

ON EXISTENCE OF INTEGRAL AND ANTI-PERIODIC DIFFERENTIAL INCLUSION OF FRACTIONAL ORDER

$$\alpha \in (4, 5]$$

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Abstract. We are concerned in this article the existence of solutions of a class of fractional differential inclusions with anti-periodic and integral boundary conditions involving the Caputo fractional derivative with order $\alpha \in (4, 5]$. These results can be considered as a generalization of previously published articles in this topic with fractional orders less than or equal 4. However, the results based on fixed-point theorems for differential inclusions. Two examples are introduced to show the applicability of such theorems.

1 Introduction

Fractional differential models has recently a wide investigations due to its extensive development and applications in several disciplines such as physics, mechanics, chemistry, engineering, etc.(see [9],[10],[12],[14],[15],[20], [22], and references therein). The fact that using the fractional-order models instead of integer-order model is due to more realistic in description of many physical phenomenons. The investigation of existence problems of fractional differential equations in general is considered as a priority for going forward in such applications (see [13], [16]-[18],[35]). Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes ([20],[36]) and have recently received considerable attention. For examples and details of anti-periodic boundary conditions (see ([1]-[8],[11],[21],[34] and the references therein). Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors (see [23],[24] and the references therein). For some recent works on differential inclusions of fractional order, we refer the reader to the references ([1],[7],[8], [19]). In this paper, we discuss some existence results for anti-periodic boundary value problems of differential inclusions of fractional order $q \in (4, 5]$ using the nonlinear alternative of the Leray-Schauder and Covitz and Nadler fixed point theorems.

Precisely, we will devote to considering the existence of solution of the integral and anti-periodic boundary value problem

$$\begin{cases} {}^c D_{t_0}^q x(t) \in F(t, x(t)), t \in J = [t_0, T], T > t_0, q \in (4, 5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^T g_k(t, x(t)) dt, k = 0, 1, 2, 3, 4, \end{cases} \tag{1.1}$$

where $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of \mathbb{R} , $g_k : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function, $\theta_k, \beta_k \in \mathbb{R}$ with $\theta_k \neq 1$ for each $k = 0, 1, 2, 3, 4$, and ${}^c D_{t_0}^q$ denotes the Caputo fractional derivative of order q which is generally defined by

$${}^c D_{t_0}^q x(t) = \left(I_{t_0}^{n-q} x^{(n)} \right) (t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} x^{(n)}(s) ds, n-1 < q < n,$$

where $n = [q] + 1$, and $[q]$ denotes the integer part of the real number q .

This paper is organized as follows: In Section 2, we introduce some well-known results in multivalued analysis. The main results of existence theorems will be given in Section 3. Finally, we give some illustrative examples to explain the theorems.

2 Preliminaries

We recall in this section some facts from multivalued mapping analysis (see [29]-[31]) that needed for the results in the sequel.

Definition 2.1. For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned} P_{cl}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \\ P_b(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}, \\ P_{cp}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \text{ and} \\ P_{cp,c}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}. \end{aligned}$$

Definition 2.2. Let $F : X \rightarrow \mathcal{P}(X)$ be a multivalued map

- (i) F is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.
- (ii) F is bounded on bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all $B \in P_b(X)$.
- (iii) F is an upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X , and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$.
- (iv) F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$.
- (v) F has a fixed point if there is $x \in X$ such that $x \in F(x)$.
- (vi) If F is completely continuous with nonempty compact values, then F is u.s.c if and only if F has a closed graph, i.e., $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

The fixed point set of the multivalued operator F will be denoted by $Fix F$.

Definition 2.3. A multivalued map $F : J \rightarrow \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \rightarrow d(y, F(t)) = \inf\{|y - z| : z \in F(t)\}$$

is measurable.

Let $L^1(J, \mathbb{R})$ be the Banach space of all measurable functions $x : J \rightarrow \mathbb{R}$ which are Lebesgue integrable endowed with the norm $\|x\|_{L^1} = \int_{t_0}^T |x(t)| dt$, and $C(J, \mathbb{R})$ denotes the Banach space of all real valued continuous functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in J\}$.

Definition 2.4. A multivalued map $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

- (i) $t \rightarrow F(t, x)$ is measurable for each $x \in \mathbb{R}$,
- (ii) $x \rightarrow F(t, x)$ is upper semi-continuous for almost all $t \in J$.

Further a Carathéodory function F is called L^1 -Carathéodory If for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in J$.

