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ON EXISTENCE OF INTEGRAL AND ANTI-PERIODIC DIFFERENTIAL INCLUSION OF FRACTIONAL ORDER $\alpha \in (4, 5]$

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Abstract. We are concerned in this article the existence of solutions of a class of fractional differential inclusions with anti-periodic and integral boundary conditions involving the Caputo fractional derivative with order $\alpha \in (4, 5]$. These results can be considered as a generalization of previously published articles in this topic with fractional orders less than or equal 4. However, the results based on fixed-point theorems for differential inclusions. Two examples are introduced to show the applicability of such theorems.

1 Introduction

Fractional differential models has recently a wide investigations due to its extensive development and applications in several disciplines such as physics, mechanics, chemistry, engineering, etc.(see [9],[10],[12],[14],[15],[20], [22], and references therein). The fact that using the fractional-order models instead of integer-order model is due to more realistic in description of many physical phenomenons. The investigation of existence problems of fractional differential equations in general is considered as a priority for going forward in such applications (see [13], [16]-[18],[35]). Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes ([20],[36]) and have recently received considerable attention. For examples and details of anti-periodic boundary conditions (see ([1]-[8],[11],[21],[34] and the references therein). Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors (see [23],[24] and the references therein). For some recent works on differential inclusions of fractional order, we refer the reader to the references ([1],[7],[8], [19]). In this paper, we discuss some existence results for anti-periodic boundary value problems of differential inclusions of fractional order $q \in (4, 5]$ using the nonlinear alternative of the Leray-Schauder and Covitz and Nadler fixed point theorems.

Precisely, we will devote to considering the existence of solution of the integral and anti-periodic boundary value problem

$$\begin{cases} {}^cD^q_{t_0} x(t) \in F(t, x(t)),\, t \in J = [t_0, T],\, T > t_0,\, q \in (4, 5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t, x(t))dt,\, k = 0, 1, 2, 3, 4, \end{cases} \tag{1.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $g_k : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\theta_k, \beta_k \in \mathbb{R}$ with $\theta_k \neq 1$ for each $k = 0, 1, 2, 3, 4$, and ${}^cD^q_{t_0}$ denotes the Caputo fractional derivative of order q which is generally defined by

$${}^cD^q_{t_0} x(t) = \left(I^{n-q}_{t_0} x^{(n)}\right)(t) = \frac{1}{\Gamma(n-q)} \int\limits_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)ds,\, n-1 < q < n,$$

where $n = [q] + 1$, and $[q]$ denotes the integer part of the real number q.

This paper is organized as follows: In Section 2, we introduce some well-known results in multivalued analysis. The main results of existence theorems will be given in Section 3. Finally, we give some illustrative examples to explain the theorems.

2 Preliminaries

We recall in this section some facts from multivalued mapping analysis (see [29]-[31]) that needed for the results in the sequel.

Definition 2.1. For a normed space $(X, \|\cdot\|)$, let

$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : \ Y \text{ is compact and convex}\}$.

Definition 2.2. Let $F : X \to \mathcal{P}(X)$ be a multivalued map
(i) F is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.
(ii) F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in P_b(X)$.
(iii) F is an upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood $N_0$of $x_0$such that $F(N_0) \subseteq N$.
(iv) F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$.
(v) F has a fixed point if there is $x \in X$ such that $x \in F(x)$.
(vi) If F is completely continuous with nonempty compact values, then F is u.s.c if and only if F has a closed graph, i.e., $x_n \to x_*, \ y_n \to y_*, \ y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

The fixed point set of the multivalued operator F will be denoted by $Fix\ F$.

Definition 2.3. A multivalued map $F : J \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, F(t)) = inf\{|y - z| : z \in F(t)\}$$

is measurable.

Let $L^1(J, \mathbb{R})$ be the Banach space of all measurable functions $x : J \to \mathbb{R}$ which are Lebesgue integrable endowed with the norm $\|x\|_{L^1} = \int_{t_0}^{T} |x(t)|dt$, and $C(J, \mathbb{R})$ denotes the Banach space of all real valued continuous functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in J\}$.

Definition 2.4. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \to F(t, x)$ is measurable for each $x \in \mathbb{R}$,
(ii) $x \to F(t, x)$ is upper semi-continuous for almost all $t \in J$.
Further a Carathéodory function F is called L^1-Carathéodory If for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in J$.

