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Existence and Uniqueness for a Ginzburg-Landau ODE

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Abstract We study the existence of solutions for a Ginzburg-Landau ODE in the half-axis. We obtain that the solution is unique in the class of non-negative finite energy solutions. As a by-product, we determine the minimizers of the corresponding Ginzburg-Landau functional.

1 Introduction

In this note, we study the existence and uniqueness of solutions for the following ODE

$$\begin{cases} -u'' = (a-|u|^2)u \quad \text{in } \mathbb{R}_+\,, \\ u(0)=0\,, \end{cases} \tag{1.1}$$

in the following class

$$\mathcal{C} = \{u\in\mathcal{H}\ :\ u\geq 0 \text{ in } \mathbb{R}_+\}\,, \tag{1.2}$$

where

$$\mathcal{H} = \{u\in H^1_{\text{loc}}(\mathbb{R}_+)\ :\ u'\in L^2(\mathbb{R}_+) \quad \text{and} \quad a-u^2\in L^2(\mathbb{R}_+)\}\,. \tag{1.3}$$

Here $a:\mathbb{R}_+\to\mathbb{R}$ is a given function. Formally, the first equation in (1.1) can be regarded as the Euler-Lagrange equation of the following functional

$$\mathcal{F}(u) = \int_0^\infty \left(|u'|^2 + \frac{1}{2}(a-|u|^2)^2\right)dx\,. \tag{1.4}$$

When $a=1$, the functional in (1.4) is the celebrated Ginzburg-Landau energy without magnetic field and the first equation in (1.1) is the corresponding Ginzburg-Landau equation.

Note that the space $\mathcal{H}$ in (1.3) is the natural energy space for the functional in (1.1). That is, we look for solutions of (1.1) that have finite energy (i.e. $\mathcal{F}(u)$ is finite).

We prove the following theorem:

Theorem 1.1. Suppose that the function a satisfies

$$\begin{cases} a\in C^2(\overline{\mathbb{R}_+})\,, \\ \exists\, c>0,\ c\leq a(x)\leq 1 \text{ in } \mathbb{R}_+\,, \\ \lim\limits_{x\to\infty} a(x) = 1\,, \\ b(x)=\sqrt{a(x)} \text{ satisfies } b''(x)\leq 0 \text{ in } \mathbb{R}_+ \quad \text{and } 1-b\in L^2(\mathbb{R}_+)\,. \end{cases} \tag{1.5}$$

There exists a unique function $u\in\mathcal{C}$ solving the equation in (1.1).

As a by-product, we get:

Corollary 1.2. Suppose that the function a satisfies the assumptions in Theorem 1.1. Let

$$\mathfrak{m} = \inf\{\mathcal{F}(u)\ :\ u\in\mathcal{H} \quad \text{and } u(0)=0\}\,. \tag{1.6}$$

Every minimizer v of the problem in (1.6) is given as follows,

$$v = e^{i\theta} u\,,$$

for some constant $\theta \in \mathbb{R}$. Here $u \in \mathcal{C}$ is the function solving (1.1).

Theorem 1.1 is well known for $a = 1$ (cf. [5, 1]). Theorems analogous to Theorem 1.1 are proved for equations of the form (cf. [2])

$$-u''(x) + \frac{p}{x}u'(x) - \frac{q}{x^2}u(x) = F(u(x)) \text{ in } \mathbb{R}_+, \quad u(0) = 0\,, \quad u(\infty) = 1\,.$$

Compared to (1.1), we take $p = q = 0$ and in the non-linear term we allow for F to be a function of $u(x)$ and x.

The rest of this note is devoted to the proofs of Theorem 1.1 and Corollary 1.2. The structure of the proof is along the ones in [5, 3] but the presence of the Dirichlet boundary condition requires some new ingredients. Let us point out that the result in Theorem 1.1 is still valid for the Robin condition, i.e. when u satisfies $u'(0) = \gamma u(0)$ for $\gamma \geq 0$. For this condition, the proof is exactly as the one in [5].

2 Existence

Note that (1.1) is the Euler-Lagrange equation for the minimization problem in (1.6). The existence of a solution $u \in \mathcal{C}$ of (1.1) is then a consequence of:

Theorem 2.1. Let $\mathfrak{m}$ be as in (1.6). Suppose that there exists $c > 0$ such that the function a satisfies $a \geq c$ in $\mathbb{R}_+$.

