgroups of symmetric groups. Here we prove that if X is an infinite set and σ is a permutation on X which can be written as an arbitrary product of disjoint infinite cycles, then the cyclic group generated by σ, $\langle\sigma\rangle$ is t-representable on X.

We need the following definition.

Definition 2.1. [2] Let G_1 and G_2 be two permutation groups on X_1 and X_2 respectively. The direct product $G_1 \times G_2$ acts on the disjoint union $X_1 \cup X_2$ by the rule

$$(g_1, g_2)(x) = \begin{cases} g_1(x) & \text{if } x \in X_1 \\ g_2(x) & \text{if } x \in X_2. \end{cases}$$

First we prove an important property of a t-representable permutation group.

Theorem 2.2. *Let X be any set and Y be a nonempty subset of X. If H is a t-representable permutation group on Y, then the permutation group $\{I_{X\setminus Y}\} \times H$ is t-representable on X where $I_{X\setminus Y}$ is the identity permutation on $X \setminus Y$.*

Proof. Let τ_1 be a topology on Y such that $H(Y,\ \tau_1) = H$. The result is trivially true if $X\setminus Y = \emptyset$. So we assume that $X \setminus Y \neq \emptyset$. Define

$$\tau' = \{(X \setminus Y) \cup U : U \in \tau_1\}$$

By using the well-ordering Theorem, well-order the set $X \setminus Y$ by the order relation $<$. Define a topology τ_2 on $X \setminus Y$ as

$$\tau_2 = \{X \setminus Y\} \cup \{\{y \in X \setminus Y : y < x\} : x \in X \setminus Y\}.$$

Let

$$\tau = \tau_2 \cup \tau'.$$

It is easy to see that τ is a topology on X.
Claim: $H(X,\ \tau) = \{I_{X\setminus Y}\} \times H$.

Let $h \in \{I_{X\setminus Y}\} \times H$. This gives that $h = (I_{X\setminus Y}, h_1)$ for some $h_1 \in H$. Let $U \in \tau$. If $U \in \tau_2$, then we have $h(U) = U$ and $h^{-1}(U) = U$ and hence $h(U),\ h^{-1}(U) \in \tau$. If $U \in \tau'$, then $U = (X\setminus Y) \cup U_1$ for some U_1 in τ_1 . Since h_1 is a homeomorphism on (Y, τ_1), $h_1(U_1) \in \tau_1$ and $h_1^{-1}(U_1) \in \tau_1$. This implies that both $h(U) = (X\setminus Y) \cup h_1(U_1)$ and $h^{-1}(U) = (X\setminus Y) \cup h_1^{-1}(U_1)$ are in τ. Since U is arbitrary, h is a homeomorphism on (X, τ). So

$$\{I_{X\setminus Y}\} \times H \subseteq H(X, \tau). \tag{2.1}$$

Conversely assume that $h \in H(X, \tau)$. First we prove that $h(x) = x$ for all $x \in X \setminus Y$. Now we consider the case $|X \setminus Y| = 1$. If $X \setminus Y = \{x\}$, then x is isolated in X and no point of Y is isolated in X, so $h(x) = x$.

Now we assume that $|X \setminus Y| \geq 2$. Let x_0 and x_1 be the first and the second elements of the set $X \setminus Y$ and $U = \{y \in X \setminus Y : y < x_1\}$. Then $U = \{x_0\}$ and $U \in \tau$. Since h is a homeomorphism, $h(U) \in \tau$ and hence $h(x_0) = x_0$. Let x_α be any element of $X \setminus Y$ such that $h(x) = x$ for all x in $X \setminus Y$ such that $x < x_\alpha$.

If x_α has an immediate successor x_β in $X \setminus Y$, consider $U = \{x \in X \setminus Y : x < x_\beta\}$, which is an open set and hence $h(U)$ is open in τ. Now

$$h(U) = \{x \in X \setminus Y : x < x_\alpha\} \cup \{h(x_\alpha)\}.$$

If $X \setminus Y \subseteq h(U)$, then x_α and x_β are both in $h(U) \setminus \{x \in X \setminus Y : x < x_\alpha\}$, which is impossible. So $h(U) = \{x \in X \setminus Y : x < z\}$ for some $z \in X \setminus Y$ and hence $h(U) = \{x \in X \setminus Y : x < x_\beta\}$. Consequently $h(x_\alpha) = x_\alpha$.

