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Some Results On Dedekind Rings

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Abstract. In this paper we establish several equivalent conditions for a commutative ring to be a Dedekind ring.

1 Introduction

Throughout this paper R denotes a commutative ring with identity. $L(R)$ denotes the lattice of all ideals of R. In this paper we establish some conditions for a quasi-regular ring R to be a Dedekind ring (see Theorem 2.9). Using this result, we establish some equivalent conditions for a quasi-regular ring R in which every regular principal ideal of R is a finite intersection of prime power ideals to be a Dedekind ring (see Theorem 2.10). Next we obtain some equivalent conditions for a quasi-regular ring R in which every regular principal ideal of R is a finite intersection of primary ideals to be a Dedekind ring (see Theorem 2.11). We also establish some equivalent conditions for a quasi-regular ring R in which every regular principal ideal of R is a finite product of primary ideals to be a Dedekind ring (see Theorem 2.12). Using these results, we characterize Dedekind rings in terms of quasi-regular weak π-rings (see Theorem 2.13).

We use $\subset$ for proper set containment. For any $A, B \in L(R)$, we denote $A \backslash B = \{x \in A \mid x \notin B\}$. For any $a \in R$, the principal ideal generated by a is denoted by (a). An element $a \in R$ is said to be *regular* (*zero divisor*) if $((0) : (a)) = (0)$ ($ra = 0$ for some $0 \neq r \in R$). An ideal I of R is *regular* (*faithful*) if it contains a regular element ($((0) : I) = (0)$). A principal ideal (a) of R is said to be a *regular principal ideal* if a is a regular element of R. For any $I \in L(R)$, we denote $\sqrt{I} = \{a \in R \mid a^n \in I$ for some positive integer $n \in \mathbb{Z}\}$. An ideal I of R is said to be a *radical ideal* if $I = \sqrt{I}$. An ideal I of R is a *semi-primary ideal* if its radical is a prime ideal. Rings in which semi-primary ideals are primary have been studied in [8] and [9] and [10]. A ring R is said to satisfy *Property* (A) if every finitely generated faithful ideal is regular. Recall that an ideal I of R is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. An ideal M of R is called a *quasi-principal ideal* [19, Exercise 10, Page 147] (or a principal element of $L(R)$ [20]) if it satisfies the following identities (i) $(A \cap (B : M))M = AM \cap B$ and (ii) $((A + BM) : M) = (A : M) + B$, for all $A, B \in L(R)$. Obviously a finite product of quasi-principal ideals is quasi-principal and every quasi-principal ideal is a multiplication ideal. It is well known that a multiplication ideal is locally principal [1, Theorem 1]. It is also known that an ideal I of R is finitely generated and locally principal if and only if I is a finitely generated multiplication ideal [1, Theorem 3]. In fact, an ideal I of R is quasi-principal if and only if it is finitely generated and locally principal (see [20, Theorem 2]). For any $A, B \in L(R)$, we say A and B are *comaximal* if $A + B = R$. A prime ideal P of R is said to be *branched* if there exists a P-primary ideal Q of R such that $Q \neq P$. P is said to be *unbranched* if P is the only P-primary ideal. A prime ideal P of R is said to be an *ℓ-prime* if the set of all P-primary ideals of R is linearly ordered. For any prime ideal P of R, we denote $P^{\nabla} = \cap \{Q \in L(R) \mid Q$ is P-primary$\}$. For any prime ideals $M, P \in L(R)$, we say M *covers* P if $P \subset M$ and there is no prime ideal P_1 of R such that $P \subset P_1 \subset M$. A non-minimal prime ideal P of R is said to be a *C-prime ideal* if P^{∇} is prime, P covers P^{∇} and any prime $Q \subset P$ implies $Q \subseteq P^{\nabla}$.