Definition 2.5. Let Y be a Banach space, Z a nonempty closed subset of Y. The multivalued operator $F : Z \to \mathcal{P}(Y)$ is said to be lower semi-continuous (l.s.c.) if the set $\{z \in Z : F(z) \cap B \neq \phi\}$ is open for any open set B in Y.

Definition 2.6. Let A be a subset of $J \times \mathbb{R}$. A is said to be $\mathcal{L} \otimes \mathcal{B}$-measurable if A belongs to the σ-algebra generated by all sets of the form $L \times B$, where L is Lebesgue measurable in J and B is Borel measurable in $\mathbb{R}$.

Definition 2.7. A subset A of $L^1(J,\mathbb{R})$ is decomposable if for all $u, v \in A$ and measurable sets $I \subset J$, the function $u\chi_I + v\chi_{J-I} \in A$, where χ_I stands for the characteristic function of I.

Definition 2.8. If $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact values and $u \in C(J,\mathbb{R})$, then the set of selections of $F(.,.)$, denoted by $S_{F,u}$, is of lower semi-continuous type if

$$S_{F,u} = \{w \in L^1(J,\mathbb{R}) : w(t) \in F(t,u(t)) \text{ for a.e. } t \in J\}$$

is lower semi-continuous with nonempty closed and decomposable values .

Definition 2.9. Let (X, d) be a metric space associated with the metric d. The Pompeiu–Hausdorff distance of the closed subsets $A,\ B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},$$

where $d^*(A,B) = \sup\{d(a,B) : a \in A\}$, and $d(x,B) = \inf_{y\in B} d(x,y)$.

Definition 2.10. A multivalued operator F on X with nonempty values in X is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(F(x), F(y)) \leq \gamma d(x,y), \text{for each } x, y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.
The following lemmas will be used in what follows.

Lemma 2.11. *([25])Let X be a Banach space. Let $F : J\times X \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let H be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\begin{aligned} \Theta \circ S_F \quad &: \quad C(J,X) \to P_{cp,c}(C(J,X)), \\ x \quad &\to \quad (H \circ S_F)(x) = H(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We close these preliminaries by introducing the following two fixed point theorems.

Lemma 2.12. *([26])Let Y be a separable metric space and let $F : Y \to \mathcal{P}(L^1(J,\mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $F(.)$ has a continuous selection, i.e., there exists a continuous mapping (single valued) $f : Y \to L^1(J,\mathbb{R})$ such that $f(y) \in F(y)$ for every $y \in Y$.*

Lemma 2.13. *([27]) Let (X, d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then $FixF \neq \phi$.*

3 Existence results

In this section, some existence results of problem (1.1) are presented that concerns with the convex and non-convex valued cases. Before going on, we recall the following result.

Lemma 3.1. *([13]) For $0 < n-1 < q < n$, we have*

$${}^cD^q_{t_0}\left(c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \ \cdots + c_{n-1}(t-t_0)^{n-1}\right) = 0,$$

where $c_k \in \mathbb{R}, k = 0, 1, 2, ..., n-1$. Moreover

$$I^q_{t_0}\,{}^cD^q_{t_0}x(t) = x(t) + c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \ \cdots + c_{n-1}(t-t_0)^{n-1}.$$

The investigations of solution existence for fractional differential equations require an equivalent integral form of equation (1.1) which can be obtained by introducing the corresponding linear form.

Lemma 3.2. *For any $y \in C(J,\mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases} {}^{c}D_{t_0}^{q}x\left(t\right)=y\left(t\right), t\in J, 4<q\leq 5, \\ x^{(k)}\left(t_0\right)-\theta_k x^{(k)}\left(T\right)=\beta_k \int\limits_{t_0}^{T} g_k(t)dt,\ k=0,1,2,3,4, \end{cases} \tag{3.1}$$

is

$$\begin{aligned} x(t) &= \int\limits_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}y\left(s\right)ds+\sum_{k=0}^{4}\frac{\theta_k}{k!\alpha_k}\lambda_k(t)\int\limits_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}y\left(s\right)ds \\ &\quad +\sum_{k=0}^{4}\frac{\beta_k}{k!\alpha_k}\lambda_k(t)\int\limits_{t_0}^{T}g_k(s)ds \end{aligned} \tag{3.2}$$