If x_α has no immediate successor, then x_α is the last element of the set $X \setminus Y$. Since $X \setminus Y \in \tau_2$, $X \setminus Y \in \tau$. Therefore $h(X \setminus Y) \in \tau$ and $h(X \setminus Y) = \{x \in X \setminus Y : x < x_\alpha\} \cup \{h(x_\alpha)\}$. For any $z \in X \setminus Y$, $\{x \in X \setminus Y : x < z\}$ is a proper subset of $h(X \setminus Y)$. This implies that $X \setminus Y \subseteq h(X \setminus Y)$ and hence $h(x_\alpha) = x_\alpha$.

Thus $h_{|X \setminus Y} = I_{X \setminus Y}$ and $h_{|Y}$ will be a homeomorphism of (Y, τ_1). Clearly $h_{|Y} \in H$ and we get $h = (I_{X \setminus Y}, h_1)$ where $h_1 = h_{|Y} \in H$. So $h \in \{I_{X \setminus Y}\} \times H$. Thus we get

$$H(X, T) \subseteq \{I_{X \setminus Y}\} \times H. \tag{2.2}$$

From equations 2.1 and 2.2, we have $H(X, \tau) = \{I_{X \setminus Y}\} \times H$. This completes the proof. □

Remark 2.3. Let H be a non-trivial permutation group on a set X. Let $Y = X \setminus \{x \in X : h(x) = x \text{ for all } h \in H\}$. Define $H' = \{h_{|Y} : h \in H\}$, which is a permutation group on Y. Note that H' moves all the elements of Y and $H = H' \times \{I_{X \setminus Y}\}$. By Theorem 2.2, it follows that, if H' is a t-representable permutation group on Y, then H is t-representable on X. So if (X, τ) is a topological space which is not rigid and $H = H(X, \tau)$ then without loss of generality we can assume that H moves all the elements of X.

Let X be the infinite set $\{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$ and σ be the infinite cycle $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ on X. Then the group generated by σ is t-representable on X by defining a topology $\tau = \{\emptyset, X\} \cup \{\{a_j : j \leq i\} : i \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integers [8]. It follows that, the permutation group generated by an infinite cycle is t-representable.

A topological space (X, τ) is called an Alexandroff discrete space if arbitrary intersections of open sets are open in X [1]. A topological space (X, τ) is Alexandroff discrete if and only if it has a minimal open neighbourhood at every point in X.

First we consider the t-representability of cyclic group generated by a permutation which is a product of two disjoint infinite cycles. Here we prove that the subgroup of $S(X)$ generated by a permutation which is a product of two disjoint infinite cycles is t-representable on X.

Theorem 2.4. *Let X be an infinite set and σ be a permutation on X which can be written as a product of two disjoint infinite cycles. Then the cyclic group generated by σ, $\langle \sigma \rangle$ is t-representable on X.*

Proof. Let $\sigma = \sigma_1 \sigma_2$ where

$$\sigma_1 = (\ldots, a_{-1}, a_0, a_1, \ldots) \text{ and } \sigma_2 = (\ldots, b_{-1}, b_0, b_1, \ldots).$$

By Theorem 2.2, without loss of generality we can assume that $X = X_1 \cup X_2$, where $X_1 = \{a_i : i \in \mathbb{Z}\}$ and $X_2 = \{b_i : i \in \mathbb{Z}\}$. Now define a base $\mathcal{B}$ by

$$\mathcal{B} = \{A_i : i \in \mathbb{Z}\} \cup \{A_i \cup B_i : i \in \mathbb{Z}\}$$

where $A_i = \{a_j \in X_1 : j \leq i\}$ and $B_i = \{b_j \in X_2 : j \leq i\}$. Let τ be the topology having base $\mathcal{B}$. Then

$$\tau = \{\emptyset, X, X_1\} \cup \{A_i : i \in \mathbb{Z}\} \cup \{A_i \cup B_j : i, j \in \mathbb{Z} \text{ and } j \leq i\}.$$