Definition 2.5. Let Y be a Banach space, Z a nonempty closed subset of Y . The multivalued operator $F : Z \rightarrow \mathcal{P}(Y)$ is said to be lower semi-continuous (l.s.c.) if the set $\{z \in Z : F(z) \cap B \neq \emptyset\}$ is open for any open set B in Y .

Definition 2.6. Let A be a subset of $J \times \mathbb{R}$. A is said to be $\mathcal{L} \otimes \mathcal{B}$ -measurable if A belongs to the σ -algebra generated by all sets of the form $L \times B$, where L is Lebesgue measurable in J and B is Borel measurable in \mathbb{R} .

Definition 2.7. A subset A of $L^1(J, \mathbb{R})$ is decomposable if for all $u, v \in A$ and measurable sets $I \subset J$, the function $u\chi_I + v\chi_{J-I} \in A$, where χ_I stands for the characteristic function of I .

Definition 2.8. If $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact values and $u \in C(J, \mathbb{R})$, then the set of selections of $F(.,.)$, denoted by $S_{F,u}$, is of lower semi-continuous type if

$$S_{F,u} = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, u(t)) \text{ for a.e. } t \in J\}$$

is lower semi-continuous with nonempty closed and decomposable values .

Definition 2.9. Let (X, d) be a metric space associated with the metric d . The Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},$$

where $d^*(A, B) = \sup\{d(a, B) : a \in A\}$, and $d(x, B) = \inf_{y \in B} d(x, y)$.

Definition 2.10. A multivalued operator F on X with nonempty values in X is called

(a) γ -Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(F(x), F(y)) \leq \gamma d(x, y), \text{ for each } x, y \in X,$$

(b) a contraction if and only if it is γ -Lipschitz with $\gamma < 1$.

The following lemmas will be used in what follows.

Lemma 2.11. ([25]) Let X be a Banach space. Let $F : J \times X \rightarrow P_{cp,c}(X)$ be an L^1 -Carathéodory multivalued map and let H be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator

$$\begin{aligned} \Theta \circ S_F & : C(J, X) \rightarrow P_{cp,c}(C(J, X)), \\ x & \rightarrow (H \circ S_F)(x) = H(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

We close these preliminaries by introducing the following two fixed point theorems.

Lemma 2.12. ([26]) Let Y be a separable metric space and let $F : Y \rightarrow \mathcal{P}(L^1(J, \mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $F(.)$ has a continuous selection, i.e., there exists a continuous mapping (single valued) $f : Y \rightarrow L^1(J, \mathbb{R})$ such that $f(y) \in F(y)$ for every $y \in Y$.

Lemma 2.13. ([27]) Let (X, d) be a complete metric space. If $F : X \rightarrow P_{cl}(X)$ is a contraction, then $Fix F \neq \emptyset$.

3 Existence results

In this section, some existence results of problem (1.1) are presented that concerns with the convex and non-convex valued cases. Before going on, we recall the following result.

Lemma 3.1. ([13]) For $0 < n - 1 < q < n$, we have

$${}^c D_{t_0}^q \left(c_0 + c_1 (t - t_0) + c_2 (t - t_0)^2 + \dots + c_{n-1} (t - t_0)^{n-1} \right) = 0,$$

where $c_k \in \mathbb{R}, k = 0, 1, 2, \dots, n - 1$. Moreover

$$I_{t_0}^q {}^c D_{t_0}^q x(t) = x(t) + c_0 + c_1 (t - t_0) + c_2 (t - t_0)^2 + \dots + c_{n-1} (t - t_0)^{n-1}.$$

The investigations of solution existence for fractional differential equations require an equivalent integral form of equation (1.1) which can be obtained by introducing the corresponding linear form.

Lemma 3.2. For any $y \in C(J, \mathbb{R})$, the unique solution of the boundary value problem

$$\begin{cases} {}^c D_{t_0}^q x(t) = y(t), t \in J, 4 < q \leq 5, \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^T g_k(t) dt, k = 0, 1, 2, 3, 4, \end{cases} \quad (3.1)$$

is

$$\begin{aligned} x(t) &= \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} y(s) ds \\ &\quad + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s) ds \end{aligned} \quad (3.2)$$

where

$$\alpha_k = \prod_{m=0}^k (1 - \theta_m), \quad \lambda_k(t) = \sum_{m=0}^k \gamma_{m,k} \binom{k}{m} (t-t_0)^m (T-t_0)^{k-m}, \quad k = 0, 1, 2, 3, 4,$$