There exists a function $v \in \mathcal{H}$ such that $v(0) = 0$ and

$$\mathcal{F}(v) = \mathfrak{m}\,.$$

Furthermore, if $v \in \mathcal{H}$ is a minimizer of the problem in (1.6), then the function $|v|$ belongs to the space $\mathcal{C}$ and is a minimizer of the problem in (1.6) too, i.e. $\mathcal{F}(|v|) = \mathfrak{m}$.

Proof. Let (v_n) be a minimizing sequence of $\mathfrak{m}$, i.e.

$$\forall\, n \in \mathbb{N}\,, \quad v_n \in \mathcal{H}\,, \quad v_n(0) = 0\,,$$

and

$$\lim_{n\to\infty} \mathcal{F}(v_n) = \mathfrak{m}\,. \tag{2.1}$$

Let $q \in \mathbb{N}$ and $K = (0, q)$. We will prove that (v_n) is bounded in the Sobolev space $H^1(K)$. The convergence in (2.1) yields the existence of a constant $M > 0$ such that,

$$\forall\, n \in \mathbb{N}\,, \quad \mathcal{F}(v_n) \leq M\,,$$

and in turn this implies boundedness of

$$\|v_n'\|_{L^2(K)} \quad \text{and} \quad \left\|a - |v_n|^2\right\|_{L^2(K)}\,.$$

Observing the simple identity ($b = \sqrt{a}$),

$$\int_K (a - |v_n|^2)^2\,dx = \int_K (b + |v_n|)^2 (b - |v_n|)^2\,dx\,,$$

and the trivial inequality $b + |v_n| > 0$, we deduce that $\big\|b - |v_n|\big\|_{L^2(K)}$ is bounded. Since the interval K is bounded, we deduce further that $\|v_n\|_{L^2(K)}$ is bounded.

Now, we have proved that (v_n) is a bounded sequence in $H^1(K)$. This is true for every interval of the form $K = (0, q)$ and $q > 0$. We can apply the Banach-Alaoglu theorem and a diagonal argument to extract a subsequence of (v_n) and a function $v \in H^1_{\rm loc}(\mathbb{R}_+)$ such that, along this subsequence, for all $q > 0$,

$$v_n \rightharpoonup v \text{ in } H^1(0,q)\,, \quad v_n \to v \text{ strongly in } L^2(0,q) \text{ and } L^4(0,q)\,.$$

The following two statements follow immediately from the aforementioned *weak* and *strong* convergences respectively:

$$\liminf_{n\to\infty} \|v_n\|_{H^1(0,q)} \geq \|v\|_{H^1(0,q)} \quad \text{and} \quad \lim_{n\to\infty} \|v_n\|_{L^p(0,q)} = \|v\|_{L^p(0,q)} \quad (p \in \{2,4\})\,.$$

This yields that,

$$\liminf_{n\to\infty} \int_0^q |v_n'|^2\,dx \geq \int_0^q |v'|^2\,dx \quad \text{and} \quad \lim_{n\to\infty} \int_0^q (a-|v_n|^2)^2\,dx = \int_0^q (a-|v|^2)^2\,dx\,.$$

Now we apply the operation $\liminf\limits_{n\to\infty}$ on both sides of the following trivial inequality:

$$\int_0^\infty \left(|v_n'|^2 + \frac{1}{2}(a-|v_n|^2)^2\right)\,dx \geq \int_0^q \left(|v_n'|^2 + \frac{1}{2}(a-|v_n|^2)^2\right)\,dx$$

and get,

$$\begin{aligned}\liminf_{n\to\infty} \int_0^\infty \left(|v_n'|^2 + \frac{1}{2}(a-|v_n|^2)^2\right)\,dx &\geq \liminf_{n\to\infty} \int_0^q \left(|v_n'|^2 + \frac{1}{2}(a-|v_n|^2)^2\right)\,dx \\ &\geq \int_0^q \left(|v'|^2 + \frac{1}{2}(a-|v|^2)^2\right)\,dx\,.\end{aligned}$$

The following inequality holds for all $q > 0$:

$$\liminf_{n\to\infty} \int_0^\infty \left(|v_n'|^2 + \frac{1}{2}(a-|v_n|^2)^2\right)\,dx \geq \int_0^q \left(|v'|^2 + \frac{1}{2}(a-|v|^2)^2\right)\,dx\,.$$

We can apply the operation $\lim\limits_{q\to\infty}$ and use the monotone convergence theorem to write,

$$\liminf_{n\to\infty} \int_0^\infty \left(|v_n'|^2 + \frac{1}{2}(a-|v_n|^2)^2\right)\,dx \geq \int_0^\infty \left(|v'|^2 + \frac{1}{2}(a-|v|^2)^2\right)\,dx\,.$$