Now we prove that $H(X, \tau) = \langle \sigma \rangle$. It is routine to verify that if $U \in \tau$, then $\sigma(U) \in \tau$ and $\sigma^{-1}(U) \in \tau$. Hence

$$\langle \sigma \rangle \subseteq H(X, \tau). \tag{2.3}$$

For the other inclusion let $h \in H(X,\tau)$. First we prove that $h(X_1) = X_1$. Suppose instead that $h(X_1) \neq X_1$. Then either $X_1 \setminus h(X_1) \neq \emptyset$ or $h(X_1) \setminus X_1 \neq \emptyset$. Assume first that $h(X_1) \setminus X_1 \neq \emptyset$ and pick $i,\, k \in \mathbb{Z}$ such that $h(a_i) = b_k$. Then A_i is the smallest open set with a_i as a member and hence $h(A_i) = A_k \cup B_k$, the smallest open set with b_k as a member. Now $h(A_{i+1}) = h(A_i \cup \{a_{i+1}\}) = A_k \cup B_k \cup h((a_{i+1})$. Now $h(a_{i+1}) \neq h(a_i) = b_k$. So the smallest open set with $h(a_{i+1})$ as a member is either A_j for some j or $A_j \cup B_j$ for some $j \neq k$. This is impossible.

Now assume that $X_1 \setminus h(X_1) \neq \emptyset$. Then $h^{-1}(X_1) \setminus X_1 \neq \emptyset$. So we get a contradiction exactly as before. Therefore $h(X_1) = X_1$ and consequently $h(X_2) = X_2$.

We have that $h(a_0) = a_j$ for some $j \in \mathbb{Z}$. We show by induction that for all $k \in \mathbb{N} \cup \{0\}$, $h(a_k) = a_{j+k}$ and $h(a_{-k}) = a_{j-k}$. So let $k \in \mathbb{N} \cup \{0\}$ and assume that $h(a_k) = a_{j+k}$ and $h(a_{-k}) = a_{j-k}$. Then $h(A_k) = A_{j+k}$ and $h(A_{-k}) = A_{j-k}$.

Let V be the smallest open set with $h(a_{-k-1})$ as a member. Then $V = h(A_{-k-1}) = h(A_{-k} \setminus \{a_{-k}\}) = h(A_{-k}) \setminus \{h(a_{-k})\} = A_{j-k} \setminus \{a_{j-k}\} = A_{j-k-1}$. So $h(a_{-k-1}) = a_{j-k-1}$.

Now pick $l \in \mathbb{Z}$ such that $h(a_{k+1}) = a_l$. Then $A_l = h(A_{k+1}) = h(A_k \cup \{a_{k+1}\}) = A_{j+k} \cup \{h(a_{k+1})\}$. Thus $A_{j+k} \subseteq A_l$ and $A_l \setminus A_{j+k} \subseteq \{h(a_{k+1})\}$. Now $h(a_{k+1}) \neq h(a_k) = a_{j+k}$. This implies that $l \neq j+k$ and hence $l = j+k+1$. Thus we get $h(a_{k+1}) = a_{j+k+1}$.

Now let $b_k \in X_2$ and let $b_m = h(b_k)$. Then $A_m \cup B_m = h(A_k \cup B_k) = h(A_k) \cup h(B_k) = A_{j+k} \cup h(B_k)$. So $j+k = m$ and $h(b_k) = b_{j+k}$. Therefore $h = \sigma^j$ for some $j \in \mathbb{Z}$. So

$$H(X,\, \tau) \subseteq \langle\sigma\rangle. \tag{2.4}$$

From equations 2.3 and 2.4, we get $H(X,\tau) = \langle\sigma\rangle$. This completes the proof. □

If σ is a permutation on X which is a product of more than two disjoint infinite cycles, we can define a topology τ on X such that $H(X,\, \tau)$ is the group generated by σ.

