Palestine Journal of Mathematics

Vol. 8(2)(2019) , 8–14

BZS RINGS

Mark Farag and Ralph Tucci

Dedicated to the memory of our friend and colleague Dr. Henry E. Heatherly.

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16U99; Secondary 16B99.

Keywords and phrases: Boolean, zero square, BZS ring.

Abstract. Let R be an associative ring, not necessarily commutative and not necessarily with identity. We will say that R is *Boolean-zero square* or *BZS* if every element of R is either idempotent or nilpotent of index 2. We investigate the structure of these rings.

1 Introduction

Boolean rings comprise the well known class of commutative rings whose multiplicative operation is idempotent (see, for example, [9] for an important early treatment). Zero square rings, on the other hand, are a less well known class of anti-commutative rings (see [8]) with multiplication such that every non-zero element is nilpotent of index 2. In what follows, we consider rings with a "blend" of the multiplicative properties of Boolean and zero square rings.

Let R be an associative ring, not necessarily commutative and not necessarily with identity. We will say that R is *Boolean-zero square* or *BZS* if every non-zero element of R is either idempotent or nilpotent of index 2. Thus, BZS rings generalize both Boolean and zero square rings. The BZS property also generalizes Malone trivial multiplications on near-rings (see [2]) and is studied in the more general setting of near-rings in [3]. A BZS ring which is neither Boolean nor zero square is called *properly BZS*.

In the next section, we show that a BZS ring with a cyclic additive group must be either the ring $\mathbb{Z}_2$ or a ring with identically zero multiplication. We also give various examples of BZS rings. In Section 3, we study the structure of a BZS ring R; in particular, we show that: a) when R is properly BZS, its additive group is isomorphic to a direct product of copies of $\mathbb{Z}_2$, and b) the set of all nilpotent elements N is always an ideal of R. In Section 4, we prove that R properly BZS implies N is the unique maximal ideal of R, N has index 2 as an ideal of R, and N is the only prime ideal of R. In the final section of the paper we show that, up to isomorphism, there are exactly two properly BZS rings on the additive group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and we specify the multiplications that determine these rings.

We note that many of the examples and results in this paper derive from our consideration of low order BZS rings in SONATA [1], a package for GAP [5].

2 BZS Rings with Cyclic Additive Groups

Throughout we denote by E the set of idempotent elements of the BZS ring R and by N the set of nilpotent elements of R. We also let e, f denote idempotent elements, and x, y denote nilpotent elements.

Proposition 2.1. *Let $(R, +, \cdot)$ be a BZS ring such that $(R, +)$ is a cyclic group of order $n \geq 2$. Then either R is isomorphic to the ring of integers modulo 2 or it is a ring with identically zero multiplication.*

Proof. Suppose that g generates R additively. Since R is BZS, either $g \cdot g = 0$ or $g \cdot g = g$. If $g \cdot g = 0$, then for all integers $0 \leq \alpha, \beta < n-1$ we have $(\alpha g) \cdot (\beta g) = 0$ since R is distributive; i.e., R is a ring with identically zero multiplication. If $g \cdot g = g$, then $((n-1)g) \cdot ((n-1)g) = (n-1)^2 g \cdot g = 1g \cdot g = g \neq 0$, so $((n-1)g) \cdot ((n-1)g) = (n-1)g = -g$ implies $g + g = 0$, so that $n = 2$ and R is the ring of integers modulo 2.

□

In the following example, which uses the right near-ring version of a construction found in [6], we see that if $(R, +)$ fails to be a cyclic group, the conclusion of Proposition 2.1 may not hold. (We refer the reader to [7] for basic definitions and results in near-ring theory.)

Example 2.2. Let R be the Malone-trivial (right) near-ring on $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with $S = \{(0,1),(1,0)\}$ with multiplication given by:

$$a \cdot b = \begin{cases} a & \text{if } b \in S \\ (0,0) & \text{if } b \notin S \end{cases}.$$

We may verify that this multiplication satisfies the left distributive law as follows. Suppose $a, b, c \in R$. Then:

a) if $b \in S$ and $c \in S$, we have $a \cdot (b + c) = (0,0) = a + a = a \cdot b + a \cdot c$,

b) if $b \in S$ and $c \notin S$, or if $b \notin S$ and $c \in S$, we have $a \cdot (b+c) = a = a + (0,0) = a \cdot b + a \cdot c$, and

c) if $b \notin S$ and $c \notin S$, we have $a \cdot (b + c) = (0,0) = (0,0) + (0,0) = a \cdot b + a \cdot c$.

