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# STABILITY OF GORENSTEIN GRADED FLAT MODULES

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**Abstract**. In this paper, we introduce second degree Gorenstein $gr$-flat modules and we show that the two-degree Gorenstein $gr$-flat modules are nothing more than that the Gorenstein $gr$-flat modules over a $GF$-$gr$-closed ring.

## 1 Introduction

The development of the Gorenstein homological algebra has reached an advanced level since the pioneering works of Auslander and Bridger [3]. The homological theory of graded rings is very important because of its applications in algebraic geometry [6]. In particular, the Gorenstein homological theory for graded rings was developed in [1, 2], where Gorenstein $gr$-projective, $gr$-injective, and $gr$-flat modules were defined and studied. More recently, L. Mao [7] studied and proved some results about strongly Gorenstein graded modules.

On the other hand Sather-Wagstaff et al. [8] proved that iterating the process used to define Gorenstein projective modules exactly gives to the Gorenstein projective modules. Also, they proved in [9] a stability of the subcategory of Gorenstein flat modules under a procedure to build $R$-modules from complete resolutions. Further, S. Bouchiba et al. in [4] proved over a left $GF$-closed ring $R$, the stability of the Gorenstein flat modules under the very process used to define these entities. Also stability property of various modules are studied by many authors, see [4, 11, 13]. Our main aim of this paper is to obtain stability of Gorenstein graded flat modules over graded rings.

This paper is divided into 5 sections. In Section 2, we recall the basic definitions and facts which will be used in further sections. In Section 3, we recall the definitions of Gorenstein $gr$-flat modules and Gorenstein $gr$-injective modules. Also we introduce second degree Gorenstein $gr$-flat modules and initiate our main theorem. $GF$-$gr$-closed rings are introduced in section 4. Last section, we introduce strongly Gorenstein $gr$-flat modules, which is different notion from L. Mao [7] and over a $GF$-$gr$-closed ring, we show that the two-degree Gorenstein $gr$-flat modules are nothing more than that the Gorenstein $gr$-flat modules.

## 2 Preliminaries

We recall graded rings and modules and list some results which are used throughout this paper. All rings are associative with identity element and the (left or right) $R$-modules are unital. By $R$-Mod we denote the Grothendieck category of all left $R$-modules. Let $G$ be a multiplicative group with identity element $e$. A graded ring [12] $R$ is a ring with identity 1, together with a direct decomposition $R = \oplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus $R_e$ is a subring of $R$, $1 \in R_e$ and for every $\sigma \in G$, $R_\sigma$ is an $R_e$-bimodule. A left graded $R$-module is a left $R$-module $M$ endowed with an internal direct sum decomposition $M = \oplus_{\sigma \in G} M_\sigma$, where each $M_\sigma$ is a subgroup of the additive group of $M$ such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For $M$ and $N$ graded left $R$-modules, we put

$Hom_{R-gr}(M,N) = \{f : M \to N |$ is $R$-linear and $f(M_\sigma) \subseteq N_\sigma, \forall \sigma \in G\}$.

$Hom_{R-gr}(M,N)$ is the group of all morphisms from $M$ to $N$ in the category $R$-$gr$ of the graded left $R$-modules ($gr$-$R$ will denote the category of the graded right $R$-modules). It is well known that $R$-$gr$ is a Grothendieck category (refer [12]). An $R$-linear $f : M \longrightarrow N$ is said to be a *graded morphism of degree* $\tau, \tau \in G$, if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree $\tau$ build an additive subgroup $HOM_R(M,N)_\tau$ of $Hom_R(M,N)$. It is clear that $HOM_R(M,N)_e = Hom_{R-gr}(M,N)$. Note that $HOM_R(M,N) = \oplus_{\tau\in G} HOM_R(M,N)_\tau$ is a graded abelian group of type $G$. We will denote $Ext^i_R$, $Ext^i_{R-gr}$, and $EXT^n_R$ as the left derived functor of $Hom_R$, $Hom_{R-gr}$, and $HOM_R$, respectively.

Given a graded left $R$-module $M$, we can define the graded character module of $M$ as [2] $M^+ = HOM_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We note that it can be seen as $M^+ = \oplus_{\sigma\in G} Hom_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let M be a graded right $R$-module and $N$ a graded left $R$-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma, \sigma \in G$, equal to the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha, y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-$gr$ thus defined will called the graded tensor product of $M$ and $N$. The projective objects of $R$-$gr$ will be called projective (resp. flat) graded modules because $M$ is $gr$-projective (rep. $gr$-flat) if and if only is a projective (resp. flat) module (refer [12], C.A, sec 1.2).