Theorem 2.5. *If σ is a permutation on X which is a product of more than two disjoint infinite cycles, then the cyclic group generated by σ, $\langle\sigma\rangle$ is t-representable on X.*

Proof. Let $\sigma = \Pi_{ß\in I} C_i$ where I is a set, $|I| > 2$ and for $i \in I$

$$C_i = (\ldots, a_{i,-2},\, a_{i,-1},\, a_{i,0},\, a_{i,1},\, a_{i,2}, \ldots)$$

which is an infinite cycle. Let X_i be the set of all terms of the cycle C_i. In view of Theorem 2.2 we can assume without loss of generality that $X = \bigcup_{i\in I} X_i$. Well order I by the relation $<$. Let i_0 be the first element of I and i_1 denote the first element of the set $I \setminus \{i_0\}$.

Define a base $\mathcal{B}$ by

$$\mathcal{B} = \{B_{i,j} : i \in I, j \in \mathbb{Z}\}$$

where for $j \in \mathbb{Z}$, $B_{i_0,j} = \{a_{i_0,j}\}$, $B_{i_1,j} = \{a_{i_0,j},\, a_{i_1,j}\}$, and for $i > i_1$, $B_{i,j} = \{a_{k,j} : k \leq i\} \cup \{a_{i_0,j-1}\}$. It is easy to verify that $\mathcal{B}$ is a base for a topology τ on X. Since for each $i \in I$ and $j \in \mathbb{Z}$, $\sigma(B_{i,j}) = B_{i,j+1}$. Thus

$$\langle\sigma\rangle \subseteq H(X,\tau). \tag{2.5}$$

For the other inclusion, let $h \in H(X,\tau)$. Note that for each $i \in I$ and $j \in \mathbb{Z}$, $B_{i,j}$ is the smallest open set with $a_{i,j}$ as a member. Given $q \in \mathbb{Z}$, there is some $f(q) \in \mathbb{Z}$ such that $h(B_{i_0,q}) = B_{i_0,f(q)}$. We shall show that for each $i \in I$ and $q \in \mathbb{Z}$, $h(a_{i,q}) = a_{i,f(q)}$ and $f(q-1) = f(q) - 1$. This will suffice for then letting $n = f(0)$, one has $h = \sigma^n$.

So let $q \in \mathbb{Z}$ and let $r = f(q)$. Then $h(a_{i_0,q}) \in B_{i_0,r}$ and so $h(a_{i_0,q}) = a_{i_0,r}$. Now $h(B_{i_1,q}) = B_{i_1,m}$ for some $m \in \mathbb{Z}$ and so $\{a_{i_0,r},\, h(a_{i_1,q})\} = h(B_{i_1,q}) = \{a_{i_0,m}, a_{i_1,m}\}$. It follows that $m = r$ and $h(a_{i_1,q}) = a_{i_1,r}$

Now let $i > i_1$ and assume that for all $k < i$, $h(a_{k,q}) = a_{k,r}$. Let $s = f(q-1)$. The smallest open set with $a_{i,q}$ as a member is $B_{i,q}$ and $h(B_{i,q}) = \{a_{k,r} : k < i\} \cup \{h(a_{i,q}), a_{i_0,s}\}$. If $k < i$, then $h(a_{i,q}) \neq h(a_{k,q}) = a_{k,r}$. So $h(B_{i,q})$ has at least three members and hence we can pick $l > i$ and $m \in \mathbb{Z}$ such that $h(B_{i,q}) = B_{l,m}$. Then $\{a_{k,r} : k < i\} \cup \{h(a_{i,q}), a_{i_0,s}\} = \{a_{k,m} : k < l\} \cup \{a_{l,m}, a_{i_0,m-1}\}$. Now $a_{i_1,r} \in \{a_{k,r} : k < i\}$ and $a_{i_1,r} \notin \{a_{l,m}, a_{i_0,m-1}\}$. So