If $\{P_\alpha\}$ is the collection of all minimal prime ideals of an ideal I of R, then by an *isolated P_α-primary component* of I we mean the intersection Q_α of all P_α-primary ideals which contain I. The $kernel$ of I is the intersection of all ${Q_\alpha}'^s$. It is well known that every ideal is equal to its kernel if and only if the semiprimary ideals are primary [10, Theorem 4]

An ideal I of R is said to be *quasi-invertible* if it is quasi-principal and faithful. I is said to be *quasi-regular*, if it contains a quasi-invertible ideal of R. If R satisfies Property (A), then by [12, Lemma 18.1, page 110], an ideal I of R is quasi-invertible (quasi-regular) if and only if I is invertible (regular). Recall that R is called a *von Neumann Regular ring*, if for each $a \in R$, there exists $x \in R$ such that $axa = a$. It is well known that R is a von Neumann Regular ring if and only if every ideal of R is a radical ideal of R. R is called a *quasi-regular ring*, if its classical ring of quotients is a von-Neumann regular ring. For various characterizations of quasi-regular rings, the reader is referred to [7] and [13]. R is a *reduced ring* if the zero element is the only nilpotent element. Note that every quasi-regular ring is a reduced ring [7, Theorem 2.2] and in reduced rings minimal prime ideals are unbranched prime ideals. R is called a *Marot ring* if every regular ideal is generated by its set of regular elements. By [7, Theorem 2.2] and [13, Theorem 2], quasi-regular rings satisfy Property (A) and non minimal prime ideals in a quasi-regular ring are regular ideals. Also by [12, Theorem 4.5, Theorem 7.2 and Theorem 7.4], quasi-regular rings are Marot rings. A ring R is said to be *arithmetical*, if its ideal lattice is distributive. R is said to satisfy the condition (*), if every regular principal ideal is a finite intersection of primary ideals. An ideal I of R is *weak invertible*, if I is quasi-principal and $((0) : I) = (e)$ for some idempotent $e \in R$. R is said to be a *WI-ring* if every finitely generated ideal is weak invertible. A reduced ring R is said to be a *Dedekind ring*, if every ideal not contained in any minimal prime ideal is a multiplication ideal. A reduced ring R is said to be an *almost Dedekind ring* if (i) every ideal not contained in any minimal prime ideal is locally principal and (ii) for every $a \in R$, the ideal $(a) + ((0) : (a))$ is a finitely generated ideal of R. Weak invertible rings, Dedekind rings and almost Dedekind rings have been studied in [16] and [17]. R is said to be a *weak π-ring* [18] if every regular principal ideal is a finite product of prime ideals. R is said to be an *almost weak π-ring* if for each regular principal ideal $(a) \in L(R)$, $(a)_M$ is a finite product of prime ideals in R_M for all maximal ideals M containing a. For more information on weak π-rings and almost weak π-rings, the reader is referred to [18]. R is a *multiplication ring* if every ideal is a multiplication ideal. R is an *almost multiplication ring* if R_M is a multiplication ring, for every maximal ideal M of R. For more information on multiplication rings and almost multiplication rings the reader may consult [4] and [21]. R is said to be a *valuation ring* if any two ideals are comparable. It is well known that R is an arithmetical ring if and only if for every maximal ideal M of R, R_M is a valuation ring. R is said to be a *discrete valuation ring* if R is a Dedekind domain with only one proper (different from (0) and (1)) prime ideal. Following [6], R is an α-ring, if R satisfies the ascending chain condition for prime ideals and every primary ideal is a power of its radical.

Throughout this paper, all ideals are assumed to be proper (i.e., $\neq R$). For general background and terminology, the reader may consult [11] and [19].

2 Dedekind rings

In this section we establish several equivalent conditions for R to be a Dedekind ring.

We now prove some useful lemmas.

Lemma 2.1. *Suppose R is a quasi-regular ring in which every regular principal ideal of R is a finite intersection of prime power ideals and for every non minimal prime ideal M of R, M^n is M-primary for every positive integer n. If P is a C-prime ideal, then rank $P = 1$.*