## 3 Gorenstein $gr$-flat modules

In this section, we first collect some known definitions and introduce two-degree Gorenstein $gr$-flat modules.

**Definition 3.1.** [2] An exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of $gr$-flat left $R$-modules in $R$-$gr$ is called a *complete gr-flat resolution* if $E \otimes_R -$ leaves the sequence exact for any $gr$-injective right $R$-module $E$. A graded left $R$-module $M$ is called Gorenstein $gr$-flat if there is a complete $gr$-flat resolution above such that $M \cong Ker(F^0 \to F^1)$.

**Definition 3.2.** [1] A graded right $R$-module $M$ is called Gorenstein $gr$-injective, if there exists an exact sequence of $gr$-injective right $R$-modules

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M \cong Ker(E^0 \to E^1)$ and such that $Hom_{R-gr}(E, -)$ leaves the sequence exact whenever $E$ is an $gr$-injective right $R$-module.

Now, we introduce the definition of two-degree Gorenstein $gr$-flat modules as follows.

**Definition 3.3.** A graded left $R$-module $M$ is said to be two-degree Gorenstein $gr$-flat if there exists an exact sequence of Gorenstein $gr$-flat left $R$-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong Im(G_0 \to G^0)$ and it remains exact after applying $H \otimes_R -$ for any Gorenstein $gr$-injective right $R$-module $H$.

Let $\mathcal{GF}_{gr}(R)$ and $\mathcal{GF}^{(2)}_{gr}(R)$ be the class of all Gorenstein $gr$-flat left and two-degree Gorenstein $gr$-flat left modules over $R$ respectively. Also denote $\mathcal{GF}^{(2)}_{i-gr}(R)$ the subcategory of $\mathcal{M}(R)$ for which there exists an exact sequence of Gorenstein $gr$-flat $R$-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong Im(G_0 \to G^0)$ and it remains exact after applying $E \otimes_R -$ for any $gr$-injective $R$-module $E$. It is routine to check that

$$\mathcal{GF}_{gr}(R) \subseteq \mathcal{GF}^{(2)}_{gr}(R) \subseteq \mathcal{GF}^{(2)}_{i-gr}(R).$$

Our main result proves that these inequalities becomes equalities when $R$ is a left $GF$-$gr$-closed ring as is stated next.
**Main Theorem:** Let $R$ be a left $GF$-$gr$-closed ring. Then

$$\mathcal{GF}_{gr}(R) = \mathcal{GF}^{(2)}_{gr}(R) = \mathcal{GF}^{(2)}_{i-gr}(R).$$

## 4 $GF$-$gr$-closed ring

In this section, we introduce left $GF$-$gr$-closed rings as follows.

**Definition 4.1.** A ring $R$ is called left $GF$-$gr$-closed if $\mathcal{GF}_{gr}(R)$ is closed under extensions.

We start with the following Lemma.

**Lemma 4.2.** *For a graded left $R$-module $M$ the following are equivalent:*

*(1) $M$ is Gorenstein $gr$-flat;*

*(2) $M$ satisfies the two following conditions:*

*(i) $Tor_i(E, M) = 0$ for all $i > 0$ and all $gr$-injective right $R$-modules $E$; and*

*(ii) There exists an exact sequence in $R$-$gr$, $0 \to M \to F^0 \to F^1 \to \cdots$, with each $F^i$ is $gr$-flat, such that $E \otimes_R -$ leaves the sequence exact whenever $E$ is $gr$-injective right $R$-module;*

*(3) There exists a short exact sequence in $R$-$gr$, $0 \to M \to F \to G \to 0$, where $F$ is $gr$-flat and $G$ is Gorenstein $gr$-flat.*

*Proof.* From the definition of Gorenstein $gr$-flat $R$ modules, the equivalence $(1) \Leftrightarrow (2)$ and $(1) \Rightarrow (2)$ are clear.