$r = m$. Consequently $\{a_{k,r} : k < i\} \cup \{h(a_{i,q}), a_{i_0,s}\} = \{a_{k,r} : k < l\} \cup \{a_{l,r}, a_{i_0,r-1}\}$. Since $h(B_{i,q}) \neq B_{i_0,r-1}$, $h(a_{i,q}) \neq a_{i_0,r-1}$. So $a_{i_0,r-1} = a_{i_0,s}$ and thus $s = r - 1$. (Note that we have established that $f(q-1) = f(q) - 1$

Now we have that $\{a_{k,r} : k < i\} \cup \{h(a_{i,q}), a_{i_0,r-1}\} = \{a_{k,r} : k \leq l\} \cup \{a_{i_0,r-1}\}$. So $h(a_{i,q}) = a_{p,r}$ for some p with $i \leq p \leq l$. Suppose that $i < l$ and pick $j \neq p$ with $i \leq j \leq l$. Then $a_{j,r} \notin \{a_{k,r} : k < i\} \cup \{h(a_{i,q}), a_{i_0,r-1}\}$, a contradiction. So $i = l$ and $h(a_{i,q}) = a_{i,r}$ as required. Hence $h \in \langle\sigma\rangle$. Thus

$$H(X, \tau) \subseteq \langle\sigma\rangle. \tag{2.6}$$

From equations 2.5 and 2.6, we get $\langle\sigma\rangle = H(X, \tau)$. Hence $\langle\sigma\rangle$ is a t-representable permutation group on X. □

We conclude this section by the following theorem.

Theorem 2.6. *Let X be an infinite set and σ be a permutation on X which can be written as an arbitrary product of disjoint infinite cycles. Then the cyclic group generated by σ, $\langle\sigma\rangle$ is t-representable on X.*

Proof. Proof follows from Theorems 2.4 and 2.5 and the paragraph after Remark 2.3. □

3 t-representability of the groups generated by a permutation which is a product of two disjoint cycles having finite length

We now turn our attention to the t-representability of cyclic groups generated by a permutation which is a product of two disjoint finite cycles. The main result in this section is, if σ is a permutation on a set X which is a product of two disjoint cycles having finite length, then the cyclic group generated by σ, $\langle\sigma\rangle$ is not t-representable on X provided the length of at least one of them is greater than two.

A topological space (X, τ) is said to be *homogeneous* if for any $x, y \in X$, there exists a homeomorphism h from (X, τ) onto itself such that $h(x) = y$ [11]. A finite topological space X is homogeneous if and only if there exist positive integers m and n such that X is homeomorphic to $D(m) \times I(n)$ where $D(k)$ and $I(k)$ denote the set $\{1, 2, 3, \ldots, k\}$ with the discrete topology and indiscrete topology respectively [5]. So if (X, τ) is a finite homogeneous space, then there exists a partition of X using sets with equal number of elements, which forms a base for the topology on X and when $|X| \geq 2$, there exists at least one transposition which is a homeomorphism of (X, T).

In [10] we proved the following theorem in the case of a group generated by a permutation which is a product of two disjoint cycles having the equal lengths.

Theorem 3.1. *[10] If σ is a permutation on X which is a product of two disjoint cycles having equal length n where $n \geq 3$, then the group generated by σ is not t-representable on X.*

Now we consider the t-representability of the permutation groups generated by a permutation which is a product of two disjoint cycles having different lengths.

Lemma 3.2. *Let (X, τ) be a topological space which is not T_0. Then there exists at least one transposition which is a homeomorphism on (X, τ).*

Proof. Let (X, τ) be a topological space which is not T_0. Then by definition there exist two distinct points a, b in X such that every open set in X either contains both a and b or else contain neither of them. Let p be the transposition (a, b). Then $p^{-1} = p$ and $p(U) = U$ for all $U \in \tau$. This implies that p is a homeomorphism on (X, τ). This completes the proof. □

Theorem 3.3. *Let X be any set such that $|X| = m_1 + m_2$ and σ be a permutation on X which is a product of two disjoint cycles having lengths m_1 and m_2 respectively where $(m_1, m_2) = 1$, then the cyclic group generated by σ is not t-representable on X.*