and

$$\begin{aligned} \gamma_{0,0} &= 1, \gamma_{1,1} = \alpha_0, \gamma_{2,2} = \alpha_1, \gamma_{3,3} = \alpha_2, \gamma_{4,4} = \alpha_3 \\ \gamma_{0,1} &= \theta_0, \gamma_{0,2} = \theta_0(\theta_1 + 1), \gamma_{0,3} = \theta_0(\theta_1\theta_2 + 2\theta_1 + 2\theta_2 + 1), \\ \gamma_{0,4} &= \theta_0(\theta_1\theta_2\theta_3 + 3\theta_1\theta_2 + 5\theta_1\theta_3 + 3\theta_2\theta_3 + 3\theta_1 + 3\theta_3 + 1), \\ \gamma_{1,2} &= 2\alpha_0\theta_1, \gamma_{1,3} = 3\alpha_0\theta_1(\theta_2 + 1), \gamma_{1,4} = 4\alpha_0\theta_1(\theta_2\theta_3 + 2\theta_2 + 2\theta_3 + 1), \\ \gamma_{2,3} &= 3\alpha_1\theta_2, \gamma_{2,4} = 6\theta_2\alpha_1(\theta_3 + 1), \gamma_{3,4} = 4\theta_3\alpha_2. \end{aligned}$$

Proof. Using Lemma 3.1, for some constants $c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$, we have

$$\begin{aligned} x(t) &= I_{t_0}^q y(t) - c_0 - c_1(t-t_0) - c_2(t-t_0)^2 - c_3(t-t_0)^3 - c_4(t-t_0)^4 \\ &= \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds \\ &\quad - c_0 - c_1(t-t_0) - c_2(t-t_0)^2 - c_3(t-t_0)^3 - c_4(t-t_0)^4. \end{aligned} \quad (3.3)$$

Applying the boundary conditions for problem (3.1) in (3.3), we find that

$$\begin{aligned} c_0 &= \frac{-\theta_0}{(1-\theta_0)} \int_{t_0}^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s) ds - \frac{\theta_0\theta_1(T-t_0)}{(1-\theta_0)(1-\theta_1)} \int_{t_0}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s) ds \\ &\quad - \frac{\theta_0\theta_2(\theta_1+1)(T-t_0)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)} \int_{t_0}^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s) ds \\ &\quad - \frac{\theta_0\theta_3(\theta_1\theta_2+2\theta_1+2\theta_2+1)(T-t_0)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)} \int_{t_0}^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s) ds \\ &\quad - \frac{\theta_0\theta_4(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1)(T-t_0)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)} \times \\ &\quad \int_{t_0}^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s) ds - \frac{\beta_0}{(1-\theta_0)} \int_{t_0}^T g_0(s, x(s)) ds \\ &\quad - \frac{\theta_0\beta_1(T-t_0)}{(1-\theta_0)(1-\theta_1)} \int_{t_0}^T g_1(s, x(s)) ds \end{aligned}$$

$$\begin{aligned}
 & -\frac{\theta_0\beta_2(\theta_1+1)(T-t_0)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)}\int_{t_0}^T g_2(s,x(s))ds \\
 & -\frac{\theta_0\beta_3(\theta_1\theta_2+2\theta_1+2\theta_2+1)(T-t_0)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^T g_3(s,x(s))ds \\
 & -\frac{\theta_0\beta_4(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1)(T-t_0)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\times \\
 & \int_{t_0}^T g_4(s,x(s))ds, \\
 c_1 = & -\frac{\theta_1}{(1-\theta_1)}\int_{t_0}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}y(s)ds - \frac{\theta_1\theta_2(T-t_0)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)ds \\
 & -\frac{\theta_1\theta_3(\theta_2+1)(T-t_0)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)ds \\
 & -\frac{\theta_1\theta_4(\theta_2\theta_3+2\theta_2+2\theta_3+1)(T-t_0)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)ds \\
 & -\frac{\beta_1}{(1-\theta_1)}\int_{t_0}^T g_1(s,x(s))ds - \frac{\theta_1\beta_2(T-t_0)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^T g_2(s,x(s))ds \\
 & -\frac{\theta_1\beta_3(\theta_2+1)(T-t_0)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^T g_3(s,x(s))ds \\
 & -\frac{\theta_1\beta_4(\theta_2\theta_3+2\theta_2+2\theta_3+1)(T-t_0)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^T g_4(s,x(s))ds, \\
 c_2 = & -\frac{\theta_2}{2(1-\theta_2)}\int_{t_0}^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)ds - \frac{\theta_2\theta_3(T-t_0)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)ds \\
 & -\frac{\theta_2\theta_4(\theta_3+1)(T-t_0)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)ds \\
 & -\frac{\beta_2}{2(1-\theta_2)}\int_{t_0}^T g_2(s,x(s))ds - \frac{\theta_2\beta_3(T-t_0)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^T g_3(s,x(s))ds \\
 & -\frac{\theta_2\beta_4(\theta_3+1)(T-t_0)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^T g_4(s,x(s))ds, \\
 c_3 = & -\frac{\theta_3}{6(1-\theta_3)}\int_{t_0}^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)ds - \frac{\theta_3\theta_4(T-t_0)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)ds
 \end{aligned}$$