where

$$\alpha_k=\prod_{m=0}^{k}\left(1-\theta_m\right),\ \lambda_k(t)=\sum_{m=0}^{k}\gamma_{m,k}\begin{pmatrix} k \\ m \end{pmatrix}(t-t_0)^m(T-t_0)^{k-m}, k=0,1,2,3,4,$$

and

$$\begin{aligned} \gamma_{0,0} &= 1, \gamma_{1,1}=\alpha_0, \gamma_{2,2}=\alpha_1, \gamma_{3,3}=\alpha_2, , \gamma_{4,4}=\alpha_3 \\ \gamma_{0,1} &= \theta_0, \gamma_{0,2}=\theta_0(\theta_1+1), \gamma_{0,3}=\theta_0\left(\theta_1\theta_2+2\theta_1+2\theta_2+1\right), \\ \gamma_{0,4} &= \theta_0\left(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1\right), \\ \gamma_{1,2} &= 2\alpha_0\theta_1, \gamma_{1,3}=3\alpha_0\theta_1(\theta_2+1), \gamma_{1,4}=4\alpha_0\theta_1\left(\theta_2\theta_3+2\theta_2+2\theta_3+1\right), \\ \gamma_{2,3} &= 3\alpha_1\theta_2, \gamma_{2,4}=6\theta_2\alpha_1\left(\theta_3+1\right), \gamma_{3,4}=4\theta_3\alpha_2. \end{aligned}$$

Proof. Using Lemma 3.1, for some constants $c_0, c_1, c_2.c_3, c_4 \in \mathbb{R}$, we have

$$\begin{aligned} x(t) &= I_{t_0}^{q}y(t)-c_0-c_1\left(t-t_0\right)-c_2\left(t-t_0\right)^2-c_3\left(t-t_0\right)^3-c_4\left(t-t_0\right)^4 \\ &= \int\limits_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}y\left(s\right)ds \\ &\quad -c_0-c_1\left(t-t_0\right)-c_2\left(t-t_0\right)^2-c_3\left(t-t_0\right)^3-c_4\left(t-t_0\right)^4. \end{aligned} \tag{3.3}$$

Applying the boundary conditions for problem (3.1) in (3.3), we find that

$$\begin{aligned} c_0 &= \frac{-\theta_0}{(1-\theta_0)}\int\limits_{t_0}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}y\left(s\right)ds-\frac{\theta_0\theta_1\left(T-t_0\right)}{(1-\theta_0)(1-\theta_1)}\int\limits_{t_0}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}y\left(s\right)ds \\ &\quad -\frac{\theta_0\theta_2(\theta_1+1)\left(T-t_0\right)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)}\int\limits_{t_0}^{T}\frac{(T-s)^{q-3}}{\Gamma(q-2)}y\left(s\right)ds \\ &\quad -\frac{\theta_0\theta_3\left(\theta_1\theta_2+2\theta_1+2\theta_2+1\right)\left(T-t_0\right)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int\limits_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y\left(s\right)ds \\ &\quad -\frac{\theta_0\theta_4\left(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1\right)\left(T-t_0\right)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\times \\ &\quad \int\limits_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y\left(s\right)ds-\frac{\beta_0}{(1-\theta_0)}\int\limits_{t_0}^{T}g_0(s,x(s))ds \\ &\quad -\frac{\theta_0\beta_1\left(T-t_0\right)}{(1-\theta_0)(1-\theta_1)}\int\limits_{t_0}^{T}g_1(s,x(s))ds \end{aligned}$$

$$-\frac{\theta_0\beta_2(\theta_1+1)\left(T-t_0\right)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds$$
$$-\frac{\theta_0\beta_3\left(\theta_1\theta_2+2\theta_1+2\theta_2+1\right)\left(T-t_0\right)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds$$
$$-\frac{\theta_0\beta_4\left(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1\right)\left(T-t_0\right)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\times$$
$$\int_{t_0}^{T} g_4(s,x(s))ds,$$