Proof. Suppose P is a C-prime ideal. Then P is non minimal, P^∇ is prime, P covers P^∇ and any prime properly contained in P is contained in P^∇. We claim that P^∇ is a minimal prime ideal. Suppose P^∇ is a non minimal prime ideal. As R is quasi-regular, it follows that P^∇ is regular. Choose a regular element $x \in P^\nabla$. As R is quasi-regular, it follows that R is a Marot ring. Since P covers P^∇, there exists a regular element $y \in P$ such that $y \notin P^\nabla$. By hypothesis, there exist prime ideals $Q_1, Q_2, \ldots, Q_n$ such that $(xy) = \bigcap_{i=1}^{n} Q_i^{\alpha_i}$. Suppose $Q_i \subseteq P^\nabla$ for $i = 1, 2, \ldots, k$ and $Q_j \nsubseteq P^\nabla$ for $j = k+1, \ldots, n$. Note that each Q_i $(1 \leq i \leq k)$ is a non minimal prime ideal, so by hypothesis, $Q_i^{\alpha_i}$ is Q_i-primary for $1 \leq i \leq k$. Again since $xy \in \bigcap_{i=1}^{k} Q_i^{\alpha_i}$ and $y \notin Q_i$ $(1 \leq i \leq k)$, it follows that $x \in \bigcap_{i=1}^{k} Q_i^{\alpha_i}$. Therefore $(xy)_P = \bigcap_{i=1}^{k} (Q_i^{\alpha_i})_P$

$= (x)_P$ (in R_P). Therefore by Nakayama's lemma, $(x)_P = (0)_P$, a contradiction as x is regular. This shows that P^∇ is a minimal prime ideal and hence rank $P = 1$. □

Lemma 2.2. *Let R satisfy the hypothesis of Lemma 2.1 and let P be a C-prime ideal. Then R_P is a discrete valuation ring.*

Proof. By Lemma 2.1, rank $P = 1$. Let P^∇ be the minimal prime ideal properly contained in P. As R is reduced, it follows that R_P is a one dimensional domain. Now we claim that P_P is principal in R_P. If $P = P^2$, then by hypothesis, $(y)_P = P_P$ (in R_P) for some regular element $y \in P \setminus P^\nabla$. As P_P is idempotent and principal, it follows that $P_P = (0)_P$ (in R_P), a contradiction. Therefore $P \neq P^2$. Choose any regular element $x \in P \setminus P^2$. Note that $x \notin P^\nabla$ as x is regular and P^∇ is a minimal prime ideal. By hypothesis $(x) = \overset{n}{\underset{i=1}{\cap}} P_i^{\alpha_i}$ for some prime ideals $P_1, P_2, \ldots, P_n$ of R. Since $x \notin P^2$, it follows that $\alpha_i = 1$ for every $P_i \subseteq P$. Therefore $(x)_P = P_P$ (in R_P). As P_P is principal in R_P and R_P is a one dimensional domain, it follows that R_P is a discrete valuation ring and the proof is complete. □

Lemma 2.3. *Let R satisfy the hypothesis of Lemma 2.1 and let M be an idempotent prime ideal. If every non minimal prime ideal, which is minimal over a finitely generated ideal, is a C-prime ideal, then M is a minimal prime ideal.*

Proof. We claim that M is a minimal prime ideal. Suppose M is not a minimal prime ideal. Then M is regular. Choose a regular element $x \in M$. Note that by hypothesis, the principal ideal (x) has only finitely many minimal primes over (x). Let $Q_1, Q_2, \ldots, Q_k$ be the minimal primes over (x) contained in M. Suppose $M = Q_i$ for some i. Then $M = Q_i$ for all i. Again since $M = M^2$, by hypothesis, $(x)_M = M_M$ (in R_M). As $M = M^2$, by Nakayama's lemma, $M_M = (0)_M$ (in R_M), so M is a minimal prime ideal, a contradiction. Therefore assume that $M \neq Q_i$ for all i. Choose any $y \in M$ such that $y \notin \overset{k}{\underset{i=1}{\cup}} Q_i$. Let $Q \subseteq M$ be a prime ideal minimal over $(x) + (y)$. As $x \in Q$, it follows that Q is non minimal. Again by hypothesis, Q is a C-prime ideal and hence by Lemma 2.1, rank $Q = 1$. This shows that Q is minimal over (x) and hence $Q = Q_i$ for some i. But this contradicts the fact that $y \in Q$. Therefore M is a minimal prime ideal and the proof is complete. □

Lemma 2.4. *Let R be a quasi-regular ring satisfying the condition (*). Suppose M is a C-prime ideal of R. Then R_M is a one dimensional domain. Further if M is a non idempotent ℓ-prime ideal and M^n is M-primary for every positive integer n, then R_M is a discrete valuation ring.*