$$-\frac{\beta_3}{6(1-\theta_3)} \int_{t_0}^T g_3(s, x(s)) ds - \frac{\theta_3 \beta_4 (T-t_0)}{6(1-\theta_3)(1-\theta_4)} \int_{t_0}^T g_4(s, x(s)) ds,$$

and

$$c_4 = -\frac{\theta_4}{24(1-\theta_4)} \int_{t_0}^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s) ds - \frac{\beta_4}{24(1-\theta_4)} \int_{t_0}^T g_4(s, x(s)) ds.$$

Substituting the values of c_0, c_1, c_2, c_3 and c_4 in (3.3), and arranging the terms into compact expression, one can obtain (3.2). This completes the proof. \square

The main results are based on the following fixed point theorems.

Theorem 3.3. [28] (Nonlinear alternative of Leray-Schauder type) Let X be a Banach space, \mathfrak{X} be a closed convex subset of X , \mathfrak{U} be an open subset of \mathfrak{X} with $0 \in \mathfrak{U}$. Suppose that $F : \overline{\mathfrak{U}} \rightarrow P_{cp,c}(\mathfrak{X})$ is an upper semicontinuous compact map. Then either F has a fixed point in $\overline{\mathfrak{U}}$ or there are $\mathfrak{x} \in \partial\mathfrak{U}$ and $\lambda \in (0, 1)$ such that $\mathfrak{x} \in \lambda F(\mathfrak{x})$.

Theorem 3.4. [28] (Covitz and Nadler) Let (X, d) be a complete metric space. If $F : X \rightarrow P_{cl}(X)$ is a contraction, then F has a fixed point.

The first existence result can now be introduced. We denote hereafter $\lambda_k = \max_{t \in J} |\lambda_k(t)|$.

Theorem 3.5. Assume that

(HA) $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is Carathéodory and has convex values,

(HB) there exists a continuous nondecreasing function $\psi : [t_0, \infty) \rightarrow (0, \infty)$ and a function $p \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq p(t)\psi(|x|),$$

for each $(t, x) \in J \times \mathbb{R}$,

(HC) there exist continuous nondecreasing functions $\psi_k : [t_0, \infty) \rightarrow (0, \infty)$ and functions $p_k \in L^1(J, \mathbb{R}^+)$ such that

$$|g_k(t, x)| \leq p_k(t)\psi_k(|x|), k = 0, 1, 2, 3, 4,$$

for each $(t, x) \in J \times \mathbb{R}$,

(HD) there exists a large number $M > 0$ such that

$$\frac{M}{\gamma_1 \psi(M) \|p\|_{L^1} + \gamma_2} > 1,$$

where

$$\gamma_1 = \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^4 \frac{|\theta_k| \lambda_k (T-t_0)^{q-k-1}}{k! \Gamma(q-k) |\alpha_k|}$$

and

$$\gamma_2 = \sum_{k=0}^4 \frac{\lambda_k |\beta_k|}{k! |\alpha_k|} \psi_k(M) \|p_k\|_{L^1}.$$

Then the boundary value problem (1.1) has at least one solution on J .

Proof. Using Lemma 3.2, we can define an operator

$$\Omega(x) = \left\{ h \in C(J, \mathbb{R}) : h(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds \right\}$$

for $f \in S_{F,x}$. We show that Ω satisfies the assumptions of the nonlinear alternative of the Leray-Schauder type. The proof consists of several steps.

Step I: We show that $\Omega(x)$ is convex for each $x \in C(J, \mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit its proof.