Thus R is a properly BZS ring since $(1,1) \cdot (1,1) = (0,0)$ and $(0,1) \cdot (0,1) = (0,1)$.

We can rewrite this example in terms of matrices.

Example 2.3. Let R be the ring in Example 2.2. Put $x = (0,0), y = (1,1), e = (0,1), f = (1,0)$. Now let S be the ring of 2×2 matrices over $\mathbb{Z}_2$ whose non-zero entries occur only in the last column. Thus,

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \middle| \, 0, 1 \in \mathbb{Z}_2 \right\}$$

The additive group of this ring is isomorphic to that of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Label these matrices as $\hat{x}, \hat{y}, \hat{e}, \hat{f}$ respectively. Then the map $g : R \to S$ given by $g(x) = \hat{x}, g(y) = \hat{y}, g(e) = \hat{e}, g(f) = \hat{f}$ is an isomorphism from R to S.

We can use the ideas in Example 2.2 and Example 2.3 to construct an example of an infinite properly BZS ring.

Example 2.4. Let α be any infinite ordinal. Let R be the set of all tuples whose order type is $\alpha + 1$ and whose entries are from $\mathbb{Z}_2$. Let $a, b \in R$. Define addition to be componentwise. Define the multiplication as follows:

$$a \cdot b = \begin{cases} a \text{ if the last entry in } b \text{ is } 1 \\ 0 \text{ otherwise} \end{cases}.$$

Then R is properly BZS. We can think of R as a set of infinite matrices over $\mathbb{Z}_2$ with non-zero entries in the last column only.

3 The Structure of R

Proposition 3.1. *Let R be a ring with identity 1. Let x be a non-zero nilpotent element of index 2. Then $1 + x$ is neither idempotent nor nilpotent of index 2.*

Proof. Note that $(1 + x)^2 = 1 + 2x$. If $1 + x$ is idempotent then $1 + 2x = 1 + x$ and $x = 0$, contradiction! If $1 + x$ is nilpotent of index 2 then $1 + 2x = 0$. Multiply this equation by x to get $x = 0$, contradiction! □

Corollary 3.2. *Let R be a BZS ring and let e be an idempotent. Then eRe has no non-zero nilpotent elements.*

Lemma 3.3. *If R is a BZS ring and if $e \in E$, then $2e = 0$.*

Proof. If $e = 0$, the result is obvious. Otherwise, we have $(-e)^2 = e^2 = e \neq 0$. Thus $(-e)^2 = -e$ and so $e = -e$. □

Lemma 3.4. *Let R be a BZS ring, let $e \neq 0$ be idempotent, and let x be nilpotent. Then the element $e + x$ is idempotent.*

Proof. If $x = 0$ then the result is trivial. Assume $x \neq 0$. The proof is by contradiction. Suppose that $e + x$ is nilpotent of index 2. Note that $(e + x)^2 = e + ex + xe$. Then

$$e + ex + xe = 0 \tag{3.1}$$

Multiply (3.1) on the left by x to get $xe = -xex$. Multiply (3.1) on the right by x to get $ex = -xex$. Thus

$$ex = xe. \tag{3.2}$$

Equation (3.1) then becomes

$$e + 2ex = 0. \tag{3.3}$$

By Lemma 3.3 we have that $2ex = 0$. Hence we have $e = 0$, contradiction!

□

Lemma 3.5. *If R is a BZS ring, then*

a) $exe = 0$;

b) $xex = 0$;

c) if $e \neq 0$ then $ex + xe = x$.