Proof. Let $\sigma_1 = (a_1, a_2, \ldots, a_{m_1})$ and $\sigma_2 = (b_1, b_2, \ldots, b_{m_2})$ and $\sigma = \sigma_1\sigma_2$. Since $(m_1, m_2) = 1$, we have

$$\langle\sigma\rangle = \langle\sigma_1\rangle \times \langle\sigma_2\rangle$$

Let $X = X_1 \cup X_2$ where X_i is the set of all elements in the cycle σ_i for $i = 1, 2$. Assume that $\langle\sigma\rangle$ is a t-representable permutation group on X and τ is the corresponding topology.

Now we have two possible cases.

Case 1: (X, τ) is a T_0 space.

In this case the corresponding pre ordered set $(X, \leq)$ is a partially ordered set. Since X is a finite non empty set, the partially ordered set $(X, \leq)$ has both maximal and minimal elements. Assume that an element x_0 in X is both minimal and maximal. Then we claim that all the elements of X are both minimal and maximal. Since $X = X_1 \cup X_2$, we have either $x_0 \in X_1$ or $x_0 \in X_2$. Suppose that $x_0 \in X_1$. Since a homeomorphism maps minimal elements to minimal elements and maximal elements to maximal elements and $\{h(x_0) : h \in \langle\sigma\rangle\} = X_1$, all the elements of X_1 are both minimal and maximal. Let $x \in X_2$. Suppose that x is not a maximal element. Then there exists at least one element x' in X such that $x < x'$. Since all the elements of X_1 are both minimal and maximal, the only possibility is $x' \in X_2$. Now $x' \in X_2$ implies that there exist some j, $1 \leq j < m_2$ such that $x' = \sigma_2^j(x)$. Now

$$\begin{aligned} x < x' = \sigma_2^j(x) &\Longrightarrow \sigma_2^j(x) < \sigma_2^{j\oplus j}(x) \\ &\Longrightarrow \sigma_2^{j\oplus j}(x) < \sigma_2^{j\oplus 2j}(x) \\ &\vdots \\ &\Longrightarrow \sigma_2^{j\oplus(m_2-2)j}(x) < \sigma_2^{j\oplus(m_2-1)j}(x) = x. \end{aligned}$$

Thus we get $x < x'$ and $x' < x$. This implies that $x = x'$, which is not possible. So x is a maximal element. Similarly if we assume that x is not a minimal element, we get a contradiction. So all the elements of X_2 are also both minimal and maximal. In this case the topology on X is discrete and hence $H(X, \tau) = S(X)$, which is not possible. So a minimal element can not be a maximal element. Then either X_1 or X_2 is the set of all minimal elements.

Assume that X_1 is the set of all minimal elements. Then there exists at least one $a_i \in X_1$ and $b_j \in X_2$ such that a_i precedes b_j. Now $a_i < b_j$ gives $p(a_i) < p(b_j)$ for all $p \in \langle\sigma\rangle$. Since $\langle\sigma\rangle = \langle\sigma_1\rangle \times \langle\sigma_2\rangle$, any $p \in \langle\sigma\rangle$ is of the form $p = (p_1, p_2)$ where $p_1 \in \langle\sigma_1\rangle$ and $p_2 \in \langle\sigma_2\rangle$. Therefore

$$\begin{aligned} a_i < b_j &\Longrightarrow (I_{X_1}, p_2)(a_i) < (I_{X_1}, p_2)(b_j) \text{ for all } p_2 \in \langle\sigma_2\rangle \\ &\Longrightarrow a_i < p_2(b_j) \text{ for all } p_2 \in \langle\sigma_2\rangle \\ &\Longrightarrow a_i < b_k \text{ for } k = 1, 2, \ldots m_2 \end{aligned}$$

So a_i precedes all the elements of X_2 and hence every element in X_1 precedes all the elements of X_2. Hence $\tau = \mathcal{P}(X_1) \cup \{X_1 \cup B : B \subseteq X_2\}$. So we get $H(X, \tau) = S(X_1) \times S(X_2)$. Since $m_1, m_2 > 1$ and $m_1 \neq m_2$, $|S(X_1) \times S(X_2)| = m_1!m_2! > m_1.m_2 = |\langle\sigma\rangle|$. This is not possible.