$$\begin{aligned}
c_1 \;=\;& -\frac{\theta_1}{(1-\theta_1)}\int_{t_0}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}y\left(s\right)ds-\frac{\theta_1\theta_2\left(T-t_0\right)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^{T}\frac{(T-s)^{q-3}}{\Gamma(q-2)}y\left(s\right)ds\\
&-\frac{\theta_1\theta_3(\theta_2+1)\left(T-t_0\right)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y\left(s\right)ds\\
&-\frac{\theta_1\theta_4\left(\theta_2\theta_3+2\theta_2+2\theta_3+1\right)\left(T-t_0\right)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y\left(s\right)ds\\
&-\frac{\beta_1}{(1-\theta_1)}\int_{t_0}^{T} g_1(s,x(s))ds-\frac{\theta_1\beta_2\left(T-t_0\right)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds\\
&-\frac{\theta_1\beta_3(\theta_2+1)\left(T-t_0\right)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds\\
&-\frac{\theta_1\beta_4\left(\theta_2\theta_3+2\theta_2+2\theta_3+1\right)\left(T-t_0\right)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,
\end{aligned}$$

$$\begin{aligned}
c_2 \;=\;& -\frac{\theta_2}{2(1-\theta_2)}\int_{t_0}^{T}\frac{(T-s)^{q-3}}{\Gamma(q-2)}y\left(s\right)ds-\frac{\theta_2\theta_3\left(T-t_0\right)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y\left(s\right)ds\\
&-\frac{\theta_2\theta_4\left(\theta_3+1\right)\left(T-t_0\right)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y\left(s\right)ds\\
&-\frac{\beta_2}{2(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds-\frac{\theta_2\beta_3\left(T-t_0\right)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds\\
&-\frac{\theta_2\beta_4\left(\theta_3+1\right)\left(T-t_0\right)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,
\end{aligned}$$

$$c_3 \;=\; -\frac{\theta_3}{6(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y\left(s\right)ds-\frac{\theta_3\theta_4\left(T-t_0\right)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y\left(s\right)ds$$

$$-\frac{\beta_3}{6(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds-\frac{\theta_3\beta_4(T-t_0)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,$$

and

$$c_4=-\frac{\theta_4}{24(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds-\frac{\beta_4}{24(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds.$$

Substituting the values of c_0, c_1, c_2, c_3 and c_4 in (3.3), and arranging the terms into compact expression, one can obtain (3.2). This completes the proof. □

The main results are based on the following fixed point theorems.

Theorem 3.3. *[28](Nonlinear alternative of Leray-Schauder type) Let X be a Banach space, $\mathfrak{X}$ be a closed convex subset of X, $\mathfrak{U}$ be an open subset of $\mathfrak{X}$ with $0\in\mathfrak{U}$. Suppose that $F:\overline{\mathfrak{U}}\to P_{cp,c}(\mathfrak{X})$ is an upper semicontinuous compact map. Then either F has a fixed point in $\overline{\mathfrak{U}}$ or there are $\mathfrak{x}\in\partial\mathfrak{U}$ and $\lambda\in(0,1)$ such that $\mathfrak{x}\in\lambda F(\mathfrak{x})$.*

Theorem 3.4. *[28](Covitz and Nadler) Let (X,d) be a complete metric space. If $F:X\to P_{cl}(X)$ is a contraction, then F has a fixed point.*

The first existence result can now be introduced. We denote hereafter $\lambda_k=\max_{t\in J}|\lambda_k(t)|$.

Theorem 3.5. *Assume that*

(HA) $F:J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is Carathéodory and has convex values,*

(HB) *there exists a continuous nondecreasing function $\psi:[t_0,\infty)\to(0,\infty)$ and a function $p\in L^1(J,\mathbb{R}^+)$ such that*

$$\|F(t,x)\|=\sup\{|v|:v\in F(t,x)\}\le p(t)\psi(|x|),$$

for each $(t,x)\in J\times\mathbb{R}$,

(HC) *there exist continuous nondecreasing functions $\psi_k:[t_0,\infty)\to(0,\infty)$ and functions $p_k\in L^1(J,\mathbb{R}^+)$ such that*

$$|g_k(t,x)|\le p_k(t)\psi_k(|x|),k=0,1,2,3,4,$$

for each $(t,x)\in J\times\mathbb{R}$,

(HD) *there exists a large number $M>0$ such that*

$$\frac{M}{\gamma_1\psi(M)\|p\|_{L^1}+\gamma_2}>1,$$

where

$$\gamma_1=\frac{(T-t_0)^{q-1}}{\Gamma(q)}+\sum_{k=0}^{4}\frac{|\theta_k|\lambda_k(T-t_0)^{q-k-1}}{k!\Gamma(q-k)|\alpha_k|}$$

and

$$\gamma_2=\sum_{k=0}^{4}\frac{\lambda_k|\beta_k|}{k!|\alpha_k|}\psi_k(M)\|p_k\|_{L^1}.$$

Then the boundary value problem (1.1) has at least one solution on J.