Recall that, our choice of the sequence (v_n) ensures that the limit on the left hand side above is equal to $\mathfrak{m}$. Thus, we have proved that $\mathcal{F}(v) \leq \mathfrak{m}$ and $v \in \mathcal{H}$. Furthermore, since $v_n(0) = 0$ for all n, then by taking $n \to \infty$ we get $v(0) = 0$. Now, the definition of the minimum $\mathfrak{m}$ yields the additional condition $\mathfrak{m} \leq \mathcal{F}(v)$ and in turn implies that $\mathcal{F}(v) = \mathfrak{m}$.

This proves that a minimizer v of the functional $\mathcal{F}$ exists. Now we show that $u = |v|$ is a minimizer too. Using the celebrated pointwise inequality $|u'| \leq |v'|$ a.e. (see [4, thrm 6.17]), we may write, $\mathcal{F}(u) \leq \mathcal{F}(v)$. Again, since $\mathcal{F}(v) = \mathfrak{m}$, this guarantees that $\mathfrak{m} \leq \mathcal{F}(u) \leq \mathcal{F}(v) = \mathfrak{m}$. Thus u is a minimizer of $\mathcal{F}$ too. □

3 *A priori* estimates

In this section, we collect useful properties satisfied by the functions in $\mathcal{C}$ that solve (1.1). Hereafter, we assume that the function a satisfies the assumptions in Theorem 1.1.

Let us start by noting the standard result:

Lemma 3.1. Let $\mathcal{H}$ be the space introduced in (1.3). If $u \in \mathcal{H}$ is a (weak) solution of (1.1), then $u \in C^3([0,\infty))$.

Next we note that every function in the space $\mathcal{H}$ in (1.3) is bounded and has a finite limit at infinity.

Lemma 3.2. Let $\mathcal{H}$ be the space introduced in (1.3). If $u \in \mathcal{H}$, then

$$u \in L^\infty(\mathbb{R}_+) \quad \text{and} \quad \lim_{x\to\infty} |u(x)| = 1\,.$$

Proof. Note that, if $u \in \mathcal{H}$, then $\mathcal{F}(u) < \infty$, $u' \in L^2(\mathbb{R}_+)$ and $b - |u| \in L^2(\mathbb{R}_+)$. Here $b = \sqrt{a}$. By the assumption in Theorem 1.1, we know that $1 - b \in L^2(\mathbb{R}_+)$. Consequently, $1 - |u| \in L^2(\mathbb{R}_+)$. Thus $1 - |u| \in H^1(\mathbb{R}_+)$. The conclusion in Lemma 3.2 now becomes a consequence of the Sobolev embedding theorem. □

The next two lemmas give us *a priori* estimates on every solution of (1.1) in the space $\mathcal{C}$.

Lemma 3.3. Let $\mathcal{C}$ be the space in (1.2). If $v \in \mathcal{C}$ is a solution of (1.1), then $v' \in L^\infty(\mathbb{R}_+)$ and $\lim_{x\to\infty} v'(x) = 0$.

Proof. Note that $v \in \mathcal{H}$, hence $v' \in L^2(\mathbb{R}_+)$, $a - |v|^2 \in L^2(\mathbb{R}_+)$ and, by Lemma 3.2, $v \in L^\infty(\mathbb{R}_+)$. Now, v satisfies the following ODE:

$$-v'' = (a - |v|^2)v \quad \text{in } \mathbb{R}_+ .$$

Squaring both sides then integrating on $\mathbb{R}_+$, we get,

$$\int_{\mathbb{R}_+} |v''|^2\,dx \le \int_{\mathbb{R}_+} (a - |v|^2)^2 v^2\,dx \le \|v\|_\infty^2 \int_{\mathbb{R}_+} (a - |v|^2)^2\,dx < \infty .$$

This implies that $v'' \in L^2(\mathbb{R}_+)$ and consequently, $v' \in H^1(\mathbb{R}_+)$. The Sobolev embedding theorem finishes the proof of Lemma 3.3. □

Lemma 3.4. Let $\mathcal{C}$ be the space in (1.2). If $v \in \mathcal{C}$ is solution of (1.1), then $0 < v < \sqrt{a}$ in $\mathbb{R}_+$.

Proof. We know that $v \ge 0$ by definition of the space $\mathcal{C}$. The proof of Lemma 3.4 is decomposed into three steps.