Case 2: The space (X, τ) is not T_0.

In this case either m_1 or $m_2 = 2$ by Lemma 3.2. Let $m_1 = 2$. Assume that $X_1 = \{a_1, a_2\}$ and $\sigma_1 = (a_1, a_2)$. Since $\langle\sigma\rangle = \langle\sigma_1\rangle \times \langle\sigma_2\rangle$, (σ_1, I_{X_2}) is a homeomorphism on X. Since the transposition (a_1, a_2) is a homeomorphism on the space (X, τ), the subspace $(X_1, \tau_{/X_1})$ has either the discrete topology or indiscrete topology.

Now if the subspace $(X_1, \tau_{/X_1})$ has the discrete topology, then there exist open sets of the form $U = U_1 \cup \{a_1\}$ and $V = V_1 \cup \{a_2\}$ where U_1 and V_1 are subsets of X_2. Since (X, τ) is not T_0, there exist two distinct points x and y such that every open set in (X, τ) contains both x and y or else contain neither of them. Since the topology on $(X_1, \tau_{/X_1})$ is discrete, we have at least one of x, y does not belongs to X_1. Hence we get a transposition (x, y) other than (a_1, a_2), which is a homeomorphism on X. This is not possible since $H(X, \tau) = \langle\sigma\rangle$.

If the subspace $(X_1, \tau_{/X_1})$ has the indiscrete topology, then every open set in (X, τ) contains either both a_1 and a_2 or else contain neither of them. Now consider the subspace $(X_2, \tau_{/X_2})$. Since $\langle\sigma_2\rangle \subseteq H(X_2, \tau_{/X_2})$, $(X_2, \tau_{/X_2})$ is a homogeneous space and hence there exists a partition $\mathcal{B} = \{B_1, B_2, \ldots B_m\}$, $|B_i| = n$ for all $i = 1, 2, \ldots, m$ and $1 \leq n \leq m_2$, of X_2, which forms a base for $(X_2, \tau_{/X_2})$. Let $n > 1$. Now choose two elements x, y in B_1 and an open set U containing x. Then $U \cap X_2 \in \tau_{/X_2}$ and $x \in U \cap X_2$. This implies that $B_1 \subseteq U \cap X_2$ and hence $y \in U$. Thus any open set containing x contains y also and vice versa. Consequently $p = (x, y)$ is a homeomorphism on (X, τ). Now suppose that $n = 1$. In this case $(X_2, \tau_{/X_2})$ is the discrete topology. Let $x \in X_2$. Then either $\{x\}$ or $\{x\} \cup X_1$ is open in X. We have $\langle\sigma\rangle = H(X, \tau)$. Therefore if $\{x\} \in \tau$, then $\{\{x\} : x \in X_2\} \subseteq \tau$. Simillarly if $\{x\} \cup X_1 \in \tau$, then $\{\{x\} \cup X_1 : x \in X_2\} \subseteq \tau$. This follows that any transposition on X_2 is a homeomorphism of (X, τ), which is a contradiction. Thus in both cases we get $\langle\sigma\rangle$ is not a t-representable permutation group on X.