Proof. Choose any regular element $a \in M$ such that $a \notin M^\nabla$. Suppose M^∇ is a non minimal prime ideal. Choose a regular element $x \in M^\nabla$. By hypothesis $(xa) = \overset{n}{\underset{i=1}{\cap}} Q_i$ for some primary ideals $Q_1, Q_2, \ldots, Q_n$ of R. Suppose $Q_i \subseteq M^\nabla$ for $i = 1, 2, \ldots, k$ and $Q_j \nsubseteq M^\nabla$ for $j = k+1, \ldots, n$. Again since $xa \in Q_i$ and $a \notin \sqrt{Q_i}$ for $i = 1, 2, \ldots, k$, it follows that $x \in \overset{k}{\underset{i=1}{\cap}} Q_i$ and hence $(x)_M = (x)_M (a)_M$ (in R_M). Now by Nakayama's lemma, $(x)_M = (0)_M$ (in R_M), a contradiction as x is regular. Therefore M^∇ is a minimal prime ideal and hence R_M is a one dimensional domain. Further if M is a non idempotent ℓ-prime ideal and M^n is M-primary for every positive integer n, then $M_M = (x)_M$ for any $x \in M \setminus M^2$ and hence R_M is a discrete valuation ring. □

Lemma 2.5. *Let R be a quasi-regular ring satisfying the condition (*). Suppose every non minimal branched prime ideal is a C-prime ideal. If the prime ideal M is unbranched, then M is a minimal prime ideal.*

Proof. Suppose the prime ideal M is unbranched. We claim that M is a minimal prime ideal. Suppose M is a non minimal prime ideal. Choose a regular element $x \in M$. By hypothesis, the principal ideal (x) has only finitely many minimal primes. Let $Q_1, Q_2, \ldots, Q_k$ be the minimal primes over (x). If $M = Q_i$ for some i, then by hypothesis, $(x)_M = {(x)^2}_M = M_M$ (in R_M) as M is unbranched. So by Nakayama's lemma, $(x)_M = (0)_M$ (in R_M), a contradiction as x is a regular element. Therefore $M \neq Q_i$ for all i. Choose any $y \in M$ such that $y \notin \overset{k}{\underset{i=1}{\cup}} Q_i$. Let Q be a minimal prime over $(x)+(y)$. If Q is unbranched, then $((x)+(y))_Q = (((x)+(y))^2)_Q = Q_Q$ (in R_Q), so by Nakayama's lemma, $Q_Q = (0)_Q$ (in R_Q) and hence Q is a minimal prime ideal of R, a contradiction. Therefore Q is a branched non minimal prime ideal. Again by hypothesis, Q is a C-prime ideal and hence by Lemma 2.4, $dim R_Q = 1$. As rank $Q = 1$, it follows that Q is a minimal prime ideal over (x), which is again a contradiction. This shows that M is a minimal prime ideal. □

For any ideal $I \in L(R)$, we denote $\theta(I) = \sum\{(I_1 : I) \mid I_1 \subseteq I$ and I_1 is a finitely generated ideal$\}$.

Lemma 2.6. *Suppose R is a quasi-regular ring in which every regular principal ideal is a finite product of primary ideals. Suppose I is a regular ideal of R such that I is locally principal and every prime minimal over I is a maximal ideal. Then I is invertible.*

Proof. By [12, Lemma 18.1, page 110], it is enough if we show that I is finitely generated. We claim that $\theta(I) = R$. Suppose $\theta(I) \neq R$. Then $\theta(I) \subseteq M$ for some maximal ideal M of R. By hypothesis, I is generated by regular elements. Again since I is locally principal, by [3, Theorem 1], I is locally completely join irreducible, so $I_M = (x)_M$ for some regular element $x \in I$. By hypothesis, there exist primary ideals $Q_1, Q_2, \ldots, Q_n$ such that $(x) = Q_1 Q_2 \cdots Q_n$. Without loss of generality, assume that $Q_i \subseteq M$ for $i = 1, 2, \ldots, k$ and $Q_j \nsubseteq M$ for $j = k+1$, $k+2, \ldots, n$. Then $I_M = (x)_M = (Q_1)_M (Q_2)_M \cdots (Q_k)_M$. Since $I_M \subseteq (Q_i)_M$, it follows that $I \subseteq Q_i$ for $i = 1, 2, \ldots, k$. Since M is minimal over I, it follows that each Q_i is M-primary and hence $Q_1 Q_2 \cdots Q_k$ is M-primary. Therefore $I \subseteq Q_1 Q_2 \cdots Q_k$. Choose elements $x_j \in Q_j$ such that $x_j \notin M$ for $j = k+1, k+2, \ldots, n$. Let $z = x_{k+1} x_{k+2} \cdots x_n$. Since $I \subseteq Q_1 Q_2 \cdots Q_k$ and $z \in Q_{k+1} Q_{k+2} \cdots Q_n$, it follows that $Iz \subseteq Q_1 Q_2 \cdots Q_n = (x)$, so $z \in ((x) : I) \subseteq \theta(I) \subseteq M$, which is a contradiction. Therefore $\theta(I) = R$ and hence $R = \sum_{i=1}^{n} (I_i : I)$, where $I_i{}'^s$ are finitely generated ideals contained in I. Therefore $I = \sum_{i=1}^{n} I_i$ and hence I is a finitely generated ideal. □

