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ON R-STAR-LINDELÖF SPACES

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Abstract. Motivated by the recent works of Lj.D.R. Kočinac, we introduce a new type of star-Lindelöfness which is termed as R-star-Lindelöfness. A topological space X will be called R-star-Lindelöf if for every sequence $\{D_n : n \in \omega\}$ of dense subsets of X and for every open cover $\mathcal{U}$ of X there exist points $x_n \in D_n$ for each $n \in \omega$ such that $St(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}) = X$. We took interest in studying the properties of this space because every R-separable space is R-star-Lindelöf space but not every R-star-Lindelöf space is R-separable.

1 Introduction

Scheepers [11] introduced a number of combinatorial properties of a topological space weaker than separability. M-separability (or selective separability) and R-separability are two of them which have many interesting properties. The topological properties star-Lindelöfness and star-compactness was introduced by van Dowen [4] in 1991. Recently Kočinac [6,7,8,9] studied a lot of selection properties by combining the concepts of Scheepers and van Dowen and got many interesting results.

For any topological space X, $\tau(X)$ will denote its topology. If $A \subseteq X$ and $\mathcal{U}$ is a collection of subsets of X, then the star of A with respect to $\mathcal{U}$ is denoted by $St(A, \mathcal{U})$ and defined as St(A, $\mathcal{U}$)=$\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We assume St^1($A$, $\mathcal{U}$)=St(A, $\mathcal{U}$) and for each $k \in \mathbb{N}$ we define St^{k+1}(A, $\mathcal{U}$) = St(St^k(A, $\mathcal{U}$), $\mathcal{U}$) [4]. Any separable space is always a star-Lindelöf space. Bhowmik et. al [3] introduced the notion of selectively star-Lindelöf space or in other word M-star-Lindelöf space and proved that any M-separable space is a M-star Lindelöf space. In this paper we introduce and study a new type of star-Lindelöf spaces, stronger than M-star-Lindelöfness and weaker than R-separablility.

2 Preliminaries

A topological space X is said to be separable if it has a countable dense subset.

Definition 2.1. [4] A topological space X is said to be star-Lindelöf if for any open cover $\mathcal{U}$ of a space X there exist a countable subset F of X such that St(F, $\mathcal{U}$)=X.

Recently the notion of selective separability has received a great attention in [2].

Definition 2.2. [11] A topological space X is said to be selectively separable if for any sequence $\{D_n : n \in \omega\}$ of dense subsets of X, there exists a family $\{F_n : n \in \omega\}$ of finite subsets of X such that $F_n \subseteq D_n$ for each $n \in \omega$ and $\cup_{n\in\omega} F_n$ is dense in X.

Definition 2.3. [11] A topological space X is said to be R-separable if for any sequence $\{D_n : n \in \omega\}$ of dense subsets of X, there are points $x_n \in D_n (n \in \omega)$ such that $\cup_{n\in\omega}\{x_n\}$ is dense in X.

Definition 2.4. [1] A topological space X has countable fan tightness if for any $x \in$X and for any sequence $\{U_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap_{n\in\omega} \overline{U_n}$, we can choose finite subsets $B_n \subseteq U_n$ such that $x \in \overline{\bigcup_{n\in\omega} B_n}$.

Definition 2.5. [3] A topological space X is said to be selectively star-Lindelöf (or M-star-Lindelöf) if for every sequence $\{D_n : n \in \omega\}$ of dense subsets of X and for every open cover $\mathcal{U}$ of X there exist a family $\{F_n : n \in \omega\}$ of finite subsets of X such that $F_n \subseteq D_n$ for each $n \in \omega$ and St($\bigcup_{n\in\omega} F_n$,$\mathcal{U}) = X$.

For different notions of topology we follow [5] and [13]. In this paper no separation axiom is considered, unless otherwise stated.

3 R-Star-Lindelöf Spaces

In this section, we introduce a star version of selective Lindelöfness and study some of its properties.