Proof. Using Lemma 3.2, we can define an operator

$$\begin{aligned}\Omega(x) &= \Bigg\{h \in C(J,\mathbb{R}) : h(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ &+ \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds\Bigg\}\end{aligned}$$

for $f \in S_{F,x}$. We show that Ω satisfies the assumptions of the nonlinear alternative of the Leray-Schauder type. The proof consists of several steps.

Step I: We show that $\Omega(x)$ is convex for each $x \in C(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit its proof.

Step II: We show that $\Omega(x)$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C(J,\mathbb{R})$. Then, for each $h \in \Omega(x),\ x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned}h(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds.\end{aligned}$$

Then for $t \in J$, we have

$$\begin{aligned}|h(t)| &\leq \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)|\,ds + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|}|\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s)|\,ds \\ &+ \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|}|\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))|\,ds \\ &\leq \psi(\|x\|)\,\|p\|_{L^1}\left\{\frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k\,|\theta_k|\,(T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|}\right\} \\ &+ \sum_{k=0}^{4} \frac{|\beta_k|\,\lambda_k}{k!\,|\alpha_k|}\psi_k(\|x\|)\,\|p_k\|_{L^1}.\end{aligned}$$

Thus,

$$\begin{aligned}\|h\| &\leq \psi(\|r\|)\,\|p\|_{L^1}\left\{\frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k\,|\theta_k|\,(T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|}\right\} \\ &+ \sum_{k=0}^{4} \frac{\lambda_k\,|\beta_k|}{k!\,|\alpha_k|}\psi_k(\|r\|)\,\|p_k\|_{L^1}.\end{aligned}$$

Step III: We show that Ω maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, and $x \in B_r$ where B_r is a bounded set of $C(J,\mathbb{R})$. In view of (HC), for each

$h \in \Omega(x)$, we obtain

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
= &\left| \int_{t_0}^{t_2} \frac{(t_2-s)^{q-1} - (t_1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s)\,ds \right. \\
&+ \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} (\lambda_k(t_2) - \lambda_k(t_1)) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\
&\left. + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} (\lambda_k(t_2) - \lambda_k(t_1)) \int_{t_0}^{T} g_k(s, x(s))ds \right|
\end{aligned}$$

$$\begin{aligned}
\leq\ &\psi(\|x\|) \int_{t_0}^{t_2} \frac{\left|(t_2-s)^{q-1} - (t_1-s)^{q-1}\right|}{\Gamma(q)} p(s)ds \\
&+\psi(\|x\|) \int_{t_1}^{t_2} \frac{|(t_2-s)^{q-1}|}{\Gamma(q)} p(s)ds \\
&+ \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t_2) - \lambda_k(t_1)|\, \psi(\|x\|) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} p(s)ds \\
&+ \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t_2) - \lambda_k(t_1)|\, \psi_k(\|x\|) \int_{t_0}^{T} p_k(s)ds
\end{aligned}$$

The functions $\lambda_k(t)$ behave like a polynomial function, particularly, $(\lambda_k(t_2) - \lambda_k(t_1)) \to 0$, as $t_2 - t_1 \to 0$. Therefore, the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω satisfies the above three assumptions, it follows by the Arzela-Ascoli theorem that $\Omega : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Step IV: We show that Ω has a closed graph. Let $x_n \to x_*$, $h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned}
h_n(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f_n(s)\,ds \\
&+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s, x_n(s))ds.
\end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in J$,

$$\begin{aligned}
h_*(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f_*(s)\,ds \\
&+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s, x_*(s))ds. \qquad (3.4)
\end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ given by

$$f \quad \to \quad \Theta(f)(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds$$
$$+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds.$$

Observe that

$$\begin{aligned} |h_n(t) - h_*(t)| \quad &\le \quad \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f_n(s) - f_*(s)|\,ds \\ &+ \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f_n(s) - f_*(s)|\,ds \\ &+ \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x_n(s)) - g_k(s,x_*(s))|\,ds. \end{aligned}$$

Thus, it follows by Lemma 2.11 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x})$, since $x_n \to x_*$, then h_* satisfying (3.4) for some $f_* \in S_{F,x_*}$.