Lemma 2.7. *Let R be a quasi-regular ring in which every regular principal ideal of R is a finite product of primary ideals. Suppose M is a C-prime ideal of R. Then R_M is a one dimensional domain. Further if M is a non idempotent ℓ-prime and M^n is M-primary for every positive integer n, then R_M is a discrete valuation ring.*

Proof. By hypothesis, M is non minimal and so M is a regular ideal. Choose any regular element $a \in M$ such that $a \notin M^{\nabla}$. Then $(a)_M$ is M_M-primary (in R_M), so ${M^{\nabla}}_M \subset (a)_M$. We claim that M^{∇} is a non regular ideal. Suppose M^{∇} is a regular ideal. Let $x \in M^{\nabla}$ be a regular element of R. By hypothesis $(x) = QA$ for some primary ideal $Q \subseteq M^{\nabla}$. Note that $Q_M = (a)_M Q_M$. Therefore $(x)_M = Q_M A_M = Q_M (a)_M A_M = (x)_M (a)_M$ (in R_M) and hence by Nakayama's lemma, $(x)_M = (0)_M$, which is a contradiction as x is a regular element. Therefore M^{∇} is a non regular ideal and so M^{∇} is a minimal prime ideal. As R is reduced, it follows that, R_M is a one dimensional domain. Further if M is a non idempotent ℓ-prime and M^n is M-primary for every positive integer n, then $M_M = (x)_M$ for any $x \in M \setminus M^2$ and hence R_M is a discrete valuation ring. □

Lemma 2.8. *Let R be a quasi-regular ring in which every regular principal ideal is a finite product of primary ideals. Suppose every non minimal branched prime ideal is a C-prime ideal. If the prime ideal M is unbranched, then M is minimal.*

Proof. By using Lemma 2.7 and by imitating the proof of Lemma 2.5, we can get the result. □

We now establish some conditions for R to be a Dedekind ring (see Theorem 2.9).

Theorem 2.9. *R is a Dedekind ring if and only if R satisfies the following conditions:*

(i) *R is a quasi-regular ring.*
(ii) *Every regular principal ideal of R is a finite intersection of prime power ideals.*
(iii) *For every non minimal prime ideal P of R, P^n is P-primary for every positive integer n.*
(iv) *Every non minimal maximal ideal is a C-prime ideal.*

Proof. Suppose R is a Dedekind ring. By [17, Theorem 1], R satisfies the conditions (i) and (ii). By [16, Theorem 3.8 and Theorem 3.13], R is an almost multiplication ring, so by [21, Theorem 1 and Theorem 4], $dimR \leq 1$ and hence cvery non minimal prime ideal is maximal. Again by [16, Theorem 3.13(v)], non minimal prime ideals are invertible prime ideals, so by [21, Lemma 21], non minimal prime ideals are C-prime ideals. Consequently, R satisfies the conditions (iii) and (iv).