Step II: We show that $\Omega(x)$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$. For a positive number r , let $B_r = \{x \in C(J, \mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C(J, \mathbb{R})$. Then, for each $h \in \Omega(x)$, $x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds.$$

Then for $t \in J$, we have

$$\begin{aligned} |h(t)| &\leq \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)| ds + \sum_{k=0}^4 \frac{|\theta_k|}{k! |\alpha_k|} |\lambda_k(t)| \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s)| ds \\ &\quad + \sum_{k=0}^4 \frac{|\beta_k|}{k! |\alpha_k|} |\lambda_k(t)| \int_{t_0}^T |g_k(s, x(s))| ds \\ &\leq \psi(\|x\|) \|p\|_{L^1} \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^4 \frac{\lambda_k |\theta_k| (T-t_0)^{q-k-1}}{k! \Gamma(q-k) |\alpha_k|} \right\} \\ &\quad + \sum_{k=0}^4 \frac{|\beta_k| \lambda_k}{k! |\alpha_k|} \psi_k(\|x\|) \|p_k\|_{L^1}. \end{aligned}$$

Thus,

$$\begin{aligned} \|h\| &\leq \psi(\|r\|) \|p\|_{L^1} \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^4 \frac{\lambda_k |\theta_k| (T-t_0)^{q-k-1}}{k! \Gamma(q-k) |\alpha_k|} \right\} \\ &\quad + \sum_{k=0}^4 \frac{\lambda_k |\beta_k|}{k! |\alpha_k|} \psi_k(\|r\|) \|p_k\|_{L^1}. \end{aligned}$$

Step III: We show that Ω maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, and $x \in B_r$ where B_r is a bounded set of $C(J, \mathbb{R})$. In view of (HC), for each

$h \in \Omega(x)$, we obtain

$$\begin{aligned}
 & |h(t_2) - h(t_1)| \\
 = & \left| \int_{t_0}^{t_2} \frac{(t_2-s)^{q-1} - (t_1-s)^{q-1}}{\Gamma(q)} f(s) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s) ds \right. \\
 & + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} (\lambda_k(t_2) - \lambda_k(t_1)) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s) ds \\
 & \left. + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} (\lambda_k(t_2) - \lambda_k(t_1)) \int_{t_0}^T g_k(s, x(s)) ds \right| \\
 \leq & \psi(\|x\|) \int_{t_0}^{t_2} \frac{|(t_2-s)^{q-1} - (t_1-s)^{q-1}|}{\Gamma(q)} p(s) ds \\
 & + \psi(\|x\|) \int_{t_1}^{t_2} \frac{|(t_2-s)^{q-1}|}{\Gamma(q)} p(s) ds \\
 & + \sum_{k=0}^4 \frac{|\theta_k|}{k! |\alpha_k|} |\lambda_k(t_2) - \lambda_k(t_1)| \psi(\|x\|) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} p(s) ds \\
 & + \sum_{k=0}^4 \frac{|\beta_k|}{k! |\alpha_k|} |\lambda_k(t_2) - \lambda_k(t_1)| \psi_k(\|x\|) \int_{t_0}^T p_k(s) ds
 \end{aligned}$$

The functions $\lambda_k(t)$ behave like a polynomial function, particularly, $(\lambda_k(t_2) - \lambda_k(t_1)) \rightarrow 0$, as $t_2 - t_1 \rightarrow 0$. Therefore, the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \rightarrow 0$. As Ω satisfies the above three assumptions, it follows by the Arzela-Ascoli theorem that $\Omega : C(J, \mathbb{R}) \rightarrow \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Step IV: We show that Ω has a closed graph. Let $x_n \rightarrow x_*$, $h_n \in \Omega(x_n)$ and $h_n \rightarrow h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F, x_n}$ such that for each $t \in J$,

$$\begin{aligned}
 h_n(t) &= \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f_n(s) ds \\
 &+ \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x_n(s)) ds.
 \end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F, x_*}$ such that for each $t \in J$,

$$\begin{aligned}
 h_*(t) &= \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f_*(s) ds \\
 &+ \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x_*(s)) ds.
 \end{aligned} \tag{3.4}$$

Let us consider the continuous linear operator $\Theta : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ given by

$$f \rightarrow \Theta(f)(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k!|\alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k!|\alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds.$$

Observe that

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f_n(s) - f_*(s)| ds \\ &\quad + \sum_{k=0}^4 \frac{|\theta_k|}{k!|\alpha_k|} |\lambda_k(t)| \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f_n(s) - f_*(s)| ds \\ &\quad + \sum_{k=0}^4 \frac{|\beta_k|}{k!|\alpha_k|} |\lambda_k(t)| \int_{t_0}^T |g_k(s, x_n(s)) - g_k(s, x_*(s))| ds. \end{aligned}$$

Thus, it follows by Lemma 2.11 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x})$, since $x_n \rightarrow x_*$, then h_* satisfying (3.4) for some $f_* \in S_{F,x_*}$.