Definition 3.1. A topological space X is said to be R-star-Lindelöf if for every sequence $\{D_n : n \in \omega\}$ of dense subsets of X and for every open cover $\mathcal{U}$ of X there are points $x_n \in D_n$ for each $n \in \omega$ such that St($\bigcup_{n\in\omega}\{x_n\}$, $\mathcal{U}$)=X.

Definition 3.2. A subset Y of a topological space X is said to be R-star-Lindelöf with respect to X if for every sequence $\{D_n : n \in \omega\}$ of subsets of X such that $Y \subseteq \bar{D}_n$ for each $n \in \omega$ and for every cover $\mathcal{U}$ of Y by sets open in X there are points $x_n \in D_n$ for each $n \in \omega$ such that $Y \subseteq St(\bigcup_{n\in\omega}\{x_n\},\mathcal{U})$.

Theorem 3.3. *If Y is an open subset of X, then Y is R-star-Lindelöf with respect to X iff Y is R-star-Lindelöf subspace of X.*

Theorem 3.4. *A topological space X is R-star-Lindelöf if and only if for every sequence $\{D_n : n \in \omega\}$ of dense subsets of X and basic open cover $\mathcal{U}_B$ there are points $x_n \in D_n$ $(n \in \omega)$ such that $St(\cup_{n\in\omega}\{x_n\},\mathcal{U}_B) = X$.*

Proof. If X is R-star-Lindelöf space, then the condition is trivial.

Conversely, let the given condition holds. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets of X and $\mathcal{U}$ be any open cover of X. Let $\mathcal{B}$ be an open base for $\tau(X)$. Let $\mathcal{U}_\mathcal{B} = \{B \in \mathcal{B} : B \subseteq U,$ for some $U \in \mathcal{U}\}$. So $\mathcal{U}_\mathcal{B}$ is an basic open cover of X. Therefore, by the given condition, there are points $x_n \in D_n$ such that $St(\cup_{n\in\omega}\{x_n\},\mathcal{U}_\mathcal{B}) = X$, i.e. $St(\cup_{n\in\omega}\{x_n\},\mathcal{U}) = X$. Hence the theorem. □

Theorem 3.5. *If there exist two open R-star-Lindelöf subspaces A and B of a space X such that $A \cup B = X$, then X is a M-star Lindelöf space.*

Proof. Let A and B be two open R-star-Lindelöf subspaces of X such that $X = A \cup B$.

Let $\mathcal{U}$ be an arbitrary open cover of X and $\{D_n : n \in \omega\}$ be any sequence of dense subsets of X. $\mathcal{U}^A$ and $\mathcal{U}^B$ be basic open covers of A and B respectively.

Clearly, $\{(D_n \cap A) : n \in \omega\}$ is a sequence of dense subsets in A and $\{(D_n \cap B) : n \in \omega\}$ is a sequence of dense subsets in B. By R-star-Lindelöfness of A and B, for every $n \in \omega$, there are points $x_n^{'} \in (D_n \cap A)$ and $x_n^{''} \in (D_n \cap B)$ such that $St(\bigcup_{n\in\omega}\{x_n^{'}\},\mathcal{U}^A) = A$ and $St(\bigcup_{n\in\omega}\{x_n^{''}\},\mathcal{U}^B) = B$. Thus, for each $n \in \omega$, $x_n^{'} \in D_n$ and $x_n^{''} \in D_n$, i.e. for each $n \in \omega$, $F_n = \{x_n^{'}, x_n^{''}\} \subseteq D_n$ and $St(\bigcup_{n\in\omega} F_n,\mathcal{U}^A \cup \mathcal{U}^B) = A \cup B = X$.

Let, $U \in \mathcal{U}$. Then either $U \subseteq A$ or $U \subseteq B$ or $U \cap A \neq \emptyset \neq U \cap B$. If $U \subseteq A$, then U can be expressed as the union of some members of $\mathcal{U}^A$. If $U \subseteq B$, then U can be expressed as the union of some members of $\mathcal{U}^B$.

Let, $U \nsubseteq A$, $U \nsubseteq B$ and $U \subseteq A \cup B$.