Now, we prove $(3) \Rightarrow (2)$. Suppose that there exists a short exact sequence in $R$-$gr$:

$$(\alpha) : 0 \to M \to F \to G \to 0$$

where $F$ is $gr$-flat and $G$ is Gorenstein $gr$-flat. Let $E$ be an $gr$-injective right $R$-module. Since $G$ is Gorenstein $gr$-flat and by the equivalence $(1) \Leftrightarrow (2)$, $Tor_{i+1}(E, G) = 0$ for all $i \geq 0$. Then, use the long exact sequence,

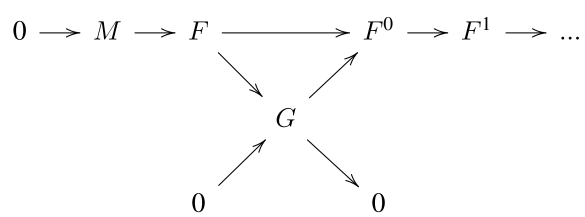
$$Tor_{i+1}(E, G) \to Tor_i(E, M) \to Tor_i(E, F),$$

to get $Tor_i(E, M) = 0$ for all $i > 0$.

On the other hand, since $G$ is Gorenstein $gr$-flat, there is an exact sequence

$$(\beta) : 0 \to G \to F^0 \to F^1 \to \cdots,$$

in $R$-$gr$ with each $F^i$ is $gr$-flat, such that $E \otimes_R -$ leaves the sequence exact whenever $E$ is an $gr$-injective right $R$-module. From the sequences $(\alpha)$ and $(\beta)$, we get the following commutative diagram:




such that $E \otimes_R -$ leaves the upper exact sequence exact whenever $E$ is an $gr$-injective right $R$-module, as desired. □

**Lemma 4.3.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $R$-$gr$. If $A$ is Gorenstein $gr$-flat and $C$ is $gr$-flat, then $B$ is Gorenstein $gr$-flat*

*Proof.* Since $A$ is Gorenstein $gr$-flat, there exists a short exact sequence $0 \to A \to F \to G \to 0$ in $R$-$gr$, with $F$ is $gr$-flat and $G$ is Gorenstein $gr$-flat. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \| & & \\
0 & \longrightarrow & F & \dashrightarrow & F' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

In the sequence $0 \to F \to F' \to C \to 0$, both $F$ and $C$ are $gr$-flat, therefore $F'$ is. Then, by the middle vertical sequence $B$ is Gorenstein $gr$-flat by Lemma 4.2, as desired. □

Next, we have the following result which is clear from the definition of Gorenstein graded flat modules.

**Theorem 4.4.** *If $R$ is a left $GF$-$gr$-closed ring, then the class $\mathcal{GF}_{gr}(R)$ is closed under direct summands.*

## 5 Stability of Gorenstein $gr$-flat modules

We need the following definitions and results to prove the main theorem of this paper. First, let us denote Gorenstein G $gr$-flat module, any element of $\mathcal{GF}^{(2)}_{i-gr}(R)$.

**Definition 5.1.** A graded left $R$-module $M$ is called a strongly Gorenstein $gr$-flat module if there exists an exact sequence of $R$-modules,

$$0 \to M \to F \to M \to 0$$

in $R$-$gr$ such that $F$ is a $gr$-flat and $E \otimes_R -$ leaves this sequence exact whenever $E$ is an $gr$-injective right module.

Next, we introduce strongly Gorenstein G $gr$-flat module.

**Definition 5.2.** A graded left $R$-module $M$ is called a strongly Gorenstein G $gr$-flat module if there exists an exact sequence $0 \to M \to G \to M \to 0$ in $R$-$gr$ such that $G$ is Gorenstein $gr$-flat over $R$ and $E \otimes_R -$ leaves the sequence exact for all $gr$-injective right $R$-module $E$.

**Proposition 5.3.** *(1) Any strongly Gorenstein G $gr$-flat module is Gorenstein G $gr$-flat.*

*(2) The family of Gorenstein G $gr$-flat modules is stable under arbitrary direct sums.*

*Proof.* (1) It is clear.
(2) It is straightforward, since any direct sum of Gorenstein $gr$-flat modules is Gorenstein $gr$-flat and since, for each positive integer $m$, $Tor_m(B, \bigoplus_i A_i) \cong \bigoplus_i Tor_m(B, A_i)$ for any family of modules $A_i$ and any right module $B$ by [10, Theorem 8.10]. □