Step V: We discuss a priori bounds on solutions. Let x be a solution of (1.1), then there exists $f \in L^1(J,\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in J$, we have

$$\begin{aligned} x(t) \quad &= \quad \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

Using the computations of the second step above, we have

$$\begin{aligned} |x(t)| \quad &\le \quad \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)|\,ds + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s)|\,ds \\ &+ \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))|\,ds \\ &\le \quad \psi(\|x\|) \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k\,|\theta_k|\,(T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|} \right\} \int_{t_0}^{T} p(s)ds \\ &+ \sum_{k=0}^{4} \frac{\lambda_k\,|\beta_k|}{k!\,|\alpha_k|} \psi_k(\|x\|) \int_{t_0}^{T} p_k(s)ds. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\gamma_1 \psi(\|x\|)\,\|p\|_{L^1} + \gamma_2} \le 1.$$

In view of (HD), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of the Leray–Schauder type, we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of problem (1.1). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of the Leray–Schauder type together with the selection theorem of Bressan and Colombo [26] for lower semi-continuous maps with decomposable values.

Theorem 3.6. *Assume that (HB), (HC), (HD), and the following condition holds*

(HE) $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
(a) $(t,x) \to F(t,x)$ *is* $\ell \otimes B$ *measurable,*
(b) $x \to F(t,x)$ *is lower semi-continuous for each* $t \in J$.
Then the boundary value problem (1.1) has at least one solution on J.

Proof. It follows from (HB) and (HE) that F is of $l.s.c.$ type. Then from Lemma 2.12, there exists a continuous function $f : C(J,\mathbb{R}) \to L_1(J,\mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in C(J,\mathbb{R})$. Consider the problem

$$\begin{cases} {}^cD^q_{t_0}x(t) = f(x(t)), t \in J, q \in (4,5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t,x(t))dt, k = 0,1,2,3,4. \end{cases} \tag{3.5}$$

Observe that if $x \in C(J,\mathbb{R})$ is a solution of (3.5), then x is a solution to problem (1.1). In order to transform problem (3.5) into a fixed point problem, we define the operator Π as

$$\begin{aligned} \Pi x(t) &= \int\limits_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int\limits_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(x(s))\,ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int\limits_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

It can easily be shown that Π is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.5, so we omit it. This completes the proof. □

Now we prove the existence of solutions for problem (1.1) with a nonconvex value by applying a fixed point theorem for a multivalued map due to Covitz and Nadler [27].

Theorem 3.7. *Assume that the following conditions hold:*

(HF) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x) : J \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$,

(HG) $d_H(F(t,x),F(t,y)) \le z(t)|x-y|$ *for almost all* $t \in J$ *and* $x,y \in \mathbb{R}$ *with* $z \in L^1(J,\mathbb{R}^+)$ *and* $d(0,F(t,0)) \le z(t)$ *for almost all* $t \in J$.

(HH) *There exist functions* $z_k \in L^1(J,\mathbb{R}^+)$ *such that*

$$|g_k(t,x) - g_k(t,y)| \le z_k(t)|x-y|,$$

for $t \in J$, $k = 0,1,2,3,4$, *and* $x,y \in \mathbb{R}$.