Step 1. Let $h(x) = b(x) - v(x)$ and $b(x) = \sqrt{a(x)}$. Suppose that

$$m = \inf\{h(x)\ :\ x \in \mathbb{R}_+\} < 0 .$$

We will derive a contradiction to obtain that $m \ge 0$ and deduce that $v \le b$ in $\mathbb{R}_+$.

There exists a sequence (x_n) in $(0,\infty)$ such that $h(x_n) \to m$. We can extract a subsequence of (x_n), denoted by (x_n), and $s \in [0,\infty]$ such that, along this subsequence, $x_n \to s$.

Two cases may occur:

Case 1: $0 \le s < \infty$. By smoothness of v, we have $b(s) - v(s) = h(s) = m < 0$. Since $b(s) > 0$ and $v(0) = 0$ by (1.1), this yields that $s \ne 0$. Similarly, if $v(s) = 0$, then $0 < b(s) = b(s) - v(s) = h(s) = m < 0$. This yields that $v(s) \ne 0$.

Now, we know that $0 < s < \infty$, $v(s) > 0$ and the function h has a local minimum at s. Consequently $h''(s) \ge 0$. But the equation in (1.1) and the assumption $b'' \le 0$ together yield

$$h''(s) = b''(s) + (a(s) - v^2(s))v(s) = b''(s) + h(s)(b(s) + v(s))v(s) < 0.$$

This is a contradiction.

Case 2: $s = \infty$. In this case, we use Lemma 3.2 and the assumption $\lim_{x\to\infty} a(x) = 1$ to obtain the following contradiction

$$0 = \lim_{n\to\infty} h(x_n) = m < 0 .$$

Step 2.

Now, we prove that $v < b$ in $\mathbb{R}_+$. Suppose that there exists $x_1 \in (0,\infty)$ such that $v(x_1) = b(x_1)$. By Step 1, x_1 becomes a minimum of the function $h(x) = b(x) - v(x)$. Using the assumption that $b''(x) \le 0$ and the equation for v, we get,

$$-h'' + ch = -b'' \ge 0 \quad \text{in } \mathbb{R}_+ ,$$

where

$$c = (b + v)v \ge 0 ,$$

and

$$h(x_1) = \min_{x\in(0,\infty)} h(x) = 0 .$$

The strong maximum principle yields that h is a constant function in $\overline{\mathbb{R}_+}$, i.e. $h \equiv h(x_1) = 0$. Consequently, $v \equiv b$. In particular $b(0) = v(0) = 0$. But, by the assumption on the function a in Theorem 1.1, $b(0) > 0$. This is a contradiction.

Step 3. Now, we prove that $v > 0$ in $\mathbb{R}_+$. Suppose that there exists $x_0 \in \mathbb{R}_+$ such that $v(x_0) = 0$. We will derive a contradiction to deduce that this is impossible. Since $v \in \mathcal{C}$, then $v \geq 0$. Thus,

$$v(x_0) = \min_{x \in (0,\infty)} v(x) = 0\,.$$

Furthermore, by Step 1, $0 \leq v \leq b = \sqrt{a}$ in $\mathbb{R}_+$, and the equation in (1.1) yields that

$$-v'' = (a - v^2)v \geq 0.$$

By the maximum principle, v becomes a constant function. Thanks to the boundary condition in (1.1), we deduce that $v = 0$. But this function does not belong to the space $\mathcal{C}$, hence we get a contradiction. □

In the next lemma, we determine the sign of the derivative of the non-negative solutions of (1.1).

Lemma 3.5. Let $\mathcal{C}$ be the space introduced in (1.2). Suppose that v is a solution of (1.1). If $v \in \mathcal{C}$, then for all $x \geq 0$, $v'(x) > 0$.

Proof. Recall that the assumption $v \in \mathcal{C}$ yields that v is real-valued, $v \geq 0$ and $v(0) = 0$, and by Lemma 3.2, $\lim\limits_{x\to\infty} v(x) = 1$. We claim that $v'(x) > 0$, for all $x \geq 0$. Suppose that the claim is false. There exists $x_0 \geq 0$ such that $v'(x_0) \leq 0$. In light of (1.1), by integrating the first equation $-v'' = (a - v^2)v$ between x_0 and x, we get

$$v'(x) = v'(x_0) + \int_{x_0}^{x} v''(t)\,dt = v'(x_0) + \int_{x_0}^{x} (|v(t)|^2 - a(t))v(t)dt \leq \int_{x_0}^{x} (|v(t)|^2 - a(t))v(t)dt\,.$$