Conversely, assume that R satisfies the conditions (i), (ii), (iii) and (iv). Let M be a maximal ideal. If M is minimal, then R_M is a field. Suppose M is a non minimal prime ideal. Then by Lemma 2.1 and Lemma 2.2, R_M is a discrete valuation ring. Therefore R is an arithmetical ring, so by [16, Theorem 3.3], R is a WI-ring. Observe that by hypothesis, R satisfies the condition (*). Let P be a regular prime ideal. By [13, Theorem 2], P is a non minimal prime ideal. Since P is locally principal, by [2, see the remark after Theorem 13], P is invertible, so by [16, Theorem 3.13(iii)], R is a Dedekind ring and the proof is complete. □

In Theorem 2.10 and Theorem 2.11, we obtain some equivalent conditions for a quasi-regular ring in which every regular principal ideal is a finite intersection of prime power ideals (primary ideals) to be a Dedekind ring.

Theorem 2.10. *Suppose R is a quasi-regular ring in which every regular principal ideal of R is a finite intersection of prime power ideals. Then the following statements on R are equivalent:*

(i) *R is a Dedekind ring.*
(ii) *Every semiprimary ideal is primary.*
(iii) *Every primary ideal is a power of its radical.*
(iv) *R is an α-ring.*
(v) *Every non minimal prime ideal is a multiplication ideal.*
(vi) *R_M is a valuation ring, for every prime ideal M of R.*

Proof. (i)⇒(ii). Suppose (i) holds. By [16, Theorem 3.8 and Theorem 3.13], R is an almost multiplication ring, so by [21, Theorem 4], every semiprimary ideal is primary. Thus (ii) holds.

(ii)⇒(iii). Suppose (ii) holds. Let Q be a primary ideal of R. If $\sqrt{Q} = M$ is a minimal prime ideal, then $Q = M$ as M is unbranched. Suppose M is non minimal. By [9, Corollary 2.2], $dimR \leq 1$ and also by [9, Corollary 2.3], every non minimal maximal prime ideal is a C-prime ideal. Therefore M is a C-prime ideal. Again by Lemma 2.3, non minimal prime ideals are non idempotent. So M is a non idempotent maximal ideal. Choose any regular element $x \in M \backslash M^2$. Then by hypothesis, $(x)_M = M_M$ (in R_M), so R_M is a discrete valuation ring. Consequently, Q is a power of M and therefore (iii) holds.

(iii)⇒(iv). Suppose (iii) holds. Let M be a non minimal maximal ideal of R. Observe that by [5, Theorem 3], each P^n ($n \in Z^+$) is P-primary, for every non minimal prime ideal P of R. Also if P is a non minimal prime ideal and minimal over a finitely generated ideal, then P is not the union of a chain of primes properly contained in P, so by [5, Corollary 1], $P \neq P^2$, and hence by [5, Theorem 3], P is a C-prime ideal. Therefore by Lemma 2.3, non minimal prime ideals are non idempotent. Again by [5, Theorem 3], non minimal prime ideals are C-prime ideals. Consequently, by Lemma 2.1, rank $M = 1$ and hence $dimR \leq 1$. Therefore R is an α-ring.

(iv)⇒(v). Suppose (iv) holds. Let P be a non minimal prime ideal of R. By the ascending chain condition for prime ideals, there exists a prime ideal P_1 such that P covers P_1. Note that P is minimal over $P_1 + (x)$ for any $x \in P \setminus P_1$, so by [5, Theorem 1 and Theorem 3], P is a C-prime ideal. Clearly, each Q^n ($n \in Z^+$) is Q-primary for every non minimal prime ideal Q of R [5, Theorem 3]. Hence by Theorem 2.9, R is a Dedekind ring. Thus (v) holds.

(v)⇒(vi). Suppose (v) holds. Let M be a maximal ideal of R. If M is minimal, then R_M is a field. Suppose M is non minimal. Then M is regular. By hypothesis, M is locally principal. Note that by [2, Lemma 1], R satisfies the condition (*). Again by [2, see the remark after Theorem 13], M is an invertible ideal and so by [21, Lemma 21], M is a C-prime ideal. Again by [2, Lemma 1] and by Lemma 2.1 and Lemma 2.2, R_M is a discrete valuation ring and therefore (vi) holds.