Then the boundary value problem (1.1) has at least one solution on J if

$$\gamma_1 \|z\|_{L^1} + \omega < 1,$$

where

$$\omega = \sum_{k=0}^{4} \frac{\lambda_k \|z_k\|_{L^1} |\beta_k|}{k!\,|\alpha_k|}.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J,\mathbb{R})$ by assumption (HF), so F has a measurable selection (see [29]: Theorem 3.6). Now we show that the operator Ω satisfies the assumptions of Theorem 2.13. To show that $\Omega(x) \in P_{cl}((C(J,\mathbb{R}))$ for each $x \in C(J,\mathbb{R})$, let $(u_n)_{n\geq 0} \in \Omega(x)$ be such that $u_n \to u$ in $C\ (J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in J$,

$$\begin{aligned} u_n(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_n(s)ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$,

$$\begin{aligned} u_n(t) &\to u(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v(s)ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

Hence, $u \in \Omega(x)$. Next we show that there exists $\tau < 1$ such that

$$d_H(\Omega(x),\Omega(y)) \leq \tau \left\|x-y\right\|,$$

for each $x, y \in C(J,\mathbb{R})$. Let $x, y \in C(J,\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists

$$v_1(t) \in F(t,x(t)),$$

such that, for each $t \in J$,

$$\begin{aligned} h_1(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_1(s)ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

By (HG), we have

$$d_H(F(t,x),F(t,y)) \leq z(t)|x(t)-y(t)|.$$

So, there exists $w_* \in F(t,y(t))$ such that

$$|v_1(t)-w_*| \leq z(t)|x(t)-y(t)|,\ t \in J.$$

Define $V : J \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w_* \in \mathbb{R} : |v_1(t)-w_*| \leq z(t)\,|x(t)-y(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t,y(t))$ is measurable ([29]: Proposition 3.4), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t,y(t))$ and for each $t \in J$, we have $|v_1(t)-v_2(t)| \leq z(t)|x(t)-y(t)|$. Define

$$\begin{aligned} h_2(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_2(s)ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

Thus, for each $t \in J$, it follows that

$$\begin{aligned}
|h_1(t) - h_2(t)| &\le \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\, ds \\
&\quad + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |v_1(s) - v_2(s)|\, ds \\
&\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s, x(s)) - g_k(s, y(s))|\, ds \\
&\le \left\{ \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k |\theta_k| (T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|} \right\} \|z\|_{L^1} \right. \\
&\quad \left. + \sum_{k=0}^{4} \frac{\lambda_k \|z_k\|_{L^1} |\beta_k|}{k!\,|\alpha_k|} \right\} \|x - y\|.
\end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le (\gamma_1 \|z\|_{L^1} + \omega)\, \|x - y\|.$$

We deduce that

$$\begin{aligned}
d_H(\Omega(x), \Omega(y)) &\le (\gamma_1 \|z\|_{L^1} + \omega)\, \|x - y\| \\
&\le \tau \|x - y\|.
\end{aligned}$$

Since Ω is a contraction, it follows by Theorem 2.13 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

We close this article by introducing a couple of examples.

Example 3.8. Consider the following fractional differential inclusion

$$\begin{cases} {}^{c}D_0^{4.2} x(t) \in F(t, x(t)),\, t \in [0,1], \\ x^{(k)}(0) = -x^{(k)}(1) + k \int_0^1 e^{-x(t)} dt,\, k = 0,1,2,3,4. \end{cases} \tag{3.6}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t,x) = \left\{ y \in \mathbb{R} : 0 \le y \le 1 + \frac{t|x|}{1 + t|x|} \right\}.$$

Observe that

$$t \to F(t,x) = \left\{ y \in \mathbb{R} : 0 \le y \le 1 + \frac{t|x|}{1 + t|x|} \right\}$$

is measurable for each $x \in \mathbb{R}$, since each lower and upper functions are both measurable on $[0,1] \times \mathbb{R}$. Now let (t_0, x_0) be any element in $[0,1] \times \mathbb{R}$. Then

$$F(t_0, x_0) = \left\{ y \in \mathbb{R} : 0 \le y \le 1 + \frac{t|x_0|}{1 + t|x_0|} \right\}$$

is a closed subset of $[0,1] \times \mathbb{R}$ and

$$\left\{ y \in \mathbb{R} : 0 \le y \le 1 + \frac{t|x_0|}{1 + t|x_0|} \right\} \ne \phi.$$

Let O be any open interval in $\mathbb{R}$ such that

$$\left\{ y \in \mathbb{R} : 0 \le y \le 1 + \frac{t|x_0|}{1 + t|x_0|} \right\} \subset O,$$

we can find an open intervals U of t_0 and V of x_0 with $[0,1] \subset U = (-\varepsilon, 1+\varepsilon)$ and $F(u,v) \subset O$ for every $u \in U$ and $v \in V$, where ε is a small positive real number. So $x \to F(t,x)$ is upper semi-continuous for all $t \in J$. Thus F is a Carathéodory and clearly has convex values. Clearly