Step V: We discuss a priori bounds on solutions. Let x be a solution of (1.1), then there exists $f \in L^1(J, \mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in J$, we have

$$\begin{aligned} x(t) &= \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k!|\alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s) ds \\ &\quad + \sum_{k=0}^4 \frac{\beta_k}{k!|\alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds. \end{aligned}$$

Using the computations of the second step above, we have

$$\begin{aligned} |x(t)| &\leq \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)| ds + \sum_{k=0}^4 \frac{|\theta_k|}{k!|\alpha_k|} |\lambda_k(t)| \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s)| ds \\ &\quad + \sum_{k=0}^4 \frac{|\beta_k|}{k!|\alpha_k|} |\lambda_k(t)| \int_{t_0}^T |g_k(s, x(s))| ds \\ &\leq \psi(\|x\|) \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^4 \frac{\lambda_k |\theta_k| (T-t_0)^{q-k-1}}{k! \Gamma(q-k) |\alpha_k|} \right\} \int_{t_0}^T p(s) ds \\ &\quad + \sum_{k=0}^4 \frac{\lambda_k |\beta_k|}{k! |\alpha_k|} \psi_k(\|x\|) \int_{t_0}^T p_k(s) ds. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\gamma_1 \psi(\|x\|) \|p\|_{L^1} + \gamma_2} \leq 1.$$

In view of (HD), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J, \mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega : \bar{U} \rightarrow \mathcal{P}(C(J, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U , there is no $x \in \partial U$ such that $x = \mu\Omega(x)$ for some $\mu \in (0, 1)$. Consequently, by the nonlinear alternative of the Leray–Schauder type, we deduce that Ω has a fixed point $x \in \bar{U}$ which is a solution of problem (1.1). This completes the proof. \square

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of the Leray–Schauder type together with the selection theorem of Bressan and Colombo [26] for lower semi-continuous maps with decomposable values.

Theorem 3.6. *Assume that (HB), (HC), (HD), and the following condition holds*

(HE) $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that

- (a) $(t, x) \rightarrow F(t, x)$ is $\ell \otimes B$ measurable,
- (b) $x \rightarrow F(t, x)$ is lower semi-continuous for each $t \in J$.

Then the boundary value problem (1.1) has at least one solution on J .

Proof. It follows from (HB) and (HE) that F is of l.s.c. type. Then from Lemma 2.12, there exists a continuous function $f : C(J, \mathbb{R}) \rightarrow L_1(J, \mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in C(J, \mathbb{R})$. Consider the problem

$$\begin{cases} {}^c D_{t_0}^q x(t) = f(x(t)), t \in J, q \in (4, 5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^T g_k(t, x(t)) dt, k = 0, 1, 2, 3, 4. \end{cases} \tag{3.5}$$

Observe that if $x \in C(J, \mathbb{R})$ is a solution of (3.5), then x is a solution to problem (1.1). In order to transform problem (3.5) into a fixed point problem, we define the operator Π as

$$\begin{aligned} \Pi x(t) &= \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s)) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(x(s)) ds \\ &\quad + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds. \end{aligned}$$

It can easily be shown that Π is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.5, so we omit it. This completes the proof. \square

Now we prove the existence of solutions for problem (1.1) with a nonconvex value by applying a fixed point theorem for a multivalued map due to Covitz and Nadler [27].

Theorem 3.7. *Assume that the following conditions hold:*

(HF) $F : J \times \mathbb{R} \rightarrow P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : J \rightarrow P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$,

(HG) $d_H(F(t, x), F(t, y)) \leq z(t)|x - y|$ for almost all $t \in J$ and $x, y \in \mathbb{R}$ with $z \in L^1(J, \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq z(t)$ for almost all $t \in J$.

(HH) There exist functions $z_k \in L^1(J, \mathbb{R}^+)$ such that

$$|g_k(t, x) - g_k(t, y)| \leq z_k(t)|x - y|,$$

for $t \in J, k = 0, 1, 2, 3, 4$, and $x, y \in \mathbb{R}$.

Then the boundary value problem (1.1) has at least one solution on J if

$$\gamma_1 \|z\|_{L^1} + \omega < 1,$$

where

$$\omega = \sum_{k=0}^4 \frac{\lambda_k \|z_k\|_{L^1} |\beta_k|}{k! |\alpha_k|}.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J, \mathbb{R})$ by assumption (HF), so F has a measurable selection (see [29]: Theorem 3.6). Now we show that the operator Ω satisfies the assumptions of Theorem 2.13. To show that $\Omega(x) \in P_{cl}((C(J, \mathbb{R}))$ for each $x \in C(J, \mathbb{R})$, let $(u_n)_{n \geq 0} \in \Omega(x)$ be such that $u_n \rightarrow u$ in $C(J, \mathbb{R})$. Then $u \in C(J, \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in J$,

$$u_n(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_n(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds.$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J, \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$,

$$u_n(t) \rightarrow u(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds.$$