□

Theorem 3.4. *Let X be a set such that $|X| = m_1 + m_2$ and σ be a permutation on X which is a product of two disjoint cycles having different lengths m_1 and m_2 respectively where $(m_1, m_2) = d > 1$, then the cyclic group generated by σ is not t-representable on X.*

Proof. Let $m_1 < m_2$. We have $(m_1, m_2) = d > 1$ and hence $m_1 = ld$ and $m_2 = kd$, where l and k are positive integers. Assume that

$$\sigma = (a_1, a_2, \ldots, a_{m_1})(b_1, b_2, \ldots, b_{m_1}, b_{m_1+1}, \ldots, b_{m_2})$$

and $\langle\sigma\rangle$ be the cyclic group generated by σ. Let $X = Y \cup Z$ where Y is the set of all terms in the cycle σ_1 and Z is the set of all terms in the cycle σ_2. Assume that $\langle\sigma\rangle$ is a t-representable permutation group on X. Note that there exist no transposition as homeomorphism on (X, τ). So by Lemma 3.2, the corresponding topology τ on X is T_0 hence the corresponding preordered set is a partially ordered set. Then by a similar argument as in Theorem 3.3, we get either Y or Z is the set of all minimal elements.

Assume that Y is the set of all minimal elements and Z is the set of all maximal elements. Then there exist at least one $a_i \in Y$ and $b_j \in Z$ such that a_i preceeds b_j. Now $a_i < b_j$ gives $f(a_i) < f(b_j)$ for all $f \in \langle\sigma\rangle$. Note that $|\langle\sigma\rangle| = n$ where n is the least common multiple of m_1 and m_2. Without loss of generality we assume that a_1 preceeds b_1. Now

$$\begin{aligned} a_1 < b_1 &\Rightarrow \sigma^h(a_1) < b_1 \text{ for all } h = 0, m_2, , \ldots (l-1)m_2 \\ &\Rightarrow a_{1\oplus_{m_1} pm_2} < b_1 \text{ for all } p = 0, 1, , \ldots (l-1) \end{aligned}$$

where $\oplus_{m_1}$ denotes addition modulo m_1. Simillarly we have

$$\begin{aligned} a_1 < b_1 &\Rightarrow a_1 < \sigma^h(b_1) \text{ for all } h = 0, m_1, , \ldots (k-1)m_1 \\ &\Rightarrow a_1 < b_{1\oplus_{m_2} pm_1} \text{ for all } p = 0, 1, , \ldots (k-1). \end{aligned}$$

This implies that there exist partitions $\{Y_1, Y_2, \ldots, Y_d\}$ and $\{Z_1, Z_, \ldots, Z_d\}$ of Y and Z respectively where $Y_i = \{a_{i\oplus_{m_1} pm_2} : p = 0, 1, \ldots (l-1)\}$ and $Z_i = \{b_{i\oplus_{m_2} pm_1} : p = 0, 1, , \ldots (k-1)\}$ for all $i = 1, 2, \ldots d$. and $y < z$ for all $y \in Y_i$ and $z \in Z_i$ for $i = 1, 2, \ldots d$.

Suppose a_1 precede q elements $b_1, b_2, \ldots, b_q$ in $Z_1, Z_2, \ldots, Z_q$ respectively where $1 \leq q \leq d$. Then all elements of Y_1 preceed every element of $Z_1, Z_2, \ldots, Z_q$. This follows that

$$\mathcal{B} = \{\{x\} : x \in Y\} \cup \{Y_i \cup Y_{i\oplus_d 1} \cup \ldots, Y_{q\oplus_d(q-1)} \cup \{b_i\}\}$$

is a base for τ. It follows that $(a_i, a_{i\oplus_{m_1} m_2}, \ldots, a_{i\oplus_{m_1}(k-1)m_2})$ is a homeomorphism on (X, τ), which is a contradiction to the fact that $\langle\sigma\rangle = H(X, \tau)$. So $\langle\sigma\rangle$ is not a t-representable permutation group on X. □

Theorem 3.5. *[9] Every permutation group of order two is t-representable.*

Combining previous results, we get the following theorem.

Theorem 3.6. *Let X be any set such that $|X| = m_1 + m_2$, σ be a permutation on X which is a product of two disjoint cycles having lengths m_1 and m_2 respectively and H be the cyclic group generated by σ. Then the group H is t-representable on X if and only if order H is less than three.*

Proof. This follows directly from the Theorems 2.2, 3.1, 3.3, 3.4 and 3.5. □

The Theorem 3.6 gives the characterization of t-representable group generated by a permutation which is a product of two disjoint finite cycles.

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