Now, we apply Lemma 3.4 to deduce that for all $x > x_0$, $v'(x) < 0$. Consequently, v is a *decreasing* function in $[x_0, \infty)$ and we should have $1 = \lim\limits_{x\to\infty} v(x) \leq v(x_0)$, contradiction to the fact that $v(x_0) < \sqrt{a} \leq 1$. □

Remark 3.6. (The case $a = 1$)[1]

In the case $a = 1$, we can prove the uniqueness of the solution by using the aforementioned lemmas and separation of variables. Let $u \in \mathcal{C}$ be a solution of (1.1). We know that u is smooth, $u(0) = 0$, $u(\infty) = 1$, $u'(\infty) = 0$, $0 < u < 1$ and $u' > 0$ in $\mathbb{R}_+$.

Multiplying the first equation in (1.1) by u' yields

$$-u'u'' = u'u - u'u^3 \quad i.e. \quad \frac{1}{2}\{(u')^2\}' = \frac{1}{4}(u^4)' - \frac{1}{2}(u^2)'\,.$$

Consequently,

$$|u'| = \sqrt{\frac{1}{2}u^4 - u^2 + C} \quad \text{in } \mathbb{R}_+\,,$$

for some constant $C \in \mathbb{R}$. The conditions $u'(\infty) = 0$, $u(\infty) = 1$ and $u' > 0$ yield $C = \frac{1}{2}$ and

$$u' = \frac{1}{\sqrt{2}}(1 - u^2)\,.$$

By separation of variables and the boundary condition $u(0) = 0$, we get

$$u(x) = \frac{e^{\sqrt{2}\,x} - 1}{e^{\sqrt{2}\,x} + 1}\,.$$

Let us note that such explicit computations can not be carried out when the function a is not constant.

[1] This trick is indicated by A. Mourad.

4 Uniqueness

Here we prove that, if $v_1 \in \mathcal{C}$ and $v_2 \in \mathcal{C}$ solve (1.1), then $v_1 = v_2$. This will be proved in several lemmas.

Lemma 4.1. Let $\mathcal{C}$ be the space in (1.2). Suppose that $v_1 \in \mathcal{C}$ and $v_2 \in \mathcal{C}$ satisfy (1.1). For all $\delta > 0$, it holds the following.

(i) $\{\lambda \in (0,1] \ : \ \lambda v_1(x) < v_2(x) \text{ in } [\delta,\infty)\} \neq \emptyset$.

(ii) Let $\lambda^*(\delta) = \sup\{\lambda \in (0,1] \ : \ \lambda v_1(x) < v_2(x) \text{ in } [\delta,\infty)\}$. If $\lambda^*(\delta) < 1$, then

a. $\inf\{v_2(x) - \lambda^*(\delta)v_1(x) \ : \ x \in [\delta,\infty)\} = 0$.

b. $\lambda^*(\delta) = \dfrac{v_2(\delta)}{v_1(\delta)}$.

Proof. Let $\delta > 0$. Lemma 3.5 yields, for all $x \geq \delta$, $v_2(\frac{\delta}{2}) < v_2(x)$. On the other hand, Lemma 3.4 yields $0 < v_2(\frac{\delta}{2}) < 1$ and for all $x \geq \delta$, $0 < v_1(x) < 1$. Thus, if we define $\lambda = v_2(\frac{\delta}{2})$, then λ satisfies $\lambda \in (0,1]$ and $\lambda v_1 < v_2$ in $[\delta,\infty)$. This finishes the proof of the first item in Lemma 4.1.

Now, let us define the function w_δ on the interval $[\delta,\infty)$ in the following way:

$$w_\delta(x) = v_2(x) - \lambda^*(\delta)v_1(x)$$

Under the assumption $\lambda^*(\delta) < 1$, one can prove that $m_\delta := \inf_{x\in[\delta,\infty)} w_\delta(x) = 0$ as follows. Suppose that $m_\delta > 0$. For all $x \in [\delta,\infty)$, $v_2(x) - \lambda^*(\delta)v_1(x) \geq m_\delta$ and $0 < v_1 < 1$ (cf. Lemma 3.4). Consequently,

$$v_2(x) \geq \lambda^*(\delta)v_1(x) + m_\delta = (\lambda^*(\delta) + m_\delta)v_1(x) + (1 - v_1(x))m_\delta \geq (\lambda^*(\delta) + m_\delta)v_1(x)\,.$$

Select $\epsilon > 0$ such that $\lambda^*(\delta) + \epsilon < 1$. Define $\lambda = \min(\lambda^*(\delta) + \epsilon, \lambda^*(\delta) + m_\delta)$. Clearly, $0 < \lambda < 1$ and $\lambda^*(\delta) < \lambda$. We get $v_2 \geq \lambda v_1$ on $[\delta,\infty)$ and $0 < \lambda^*(\delta) < \lambda < 1$. This contradicts the definition of $\lambda^*(\delta)$. Therefore, the item (2)-(a) in Lemma 4.1 is true.