**Proposition 5.4.** *Let $M$ be a graded left $R$-module. Then the following hold.*

*(1) Given an exact sequence*

$$0 \to S \to G_1 \to G_2 \to \cdots \to G_m \to M \to 0$$

*in $R$-$gr$ such that $G_1, G_2, \cdots, G_m$ are Gorenstein $gr$-flat modules, then*

$$Tor_{m+i}(E, M) \cong Tor_i(E, S)$$

*for each $gr$-injective right $R$-module $E$ and each integer $i \geq 1$.*

*(2) If $M$ is a Gorenstein G $gr$-flat $R$-module, then $Tor_i(E, M) = 0$ for each $gr$-injective right $R$-module $E$ and each integer $i \geq 1$.*

*Proof.* (1) It suffices to prove the case $m = 1$. So, let $0 \to S \to G \to M \to 0$ be an exact sequence such that $G$ is Gorenstein $gr$-flat. Let $E$ be a right $gr$-injective $R$-module. Applying $E \otimes_R -$ to this sequence gives the following exact sequence:

$$Tor_{i+1}(E, G) = 0 \to Tor_{i+1}(E, M) \to Tor_i(E, S) \to Tor_i(E, G) = 0$$

for each integer $i \geq 1$. This gives that

$$Tor_{i+1}(E, M) \cong Tor_i(E, S)$$

for each integer $i \geq 1$, as required.

(2) Let $M$ be a Gorenstein G $gr$-flat module. Then there exists a short exact sequence $0 \to K \to G \to M \to 0$ such that $G$ is a Gorenstein $gr$-flat module, $K$ is a Gorenstein G $gr$-flat module and

$$0 \to E \otimes S \to E \otimes G \to E \otimes M \to 0$$

is exact whenever $E$ is an $gr$-injective right $R$-module. Hence, $Tor_1(E, M) = 0$ for each $gr$-injective right module $E$. Reiterating this process and using (1), we get $Tor_i(E, M) = 0$ for each $gr$-injective right module $E$ and each integer $i \geq 1$. □

The next two results gives a graded version of Proposition 2.4 in [4].

**Proposition 5.5.** *Let $M$ be a graded left $R$-module. Then the following statements are equivalent:*

*(1) $M$ is a strongly Gorenstein G $gr$-flat module.*

*(2) There exists an exact sequence $0 \to M \to G \to M \to 0$ in $R$-$gr$ such that $G$ is a Gorenstein $gr$-flat module, and $Tor_1(E, M) = 0$ for any $gr$-injective right $R$-module $E$.*

*(3) There exists an exact sequence $0 \to M \to G \to M \to 0$ in $R$-$gr$ such that $G$ is a Gorenstein $gr$-flat module and, for any right $gr$-injective $R$-module $E$, the following sequence is exact*

$$0 \to E \otimes M \to E \otimes G \to E \otimes M \to 0.$$

*Proof.* (1) $\Rightarrow$ (2) holds by Proposition 5.4
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are straightforward, this completes the proof. □

**Proposition 5.6.** *Let $R$ be a graded ring and let $M$ be a Gorenstein G $gr$-flat $R$-module. Then $M$ is a direct summand of a strongly Gorenstein G $gr$-flat module.*

*Proof.* Let $M$ be a Gorenstein G $gr$-flat module and $\mathbf{G} = \cdots \longrightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} G_{-2} \xrightarrow{d_{-2}} \cdots$ be a complete Gorenstein $gr$-flat resolution such that $M = Im(d_0)$. Let $M_i := Im(d_i)$ for each integer $i$. It is easily seen that the following sequence is a complete Gorenstein $gr$-flat resolution:

$$\mathbf{G}' = \cdots \to \bigoplus_{i \in \mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i \in \mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i \in \mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \cdots$$

such that $Im(\bigoplus_i d_i) = \bigoplus_i M_i$ since $\mathcal{GF}_{gr}(R)$ is stable under direct sums. Then $\bigoplus_i M_i$ is a strongly Gorenstein G $gr$-flat module so that $M$ is a direct summand of a strongly Gorenstein G $gr$-flat module, as contended. □

For easiness, we adopt the following definition.

**Definition 5.7.** Let $M$ be a strongly Gorenstein G $gr$-flat module. An $R$-$gr$-module $J$ is called an $M_{gr}$-type module if there exists an exact sequence $0 \to M \to J \to H \to 0$ in $R$-$gr$ such that $H$ is a Gorenstein $gr$-flat module.