Proof. If $x = 0$ then the results are trivial. If $e = 0$, then a) and b) are trivial. Otherwise, $e + x$ is idempotent by Lemma 3.4, and we get

$$e + ex + xe = e + x,$$

yielding

$$ex + xe = x. \tag{3.4}$$

Multiply (3.4) by e on the right and simplify to get $exe = 0$. Multiply (3.4) on the left by x to get $xex = 0$. □

Corollary 3.6. *If R is a properly BZS ring and $x \in N$, then $2x = 0$.*

Proof. By Lemma 3.5 c) and Lemma 3.3 we have $2x = 2ex + 2xe = 0$. □

Corollary 3.7. *If R is a properly BZS ring, then the additive group of R is isomorphic to a direct product of copies of $\mathbb{Z}_2$.*

Proof. By Lemma 3.3 and Corollary 3.6, we have that every non-zero element of R is of order 2; i.e., $(R, +)$ is an elementary abelian 2-group. The result now follows from Theorem 8.5 of [4]. □

Corollary 3.8. *If R is a finite properly BZS ring then R has order 2^n for some integer $n \geq 2$.*

Corollary 3.9. *If R is a properly BZS ring, then the center of R is trivial.*

Proof. Let e be a non-zero idempotent element of R and let x be a non-zero nilpotent element of R. If $ex = xe$, then by Lemma 3.5 c), $2ex = x \neq 0$, contradiction! □

Corollary 3.10. *If a BZS ring R is commutative then R is not properly BZS.*

Lemma 3.11. *If R is a BZS ring then N is closed under subtraction.*

Proof. If $x - y \notin N$ then $0 \neq (x - y) \in E$. But then by Lemma 3.4 $((x - y) + y) = x$ is idempotent, contradiction! □

Lemma 3.12. *If R is a BZS ring then N is closed under multiplication by any element of R.*

Proof. If R is Boolean or if R consists only of nilpotent elements then the result is trivial. Otherwise, pick non-zero elements $e \in E$ and $x \in N$. By Lemma 3.5 we have that $(ex)^2 = 0$ and $(xe)^2 = 0$. Hence $EN \subseteq N$ and $NE \subseteq N$.

Now let $x, y \in N$. Since Lemma 3.11 implies that $x + y \in N$, $0 = (x + y)^2 = xy + yx$. Multiply on the right by xy to get $(xy)^2 = 0$. □

Theorem 3.13. *If R is a BZS ring then N is an ideal.*

Proof. This follows from Lemmas 3.11 and 3.12 and the fact that $0 \in N$. □

Proposition 3.14. *If R is a properly BZS ring and if $x, y \in N$ then $xy = yx$.*

Proof. Since N is closed under multiplication we have $0 = (x+y)^2 = xy+yx$. Thus $xy = -yx$. By Corollary 3.6 we have $-yx = yx$, and the result follows. □

The ring given in Examples 2.2 and 2.3 shows that, in general, idempotents of (properly) BZS rings need not commute.

Proposition 3.15. *If R is a properly BZS ring and if $e, f \in E \setminus \{0\}$, then $e + f \in N$.*

Proof. Let $e, f \in E \setminus \{0\}$. If $e = f$ then $e + f = 0$. Assume then that $e \neq f$. Suppose that $e + f$ is a non-zero idempotent. Pick $0 \neq x \in N$. Then $e + f + x$ is idempotent by Lemma 3.4. So

$$e + f + x = (e + f + x)^2 = (e + f)^2 + ex + fx + xe + xf$$

By Lemma 3.5 c) we have that

$$x = ex + xe = fx + xf.$$

Hence

$$e + f + x = (e + f)^2 + x + x = e + f,$$

which implies that $x = 0$, contradiction! Thus $e + f$ is nilpotent. □

Proposition 3.16. *If R is a BZS ring then E is closed under multiplication.*

Proof. If R is Boolean or zero square, the result is trivial. So suppose R is a properly BZS ring. Let $e, f \in E, e \neq f$. Note that

$$(e + f)^2 = e + ef + fe + f \tag{3.5}$$

By Proposition 3.15 we have that $e + f$ is nilpotent. Thus

$$e + ef + fe + f = 0,$$

so that

$$e + f = ef + fe. \tag{3.6}$$

Multiply (3.6) on the left by f and on the right by e and simplify to get $fe = fefe$. □

Example 3.17. Let R be the ring in Example 2.3. The idempotent elements are $\hat{e}$ and $\hat{f}$, and $\hat{e} + \hat{f} = \hat{y}$, which is nilpotent. Also, $\hat{e}\hat{f} = \hat{e}$ and $\hat{f}\hat{e} = \hat{f}$, which are idempotent. The set of nilpotent elements, $\{\hat{x}, \hat{y}\}$, is an ideal.