Now we prove the item (2)-(b) in Lemma 4.1. Suppose that $\delta > 0$ and $0 < \lambda^*(\delta) < 1$. Using the ODEs satisfied by v_1 and v_2, we have for all $x \in [\delta,\infty)$,

$$-v_2'' = (a - |v|_2^2)v_2\,,$$

and

$$\begin{aligned} -\lambda^*(\delta)v_1'' &= (a - v_1^2)\lambda^*(\delta)v_1 \\ &= (a - |\lambda^*(\delta)v_1|^2)\lambda^*(\delta)v_1 + (|\lambda^*(\delta)|^2 - 1)\lambda^*(\delta)|v_1|^2 v_1 \\ &\leq (a - |\lambda^*(\delta)v_1|^2)\lambda^*(\delta)v_1\,. \end{aligned}$$

Therefore the function w_δ satisfies

$$-w_\delta'' + cw_\delta \geq a\,w_\delta \quad \text{in } [\delta,\infty)\,,$$

where

$$c = v_2^2 + \lambda^*(\delta)v_2v_1 + \lambda^{*2}(\delta)v_1^2 \geq 0\,.$$

Now that we have $\inf w_\delta = 0$ on $[\delta,\infty)$, there exists a sequence $(x_n) \subset [\delta,\infty)$ and a number $s \in [\delta,\infty]$ such that (x_n) converges to s and $w_\delta(x_n)$ converges to zero. Three cases may occur:

Case 1. $s = \infty$.

Here we have a contradiction that follows simply by using Lemma 3.2 to write $w_\delta(x_n) = v_2(x_n) - \lambda^*(\delta)v_1(x_n) \to 1 - \lambda^*(\delta) > 0$. This is impossible since $w_\delta(x_n) \to 0$ by the assumption on the sequence (x_n) and the conclusion of the item (2)-(a).

Case 2. $s \in (\delta,\infty)$.

Here $w_\delta(s) = \inf_{[\delta,\infty)} w_\delta(x) \leq 0$. The strong maximum principle yields

$$w_\delta(x) = w_\delta(s) = 0 \quad \text{for all } x \in [\delta, \infty).$$

By the definition of w_δ, we get that $v_2 = \lambda^*(\delta) v_1$ in $[\delta, \infty)$. Using the ODEs satisfied by v_1 and v_2, and the fact that $v_1 > 0$, this contradicts the assumption that $\lambda^*(\delta) < 1$.

Case 3. $s = \delta$.

Here $\inf_{x\in[\delta,\infty)} w_\delta(x) = w_\delta(\delta) = 0$. This yields $\lambda^*(\delta) = \dfrac{v_2(\delta)}{v_1(\delta)}$.

□

The next lemma compares two solutions v_1 and v_2 away from 0.

Lemma 4.2. Let $\mathcal{C}$ be the space in (1.2), $v_1 \in \mathcal{C}$ and $v_2 \in \mathcal{C}$. If v_1 and v_2 satisfy (1.1), then

$$\exists\, \delta_1 > 0\,, \quad v_1(x) \leq v_2(x) \quad \text{in } [\delta_1, \infty)\,.$$

Proof. For all $\delta > 0$, recall that $\lambda^*(\delta) = \sup\{\lambda \in (0,1] \;:\; \lambda v_1(x) < v_2(x) \text{ in } [\delta,\infty)\}$. It suffices to prove that, $\exists\, \delta_1 > 0$, $\lambda^*(\delta_1) = 1$. We will prove this by contradiction. Note for all $\delta > 0$, $0 < \lambda^*(\delta) \leq 1$. Suppose that for all $\delta > 0$, $\lambda^*(\delta) < 1$. Lemma 4.1 yields,

$$\forall\, \delta > 0\,, \quad \lambda^*(\delta) = \frac{v_2(\delta)}{v_1(\delta)}\,. \tag{4.1}$$

Furthermore, since $v_1 > 0$ on $\mathbb{R}_+$, then the function $\lambda^*(\cdot)$ is a smooth function on $\mathbb{R}_+$. By definition of $\lambda^*(\cdot)$, we see that it is an increasing function.