**Proposition 5.8.** *Let $M$ be a strongly Gorenstein G $gr$-flat module and $J$ an $M_{gr}$-type module. Then,*

*(1) $Tor_i(E, J) = 0$ for each $gr$-injective right $R$-module $E$ and for each integer $i \geq 1$.*

*(2) If $R$ is a left $GF$-$gr$-closed ring, then there exists an exact sequence $0 \to J \to F \to L \to 0$ in $R$-$gr$ such that $F$ is an $gr$-flat and $L$ is an $M_{gr}$-type module.*

*Proof.* (1) If $0 \to M \to J \to H \to 0$ is an exact sequence such that $H$ is a Gorenstein $gr$-flat $R$-module, then, by considering the corresponding long exact sequence and by Proposition 5.4, we have $Tor_i(E, J) \cong Tor_i(E, M) = 0$ for each $gr$-injective right module $E$ and each integer $i \geq 1$.

(2) Assume that $R$ is a left $GF$-$gr$-closed ring. Let $0 \to M \to G \to M \to 0$ and $0 \to M \to J \to H \to 0$ be exact sequences such that $G$ and $H$ are Gorenstein $gr$-flat $R$-modules. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \| & & \\
0 & \longrightarrow & J & \dashrightarrow & T & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since $G$ and $H$ are Gorenstein $gr$-flat modules, we get, as $R$ is left $GF$-$gr$-closed, $T$ is Gorenstein $gr$-flat. Then there exists a short exact sequence $0 \to T \to F \to S \to 0$ in $R$-$gr$ such that $F$ is a $gr$-flat and $S$ is a Gorenstein $gr$-flat $R$-module. Hence, we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & T & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \vdots & & \\
0 & \longrightarrow & J & \longrightarrow & F & \dashrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S & = & S & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

as desired. □

**Corollary 5.9.** *Let $R$ be a left $GF$-$gr$-closed ring. Let $M$ be a strongly Gorenstein G $gr$-flat module and $J$ an $M_{gr}$-type module. Then $J$ is a Gorenstein $gr$-flat $R$-module.*

*Proof.* We observe that by Proposition 5.8 there exist a $R$-$gr$ flat module $F_0$ and an $M_{gr}$-type module $L$ such that the following sequence $0 \to J \to F_0 \to L \to 0$ is exact and stays exact after applying the functor $E \otimes_R -$ for each $gr$-injective right module $E$. Then, it suffices to iterate Proposition 5.8(2) to get a resolution $0 \to J \to F_0 \to F_1 \to F_2 \to \cdots$ of $gr$-flat modules, which remains exact after applying the functor $E \otimes_R -$ for each $gr$-injective right $R$-module $E$. By Proposition 5.8(1) completes the proof. □

Now, we are ready to prove our main result.

**Proof of the main theorem.** Clearly we have $\mathcal{GF}_{gr}(R) \subseteq \mathcal{GF}^{(2)}_{gr}(R) \subseteq \mathcal{GF}^{(2)}_{i-gr}(R)$, it suffices to prove that $\mathcal{GF}^{(2)}_{i-gr}(R) \subseteq \mathcal{GF}_{gr}(R)$. Since $R$ is left $GF$-$gr$-closed, by Corollary 4.4, $\mathcal{GF}_{gr}(R)$ is stable under direct summands. Thus, it enough, by Proposition 5.6, to prove that any strongly Gorenstein G $gr$-flat module is Gorenstein $gr$-flat. By Proposition 5.5, there exists an exact sequence $0 \to M \to G \to M \to 0$ in $R$-$gr$ such that $G$ is a Gorenstein $gr$-flat module and $Tor_i(E, M) = 0$ for each $gr$-injective right module $E$ and each integer $i \geq 1$ where $M$ is a strongly Gorenstein G $gr$-flat module. As $G$ is Gorenstein $gr$-flat, there exists an exact sequence $0 \to G \to F \to G_1 \to 0$ in $R$-$gr$ such that $F$ is a $gr$-flat module and $G_1$ is a Gorenstein $gr$-flat module. Then we have the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F & \dashrightarrow & M_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_1 & = & G_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Hence, $M_1$ is an $M_{gr}$-type $R$-module. Therefore $M_1$ is a Gorenstein $gr$-flat module by Corollary 5.9. As $R$ is left $GF$-$gr$-closed and $G_1$ is Gorenstein $gr$-flat, we get $M$ is a Gorenstein $gr$-flat $R$-module, as desired. □

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