(vi)⇒(i). Suppose (vi) holds. Observe that if P is a non minimal prime ideal, then by [6, Theorem 4.19], P^n is P-primary for every positive integer n. Also if P is non minimal and minimal over a finitely generated ideal, then by [14, Lemma 7], P is a C-prime ideal. Therefore by Lemma 2.3, non minimal prime ideals are non idempotent and hence non minimal prime ideals are branched prime ideals. Again by [14, Lemma 8], non minimal prime ideals of R are C-prime ideals. Now by Theorem 2.9, R is a Dedekind ring and the proof is complete. □

Theorem 2.11. *Suppose R is a quasi-regular ring which satisfies the condition (*). Then the following statements on R are equivalent:*

(i) *R is a Dedekind ring.*
(ii) *Every maximal ideal is locally principal.*
(iii) *Every non minimal maximal ideal is a finitely generated ℓ-prime ideal.*
(iv) *Every primary ideal is a power of its radical.*
(v) *Every idempotent maximal ideal of R is unbranched and any two incomparable primary ideals are comaximal.*
(vi) *Every idempotent maximal ideal of R is unbranched, every non minimal branched prime ideal is a C-prime ideal and every maximal ideal is an ℓ-prime ideal.*

Proof. (i)⇒(ii). Suppose (i) holds. Let M be a maximal ideal of R. If M is minimal, then R_M is a field. If M is non minimal, then by (i), M is a multiplication ideal and hence M is locally principal. So (ii) holds.

(ii)⇒(iii). Suppose (ii) holds. Let M be a non minimal maximal ideal. Then M is regular, so by hypothesis and [2, see the remark after Theorem 13], M is an invertible ideal. Now the result follows from [21, Lemma 21].

(iii)⇒(iv). Suppose (iii) holds. Let Q be P-primary. If P is a minimal prime ideal, then $Q = P$ as R is a reduced ring. Suppose P is a non minimal prime ideal of R. Suppose $P \subseteq M$ for some maximal ideal M of R. Then by [14, Lemma 7], M is a C-prime ideal. Again by Lemma 2.4, R_M is a discrete valuation ring. Consequently, $P = M$ and hence Q is a power of P. Therefore (iv) holds.

(iv)⇒(v) follows from Theorem 2.10(vi) and (v)⇒(vi) follows from [14, Lemma 8].

(vi)⇒(i). Suppose (vi) holds. Let M be a maximal ideal of R. If $M = M^2$, then by (vi), M is unbranched, so by Lemma 2.5, M is a minimal prime ideal and hence R_M is a field. Suppose $M \neq M^2$. Since in reduced rings minimal prime ideals are unbranched, it follows that M is a non minimal prime ideal as M is a branched prime ideal. Therefore by Lemma 2.4, R_M is a discrete valuation ring. Consequently, every primary ideal is a power of its radical. Again by Theorem 2.10, R is a Dedekind ring and the proof is complete. □

Next we establish some equivalent conditions for a quasi-regular ring in which every regular principal ideal is a finite product of primary ideals to be a Dedekind ring (see Theorem 2.12).

Theorem 2.12. *Suppose R is a quasi-regular ring in which every regular principal ideal is a finite product of primary ideals. Then the following conditions on R are equivalent:*

(i) *R is a Dedekind ring.*
(ii) *Every maximal ideal is locally principal.*
(iii) *Every non minimal maximal ideal is a finitely generated ℓ-prime.*
(iv) *Every primary ideal is a power of its radical.*
(v) *Every idempotent maximal ideal of R is unbranched and any two incomparable primary ideals are comaximal.*

Proof. (i)⇒(ii) is well known.

(ii)⇒(iii) follows from Lemma 2.6 and [21, Lemma 21].

(iii)⇒(iv). Suppose (iii) holds. Let M be a maximal ideal of R. Suppose M is non minimal. Then M is non idempotent. Also by Lemma 7 of [14], M is a C-prime ideal and hence by Lemma 2.7, R_M is a discrete valuation ring. If M is minimal, then R_M is a field. Consequently, R is an almost multiplication ring and hence (iv) holds [21, Theorem 4].

(iv)⇒(v). Clearly, every idempotent maximal ideal is unbranched. Let M be a maximal ideal. If M is minimal, then R_M is a field. Suppose M is non minimal. By [5, Theorem 3], every non minimal branched prime ideal is a C-prime ideal. So by Lemma 2.8, M is branched and also by [5, Theorem 3], M is a non idempotent C-prime ideal. Therefore by Lemma 2.7, R_M is a discrete valuation ring and hence (v) holds.