We will prove that $\lambda^*(\cdot)$ is a constant function. We insert $v_2 = \lambda^* v_1$ into the equation $-v_2'' = (a - v_2^2)v_2$ then we multiply both sides of the resulting equation by v_1 to get

$$-\Big((\lambda^*)' v_1^2\Big)' = (1 - \lambda^{*2})\lambda^* v_1^4.$$

Let $x \in \mathbb{R}_+$. For all $\alpha \in (0, x)$,

$$\int_\alpha^x \Big(v_1^2(t)\lambda^{*'}(t)\Big)' \,dt = -\int_\alpha^x \Big(1 - \lambda^{*2}(t)\Big)\lambda^*(t) v_1^4(t)\,dt \leq 0\,.$$

Hence, integration of the left hand side above yields,

$$v_1^2(x)\lambda^{*'}(x) \leq v_1^2(\alpha)\lambda^{*'}(\alpha)\,. \tag{4.2}$$

In what follows, we will establish that

$$\lim_{\alpha\to 0_+} v_1(\alpha)\lambda^{*'}(\alpha) = 0\,.$$

By Lemma 3.5, we have $v_1'(0) > 0$ and $v_2'(0) > 0$. Using Taylor's formula and the initial conditions $v_1(0) = v_2(0) = 0$ we can easily prove that

$$\lim_{\alpha\to 0_+} \lambda^*(\alpha) = \lim_{\alpha\to 0_+} \frac{v_2(\alpha)}{v_1(\alpha)} = \frac{v_2'(0)}{v_1'(0)}\,. \tag{4.3}$$

On the other hand, differentiation of the relation $v_2 = \lambda^* v_1$ yields

$$v_2'(\alpha) = v_1(\alpha)\lambda^{*'}(\alpha) + v_1'(\alpha)\lambda^*(\alpha). \tag{4.4}$$

By letting α tend to zero in (4.4) and by using (4.3), we establish the result concerning the limit of $v_1(\alpha)\lambda^{*'}(\alpha)$.

We come back to (4.2) and we take $\alpha \to 0_+$ to obtain

$$v_1^2(x)\lambda^{*'}(x) \leq 0, \quad \text{for all } x \in \mathbb{R}_+.$$

We get further,

$$\lambda^{*'}(x) \le 0 \quad \text{for all } x \in \mathbb{R}_+ .$$

But, $\lambda^*(\cdot)$ is an increasing (and smooth) function on $\mathbb{R}_+$, hence

$$\lambda^{*'}(x) \ge 0, \quad \text{for all } x \in \mathbb{R}_+ .$$

This proves that the function $\lambda^*(\cdot)$ is constant on $\mathbb{R}_+$.

Now we observe that $\lambda^* = 1$. This is true because, the fact that λ^* is constant, the definition of λ^* and the conclusion in Lemma 3.2 altogether yield,

$$\forall\, x > 0\,, \quad \lambda^*(x) = \frac{v_2(x)}{v_1(x)} = \lim_{x\to\infty} \frac{v_2(x)}{v_1(x)} = 1\,.$$

□

The next lemma is the last piece we need in the proof of the uniqueness of the non-negative solution of (1.1).

Lemma 4.3. Suppose that $-\infty < c < d < \infty$, $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$. Let v_1 and v_2 be solutions to the following problem:

$$\begin{cases} -v'' = (a - |v|^2)v, & \text{in } (c,d)\,, \\ v(d) = c_1, & \\ v'(d) = c_2, & \\ 0 \le v(x) \le 1 & \text{in } [c,d]\,. \end{cases} \tag{4.5}$$

If $|c-d| \le \frac{\sqrt{8}}{8}$, then $v_1 = v_2$ in $[c,d]$.

Proof. Let v be a solution of (4.5) and $t \in [c,d]$. We have

$$v'(t) = c_2 + \int_t^d \big(a(s) - |v|^2(s)\big)v(s)\,ds\,.$$

Integrating the equation above from $t = 0$ to $t = x \in [c,d]$, we get,

$$v(x) = c_1 + c_2 x - c_2 d - \int_x^d \int_t^d \big(a(s) - |v|^2(s)\big)v(s)\,ds \qquad \text{for all } x \in [c,d]\,. \tag{4.6}$$

For every $g \in \mathcal{C}([c,d])$, let us introduce the following norm:

$$\|g\|_1 = \int_c^d |g(x)|dx.$$

We will prove that

$$\|v_1 - v_2\|_1 \le \frac{1}{2}\|v_1 - v_2\|_1\,.$$

This yields that $v_1 = v_2$ on $[c,d]$.