Example 3.18. Let R be the ring in Example 2.4. The idempotent elements are precisely those tuples whose last entry is 1. Since the sum of two idempotents has 0 in the last position, this sum is nilpotent. Also, the product of two idempotents has 1 in the last position, so that the product of two idempotents is idempotent. A similar argument can be used to show that the set of nilpotent elements is an ideal.

Theorem 3.19. *If R is a BZS ring then one of the following holds.*
a) R is Boolean;
b) R is zero square;
c) R is an extension of a nil ring N whose elements are square zero by $\mathbb{Z}_2$.

Proof. This follows from Proposition 3.15. □

4 Maximal, Prime, and One-sided Ideals of Properly BZS Rings

Proposition 4.1. *If R is a properly BZS ring, then N is a maximal ideal of R of index 2. Further, N is the unique maximal ideal of R.*

Proof. It follows from Lemma 3.4 and Proposition 3.15 that the factor ring R/N has two distinct cosets: $0+N$ and $e+N$ for some $e \in E \setminus \{0\}$. Since N has index 2 as an ideal of R, N is a maximal ideal of R.

To show that N is the unique maximal ideal, let I be any ideal in R. If $I \not\subseteq N$ then I contains a non-zero idempotent e. Let $x \in N$. Then $x = ex + xe \in I$ by 3.5 c). Thus $N \subsetneq I$. Hence $I = R$ and the result follows. □

Corollary 4.2. *If R is a properly BZS ring, then $|E \setminus \{0\}| = |N|$.*

Proof. The result follows from Proposition 4.1 and the fact that $R = E \cup N$ with $E \cap N = \{0\}$. □

In a Boolean ring, every prime ideal must be maximal. Properly BZS rings share this special property.

Lemma 4.3. *If R is a properly BZS ring, then for any $x \in N$, $(x)(x) = (0)$, where (x) is the principal two-sided ideal generated by x.*

Proof. Every element of (x) is a finite sum of terms of the form x, ax, xb, or axb, for $a, b \in R$. It follows that every element of $(x)(x)$ is a sum of elements such that each summand either contains a factor of $x^2 = 0$ or contains a factor of the form xcx for some $c \in R$. If $c \in E$, then Lemma 3.5 b) implies that $xcx = 0$. If $c \in N$, then Proposition 3.14 implies that $xcx = cx^2 = 0$. So each summand in an element of $(x)(x)$ is zero, and the result follows. □

Theorem 4.4. *If R is a properly BZS ring, then every prime ideal is a maximal ideal.*

Proof. The result is trivial if R is the ring $\mathbb{Z}_2$, so we assume $|R| > 2$ (and hence $|N| \geq 2$). If P is a prime ideal of R that is not maximal, P must be properly contained in the unique maximal ideal N. Thus there is some non-zero element $x \in N \setminus P$. From Lemma 4.3, we have $(0) = (x)(x) \subseteq P$, whereas $(x) \not\subseteq P$, contradiction! □

Since they are always commutative, Boolean rings all have the property that every one-sided ideal is also a two-sided ideal. This property does not hold in general for BZS rings, as the following example shows.

Example 4.5. Let R be the ring from Example 2.2 and let $I = \{(0,0),(0,1)\}$. Then $(I,+)$ is clearly a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and since $(0,0)\cdot r$ and $(0,1)\cdot r$ are both in I for every $r \in R$, I is a right ideal of R. However, I is not a left ideal of R since $(1,0)\cdot(0,1) = (1,0) \notin I$.

5 The Properly BZS Rings on $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

In the final section of this paper, we show that, up to isomorphism, there are exactly two properly BZS rings with additive group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We also specify the multiplications that determine these properly BZS rings.

Example 5.1. Suppose that $(R,+,\cdot)$ is a ring with additive group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If a and b are distinct non-zero elements of R, then $a+b$ is the third non-zero element of R, so that $R = \{0, a, b, a+b\}$. If R is properly BZS, then by Proposition 4.1 we may assume without loss of generality that $a^2 = a$, $b^2 = b$, and $(a+b)^2 = 0$. The last equality implies that

$$a + ba + ab + b = 0. \tag{5.1}$$

Then:

<u>Case 1:</u> If $ba = ab$, Equation (5.1) implies that $a + 2ab + b = 0$, so that $a = -b$, contradiction!