Hence, $u \in \Omega(x)$. Next we show that there exists $\tau < 1$ such that

$$d_H(\Omega(x), \Omega(y)) \leq \tau \|x - y\|,$$

for each $x, y \in C(J, \mathbb{R})$. Let $x, y \in C(J, \mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists

$$v_1(t) \in F(t, x(t)),$$

such that, for each $t \in J$,

$$h_1(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_1(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds.$$

By (HG), we have

$$d_H(F(t, x), F(t, y)) \leq z(t) |x(t) - y(t)|.$$

So, there exists $w_* \in F(t, y(t))$ such that

$$|v_1(t) - w_*| \leq z(t) |x(t) - y(t)|, \quad t \in J.$$

Define $V : J \rightarrow \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w_* \in \mathbb{R} : |v_1(t) - w_*| \leq z(t) |x(t) - y(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, y(t))$ is measurable ([29]: Proposition 3.4), there exists a function $v_2(t)$ which is a measurable selection for V . So $v_2(t) \in F(t, y(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq z(t) |x(t) - y(t)|$. Define

$$h_2(t) = \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s) ds + \sum_{k=0}^4 \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_2(s) ds + \sum_{k=0}^4 \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^T g_k(s, x(s)) ds.$$

Thus, for each $t \in J$, it follows that

$$\begin{aligned}
 |h_1(t) - h_2(t)| &\leq \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)| ds \\
 &\quad + \sum_{k=0}^4 \frac{|\theta_k|}{k! |\alpha_k|} |\lambda_k(t)| \int_{t_0}^T \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |v_1(s) - v_2(s)| ds \\
 &\quad + \sum_{k=0}^4 \frac{|\beta_k|}{k! |\alpha_k|} |\lambda_k(t)| \int_{t_0}^T |g_k(s, x(s)) - g_k(s, y(s))| ds \\
 &\leq \left\{ \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^4 \frac{\lambda_k |\theta_k| (T-t_0)^{q-k-1}}{k! \Gamma(q-k) |\alpha_k|} \right\} \|z\|_{L^1} \right. \\
 &\quad \left. + \sum_{k=0}^4 \frac{\lambda_k \|z_k\|_{L^1} |\beta_k|}{k! |\alpha_k|} \right\} \|x - y\|.
 \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq (\gamma_1 \|z\|_{L^1} + \omega) \|x - y\|.$$

We deduce that

$$\begin{aligned}
 d_H(\Omega(x), \Omega(y)) &\leq (\gamma_1 \|z\|_{L^1} + \omega) \|x - y\| \\
 &\leq \tau \|x - y\|.
 \end{aligned}$$

Since Ω is a contraction, it follows by Theorem 2.13 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. \square

We close this article by introducing a couple of examples.

Example 3.8. Consider the following fractional differential inclusion

$$\begin{cases}
 {}^c D_0^{4.2} x(t) \in F(t, x(t)), t \in [0, 1], \\
 x^{(k)}(0) = -x^{(k)}(1) + k \int_0^1 e^{-x(t)} dt, k = 0, 1, 2, 3, 4.
 \end{cases} \quad (3.6)$$

where $F : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t, x) = \left\{ y \in \mathbb{R} : 0 \leq y \leq 1 + \frac{t|x|}{1+t|x|} \right\}.$$

Observe that

$$t \rightarrow F(t, x) = \left\{ y \in \mathbb{R} : 0 \leq y \leq 1 + \frac{t|x|}{1+t|x|} \right\}$$

is measurable for each $x \in \mathbb{R}$, since each lower and upper functions are both measurable on $[0, 1] \times \mathbb{R}$. Now let (t_0, x_0) be any element in $[0, 1] \times \mathbb{R}$. Then

$$F(t_0, x_0) = \left\{ y \in \mathbb{R} : 0 \leq y \leq 1 + \frac{t|x_0|}{1+t|x_0|} \right\}$$

is a closed subset of $[0, 1] \times \mathbb{R}$ and

$$\left\{ y \in \mathbb{R} : 0 \leq y \leq 1 + \frac{t|x_0|}{1+t|x_0|} \right\} \neq \phi.$$