(v)⇒(i). Suppose (v) holds. Let M be a maximal ideal of R. If $M = M^2$, then by [14, Lemma 8] and Lemma 2.8, R_M is a field. Suppose $M \neq M^2$. Since in reduced rings, minimal prime ideals are unbranched prime ideals, it follows that M is non minimal. So by Lemma 2.7 and [14, Lemma 8], R_M is a discrete valuation ring. Therefore R is an almost multiplication ring and hence by [21, Theorem 3 and Theorem 4], every ideal is equal to its kernal. Therefore R satisfies the condition (*). Now the result follows from Theorem 2.11 and the proof is complete. □

We now characterize Dedekind rings in terms of quasi-regular weak π-rings (see Theorem 2.13).

Theorem 2.13. *The following statements on R are equivalent:*

(i) *R is a Dedekind ring.*

(ii) *R is a quasi-regular weak π-ring in which primary ideals are powers of its radicals.*

(iii) *R is a quasi-regular weak π-ring in which any two incomparable primary ideals are comaximal.*

(iv) *R is a quasi-regular weak π-ring in which every maximal ideal is an ℓ-prime ideal and the ascending chain condition (a. c. c) for prime ideals is valid.*

(v) *R is a quasi-regular weak π-ring in which non minimal prime ideals are C-prime ideals.*

Proof. (i)⇒(ii). Suppose (i) holds. By [16, Theorem 3.8 and Theorem 3.13], R is a quasi-regular weak π-ring. So by Theorem 2.12, every primary ideal is a power of its radical.

(ii)⇒(iii) follows from Theorem 2.12.

(iii)⇒(iv). Suppose (iii) holds. By hypothesis and [14, Lemma 8], every non minimal branched prime ideal is a C-prime ideal, so by Lemma 2.8, non minimal prime ideals are C-prime ideals. Therefore by Lemma 2.7, every non minimal maximal ideal is a rank one prime ideal and hence (iv) holds.

(iv)⇒(v). Suppose (iv) holds. Let M be a non minimal maximal ℓ-prime ideal. By the a. c. c for prime ideals and by immitating the proof of [14, Lemma 7], it can be easily shown that M is a C-prime ideal. By Lemma 2.7, M is a rank one prime ideal. Consequently, non minimal prime ideals are C-prime ideals. Therefore (v) holds.

(v)⇒(i). Suppose (v) holds. Observe that by hypothesis and Lemma 2.7, non minimal maximal ideals are rank one prime ideals. Again since R is a quasi-regular weak π-ring and any factor of an invertible ideal is invertible, it follows that, non minimal maximal ideals are rank one invertible prime ideals. Let I be an ideal not contained in any minimal prime ideal. As R is quasi-regular, it follows that I is a regular ideal. Note that every prime ideal minimal over I is non minimal. Therefore every prime ideal minimal over I is a rank one invertible maximal ideal. By [15, Lemma 5], I has only finitely many minimal primes. Now we show that I is a finite product of invertible maximal ideals. Let $M_1, M_2, \ldots, M_n$ be the distinct prime ideals minimal over I. Note that $M_1, M_2, \ldots, M_n$ are rank one invertible maximal ideals. So by [21, Lemma 21], there exist positive integers k_i for $i = 1, 2, \ldots, n$ such that $I \subseteq {M_i}^{k_i}$ and $I \nsubseteq {M_i}^{k_i+1}$. So $I \subseteq \bigcap_{i=1}^{n} {M_i}^{k_i} = {M_1}^{k_1}{M_2}^{k_2}\cdots{M_n}^{k_n}$ as powers of M_i's are pairwise comaximal ideals. Also by [15, Lemma 5], I contains a finite product of primes which are minimal over I. Suppose $J = {M_1}^{\alpha_1}{M_2}^{\alpha_2}\cdots{M_s}^{\alpha_s} \subseteq I$. As powers of M_i's are invertible, it can be easily shown that $s = n$ and $\alpha_i = k_i$ for $i = 1, 2, \ldots, n$. Therefore $I = {M_1}^{k_1}{M_2}^{k_2}\cdots{M_n}^{k_n}$. Again by [16, Theorem 3.13(v)], R is a Dedekind ring. This completes the proof of the theorem. □

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