<u>Case 2:</u> If $ba = 0$, then:

Case 2(i): if $ab = a$, then Equation (5.1) implies that $2a + b = 0$, so that $b = 0$, contradiction!

Case 2(ii): if $ab = b$, then Equation (5.1) implies that $a + 2b = 0$, so that $a = 0$, contradiction!

Case 2(iii): if $ab = a + b$, then $a(b \cdot b) = ab = a + b$, while $(ab)b = (a + b)b = a + 2b = a$, so that associativity fails to hold, contradiction!

<u>Case 3:</u> If $ba = a$, then:

Case 3(i): if $ab = 0$, then Equation (5.1) implies that $2a + b = 0$, so that $b = 0$, contradiction!

Case 3(ii): if $ab = b$, then $(R, +, \cdot)$ is the opposite ring of the Malone trivial (near-)ring with multiplication:

$$r \cdot s = \begin{cases} r & \text{if } s \in \{a, b\} \\ 0 & \text{if } s \notin \{a, b\} \end{cases} .$$

Case 3(iii): if $ab = a + b$, then Equation (5.1) implies that $3a + 2b = 0$, so that $a = 0$, contradiction!

<u>Case 4:</u> If $ba = b$, then:

Case 4(i): if $ab = 0$, then Equation (5.1) implies that $a + 2b = 0$, so that $a = 0$, contradiction!

Case 4(ii): if $ab = a$, then $(R, +, \cdot)$ is the Malone trivial (near-)ring with multiplication:

$$r \cdot s = \begin{cases} r & \text{if } s \in \{a, b\} \\ 0 & \text{if } s \notin \{a, b\} \end{cases} .$$

Case 4(iii): if $ab = a + b$, then Equation (5.1) implies that $2a + 3b = 0$, so that $b = 0$, contradiction!

<u>Case 5:</u> If $ba = a + b$, then:

Case 5(i): if $ab = 0$, then $b(a \cdot a) = ba = a + b$, while $(ba)a = (a + b)a = 2a + b = b$, so that associativity fails to hold, contradiction!

Case 5(ii): if $ab = a$, then Equation (5.1) implies that $3a + 2b = 0$, so that $a = 0$, contradiction!

Case 5(iii): if $ab = b$, then Equation (5.1) implies that $2a + 3b = 0$, so that $b = 0$, contradiction!

Thus, there are exactly two isomorphism classes of properly BZS rings on the additive group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$: one represented by the ring in Example 2.2 and one represented by its opposite ring.

References

[1] E. Aichinger, F. Binder, J. Ecker, R. Eggetsberger, P. Mayr, C. Nöbauer, SONATA - System of Near-Rings and Their Applications, Package for the group theory system GAP4, Division of Algebra, Johannes Kepler University, Linz, Austria, 1999.

[2] G. A. Cannon, M. Farag, L. Kabza, and K. M. Neuerburg, Centers and generalized centers of near-rings without identity defined via Malone-like multiplications, *Math. Pannonica*, 25/2, (2014 - 2015), pp. 3−23.

[3] M. Farag, BZS near-rings, (preprint).

[4] L. Fuchs, *Infinite Abelian Groups, Vol. I*, Academic Press, Inc., New York, NY, 1970.

[5] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.8.6*; 2016, (`https://www.gap-system.org`).

[6] J. J. Malone, Jr., Near-rings with trivial multiplications, *Amer. Math. Monthly*, 74, No. 9, (1967), pp. $1111 - 1112$.

[7] G. F. Pilz, *Near-rings*, 2^{nd} ed., North-Holland Publishing Company, Amsterdam, New York, Oxford, 1983.

[8] R. P. Stanley, Zero square rings, *Pacific J. of Math.*, 30, No. 3, (1969), pp. $811 - 824$.

[9] M. H. Stone, The theory of representation for Boolean algebras, *Trans. of the Amer. Math. Soc.*, 40, No. 1, (Jul., 1936), pp. $37 - 111$.

Author information

Mark Farag, Department of Mathematics, Fairleigh Dickinson University, Teaneck, NJ 07666, USA.
E-mail: `mfarag@fdu.edu`

Ralph Tucci, Department of Mathematics and Computer Science, Loyola University New Orleans, New Orleans, LA 70118, USA.
E-mail: `tucci@loyno.edu`

Received: September 18, 2018.
Accepted: Februery 28 , 2019