Let O be any open interval in \mathbb{R} such that

$$\left\{ y \in \mathbb{R} : 0 \leq y \leq 1 + \frac{t|x_0|}{1+t|x_0|} \right\} \subset O,$$

we can find an open intervals U of t_0 and V of x_0 with $[0, 1] \subset U = (-\varepsilon, 1 + \varepsilon)$ and $F(u, v) \subset O$ for every $u \in U$ and $v \in V$, where ε is a small positive real number. So $x \rightarrow F(t, x)$ is upper semi-continuous for all $t \in J$. Thus F is a Carathéodory and clearly has convex values. Clearly

$$\|F(t, x)\| = \sup\{|y| : y \in F(t, x)\} \leq 2,$$

and

$$|g_k(t, x)| \leq k, k = 0, 1, 2, 3, 4,$$

for each $(t, x) \in [0, 1] \times \mathbb{R}$. Assume that $p(t) = 1, p_k(t) = k, \psi(|x|) = 2,$ and $\psi_k(|x|) = 1$ for $k = 0, 1, 2, 3, 4$. Simple calculations lead to

$$\begin{aligned} \lambda_0(t) &= 1 \\ \lambda_1(t) &= -1 + 2t \\ \lambda_2(t) &= -8t + 4t^2 \\ \lambda_3(t) &= 2 - 36t^2 + 8t^3 \\ \lambda_4(t) &= -5 + 64t - 128t^3, \end{aligned}$$

$$\begin{aligned} \gamma_1 &= \frac{1}{7.76} + \frac{1}{2(7.76)} + \frac{1}{4(2.424)} + \frac{4}{2(1.1)8} + \frac{26}{6(0.918)16} + \frac{69}{24(4.59)32} \\ &= 0.129 + 0.0645 + 0.103 + 0.227 + 0.295 + 0.02 = 0.839 \end{aligned}$$

and

$$\gamma_2 = \left(\frac{1}{4} + \frac{8}{16} + \frac{78}{96} + \frac{276}{768}\right) = 0.359375 + 0.8125 = 1.922.$$

Furthermore, let M be any number satisfying

$$M > \gamma_1 \psi(|x|) \|p\|_{L^1} + \gamma_2 = 3.6.$$

Clearly, all the conditions of Theorem 3.5 are satisfied. So there exists at least one solution of problem (3.6) on $[0, 1]$.

Example 3.9. Consider the following fractional differential inclusion

$$\begin{cases} {}^c D_0^{\frac{10}{4}} x(t) \in F(t, x(t)), t \in [0, 1], \\ x^{(k)}(0) = -x^{(k)}(1), k = 0, 1, 2, 3, 4, \end{cases} \tag{3.7}$$

where $F : [0, 1] \times [0, \frac{\pi}{2}] \rightarrow \mathbb{R}^+$ is a multivalued map given by

$$F(t, x) = \left[0, \frac{\sin x}{(2 + t)^4}\right].$$

Now

$$\begin{aligned} \sup\{|y| : y \in F(t, x)\} &\leq \frac{\sin x}{(2 + t)^4} \\ &\leq \frac{1}{16} \text{ for each } (t, x) \in [0, 1] \times \left[0, \frac{\pi}{2}\right], \end{aligned}$$

and

$$\begin{aligned}
 d_H(F(t, x), F(t, y)) &= d_H \left(\left[0, \frac{\sin x}{(2+t)^4} \right], \left[0, \frac{\sin y}{(2+t)^4} \right] \right) \\
 &= \max \left\{ d^* \left(\left[0, \frac{\sin x}{(2+t)^4} \right], \left[0, \frac{\sin y}{(2+t)^4} \right] \right), \right. \\
 &\quad \left. d^* \left(\left[0, \frac{\sin y}{(2+t)^4} \right], \left[0, \frac{\sin x}{(2+t)^4} \right] \right) \right\} \\
 &= \max \left\{ \sup \left\{ d \left(a, \left[0, \frac{\sin y}{(2+t)^4} \right] \right) : a \in \left[0, \frac{\sin x}{(2+t)^4} \right] \right\}, \right. \\
 &\quad \left. \sup \left\{ d \left(\left[0, \frac{\sin x}{(2+t)^4} \right], b \right) : b \in \left[0, \frac{\sin y}{(2+t)^4} \right] \right\} \right\} \\
 &\leq \frac{1}{(2+t)^4} |x - y|.
 \end{aligned}$$

Here $z(t) = \frac{1}{(2+t)^4}$, with $\|z\|_{L^1} \approx 0.017$, and observe that $\omega = 0$, because of the absence of the integral boundary condition which implies that $\gamma_1 \|z\|_{L^1} + \omega < 1$. The compactness of F together with the above calculations lead to the existence of solution of the problem (3.7) by Theorem 3.7.

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