Using (4.6) for $v = v_1$ and $v = v_2$ respectively, we may write, for all $x \in [c,d]$,

$$v_1(x) - v_2(x) = \int_x^d \int_t^d \big(v_2(s) - v_1(s)\big)\big(a(s) - v_1^2(s) - v_2^2(s) - v_1(s)v_2(s)\big)\,ds\,.$$

This yields the following inequality (since $0 \le a, v_1, v_2 \le 1$):

$$\begin{aligned} |v_1(x) - v_2(x)| &\le 4\int_x^d \int_t^d |v_1(s) - v_2(s)|\,ds \\ &\le 4\int_x^d \int_c^d |v_1(s) - v_2(s)|\,ds \\ &\le 4(d-c)\|v_1 - v_2\|_1. \end{aligned}$$

If $|c-d| \leq \frac{\sqrt{8}}{8}$, then we get further,

$$\int_c^d |v_1(x) - v_2(x)|dx \leq \frac{1}{2}\|v_1 - v_2\|_1 .$$

□

Proof of Theorem 1.1: Uniqueness.

Let $v_1 \in \mathcal{C}$ and $v_2 \in \mathcal{C}$ satisfy (1.1). We will prove that $v_1 = v_2$ on $\mathbb{R}_+$. By Lemma 4.2 there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that,

$$v_1 \leq v_2 \quad \text{in } [\delta_1, \infty) \quad \text{and} \quad v_2 \leq v_1 \quad \text{in } [\delta_2, \infty) .$$

Let $\overline{\delta} = \max(\delta_1, \delta_2)$. We have $v_1 = v_2$ on $[\overline{\delta}, \infty)$. We will prove that $v_1 = v_2$ on $[0, \overline{\delta}]$. Select $n \in \mathbb{N}$ such that $n \geq \sqrt{8}\,\overline{\delta}$. For all $k \in \{1, \cdots, n\}$, define,

$$c_k = (k-1)\frac{\overline{\delta}}{n}, \quad d_k = k\frac{\overline{\delta}}{n} .$$

That way, we split the interval $[0, \overline{\delta}]$ into n sub-intervals,

$$[0, \overline{\delta}] = \bigcup_{k=1}^{n} [c_k, d_k] ,$$

such that, for all k, $|c_k - d_k| = \frac{\overline{\delta}}{n} \leq \frac{\sqrt{8}}{8}$ and we can apply Lemma 4.3 in $[c_k, d_k]$.

Clearly, v_1 and v_2 satisfy (4.5) in $[c_n, d_n]$. Thus, we deduce that $v_1 = v_2$ on $[c_n, d_n]$. Now, v_1 and v_2 satisfy (4.5) in $[c_{n-1}, d_{n-1}]$ and we deduce that $v_1 = v_2$ in $[c_{n-1}, d_{n-1}]$. Repeating this proof in $[c_k, d_k]$ for $k = n-2, \cdots, 1$, we get that $v_1 = v_2$ on every $[c_k, d_k]$. This proves that $v_1 = v_2$ on $[0, \overline{\delta}]$. □

5 Proof of Corollary 1.2

Let v be a minimizer of the problem in (1.6). By Theorem 2.1, $w = |v|$ is a minimizer of (1.6). Hence, $w \in \mathcal{C}$ and w is a solution of (1.1).

Now Theorem 1.1 yields that $w = u$ in $\mathbb{R}_+$, where $u \in \mathcal{C}$ is the unique solution of (1.1). Thus,

$$|v(x)| = u(x) \quad \text{for all } x \in \mathbb{R}_+ .$$

Consequently, there exists a function $\alpha : \mathbb{R}_+ \to \mathbb{C}$ such that,

$$v(x) = \alpha(x)u(x) \quad \text{and} \quad |\alpha(x)| = 1 \quad \text{for all } x \in \mathbb{R}_+ .$$

Since $u > 0$, the function α inherits the smoothness from u and v. Inserting $v = \alpha u$ into the equation $-v'' = (a - |v|^2)v$, multiplying both sides by u then using that $-u'' = (a - u^2)u$, we get

$$\frac{d}{dx}(\alpha' u^2) = 0 .$$

Since $u(0) = 0$ and $u > 0$, this gives us that $\alpha' = 0$ and the function α becomes a constant function. Since $|\alpha| = 1$, we get that $\alpha = e^{i\theta}$ for some constant $\theta \in \mathbb{R}$.

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