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq 2,$$

and

$$|g_k(t,x)| \leq k, k = 0,1,2,3,4,$$

for each $(t,x) \in [0,1] \times \mathbb{R}$. Assume that $p(t) = 1$, $p_k(t) = k$, $\psi(|x|) = 2$, and $\psi_k(|x|) = 1$ for $k = 0,1,2,3,4$. Simple calculations lead to

$$\begin{aligned} \lambda_0(t) &= 1 \\ \lambda_1(t) &= -1+2t \\ \lambda_2(t) &= -8t+4t^2 \\ \lambda_3(t) &= 2-36t^2+8t^3 \\ \lambda_4(t) &= -5+64t-128t^3, \end{aligned}$$

$$\begin{aligned} \gamma_1 &= \frac{1}{7.76} + \frac{1}{2(7.76)} + \frac{1}{4(2.424)} + \frac{4}{2(1.1)8} + \frac{26}{6(0.918)16} + \frac{69}{24(4.59)32} \\ &= 0.129 + 0.0645 + 0.103 + 0.227 + 0.295 + 0.02 = 0.839 \end{aligned}$$

and

$$\gamma_2 = \left(\frac{1}{4} + \frac{8}{16} + \frac{78}{96} + \frac{276}{768}\right) = 0.359375 + 0.8125 = 1.922.$$

Furthermore, let M be any number satisfying

$$M > \gamma_1 \psi(|x|) \|p\|_{L^1} + \gamma_2 = 3.6.$$

Clearly, all the conditions of Theorem 3.5 are satisfied. So there exists at least one solution of problem (3.6) on $[0,1]$.

Example 3.9. Consider the following fractional differential inclusion

$$\begin{cases} {}^cD_0^{\frac{19}{4}} x(t) \in F(t, x(t)), t \in [0,1], \\ x^{(k)}(0) = -x^{(k)}(1), k = 0,1,2,3,4, \end{cases} \tag{3.7}$$

where $F : [0,1] \times [0, \frac{\pi}{2}] \to \mathbb{R}^+$ is a multivalued map given by

$$F(t,x) = \left[0, \frac{\sin x}{(2+t)^4}\right].$$

Now

$$\begin{aligned} \sup\{|y| &: y \in F(t,x)\} \\ &\leq \frac{\sin x}{(2+t)^4} \\ &\leq \frac{1}{16} \text{ for each } (t,x) \in [0,1] \times \left[0, \frac{\pi}{2}\right], \end{aligned}$$

and

$$
\begin{aligned}
d_H(F(t,x),F(t,y)) &= d_H\left(\left[0,\frac{\sin x}{(2+t)^4}\right],\left[0,\frac{\sin y}{(2+t)^4}\right]\right)\\
&= \max\left\{d^*\left(\left[0,\frac{\sin x}{(2+t)^4}\right],\left[0,\frac{\sin y}{(2+t)^4}\right]\right)\right.\\
&\qquad \left.,d^*\left(\left[0,\frac{\sin y}{(2+t)^4}\right],\left[0,\frac{\sin x}{(2+t)^4}\right]\right)\right\}\\
&= \max\left\{\sup\left\{d\left(a,\left[0,\frac{\sin y}{(2+t)^4}\right]\right) : a\in\left[0,\frac{\sin x}{(2+t)^4}\right]\right\},\right.\\
&\qquad \left.\sup\left\{d\left(\left[0,\frac{\sin x}{(2+t)^4}\right],b\right) : b\in\left[0,\frac{\sin y}{(2+t)^4}\right]\right\}\right\}\\
&\leq \frac{1}{(2+t)^4}|x-y|.
\end{aligned}
$$

Here $z(t)=\frac{1}{(2+t)^4}$, with $\|z\|_{L^1}\approx 0.017$, and observe that $\omega=0$, because of the absence of the integral boundary condition which implies that $\gamma_1\|z\|_{L^1}+\omega<1$. The compactness of F together with the above calculations lead to the existence of solution of the problem (3.7) by Theorem 3.7.

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