Then, $U \cap A$ can be expressed as the union of some members of $\mathcal{U}^A$ and $U \cap B$ can be expressed as the union of some members of $\mathcal{U}^B$. Thus $U = U \cap X = U \cap (A \cup B) = (U \cap A) \cup (U \cap B)$ can be expressed as the union of some members of $(\mathcal{U}^A \cup \mathcal{U}^B)$.

Therefore, every element of $\mathcal{U}$ contains some members of $(\mathcal{U}^{\mathcal{A}} \cup \mathcal{U}^{\mathcal{B}})$.
Therefore, $St(\bigcup_{n\in\omega} F_n, \mathcal{U}) = X$.
Hence the theorem. □

Corollary 3.6. *If there exist finite number of open R-star-Lindelöf subspaces $A_1, A_2, A_3, ...A_k$, such that $\bigcup_{i=1}^{k} A_k = X$, then X is an M-star-Lindelöf space.*

Theorem 3.7. *Every clopen subspace of a R-star-Lindelöf space is a R-star-Lindelöf space.*

Proof. Let X be a R-star-Lindelöf space and Y be a clopen subspace of X. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets of Y and $\mathcal{U}$ be an open cover of Y in the subspace Y. Now $\{D_n \cup (X - Y) : n \in \omega\}$ is a sequence of dense subsets in X and $\mathcal{U} \cup \{X - Y\}$ is an open cover of X. By R-star-Lindelöfness of X there are points $x_n \in D_n \cup \{X - Y\} (n \in \omega)$ such that St($\bigcup_{n\in\omega}\{x_n\}, \mathcal{U} \cup \{X - Y\}$)=$X$.

Now, we choose those x_n which belongs to D_n for each $n \in \omega$. Then St($\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}) = Y$, i.e. Y is a R-star-Lindelöf space. □

Theorem 3.8. *Every Lindelöf space is a R-star-Lindelöf space.*

Proof. Let, X be a Lindelöf space. $\{D_n : n \in \omega\}$ be a sequence of dense subsets of X and $\mathcal{U} = \{U_\alpha : \alpha \in \wedge\}$ be an open cover of X. So, there exists a countable subcover $\mathcal{U}' = \{U_n : n \in \omega\}$ of $\mathcal{U}$. Without loss of generality we can suppose each U_n is non-empty. For each $n \in \omega$, we choose a $x_n \in D_n \cap U_n$.

$\therefore$, St($\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}'$) $= X$ i.e., St($\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}) = X$.
Hence X is R-star-Lindelöf space. □

Corollary 3.9. *Every σ-compact space is a R-star-Lindelöf space.*

Corollary 3.10. *Every compact space is a R-star-Lindelöf space.*

Theorem 3.11. *Every R-separable space is a R-star-Lindelöf space.*

Proof. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets of X and $\mathcal{U}$ be an open cover of X. Since X is R-separable, there are points $x_n \in D_n (n \in \omega)$ such that $\bigcup_{n\in\omega}\{x_n\}$ is dense in X. Hence St($\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}$)=$X$, i.e. X is R-star-Lindelöf. □

Example 3.12. The converse of Theorem 3.11 may not be true.

Let $|X| \geq \omega_1$ and X is equipped with the co-countable topology. Then X has no countable dense subset, hence it can not be R-separable.

Let $\mathcal{U}$ be an open cover of X and $\{D_n : n \in \omega\}$ a sequence of dense subsets of X. First we take $U_0 \in \mathcal{U}$. Clearly U_0 is of the form $U_o = X \setminus C$, where $C = \{y_i : i \in \omega\}$ is a countable subset of X. Now, $U_0 \cap D_0 \neq \emptyset$. We select $x_0 \in U_0 \cap D_0$.

Since $\mathcal{U}$ is an open cover of X, there exists $V_n \in \mathcal{U}$ such that $y_n \in V_n \in \mathcal{U}$ for each $n \in \omega$. Since each D_n is dense in X we have $V_n \cap D_{n+1} \neq \emptyset$ for each $n \in \omega$. We select $x_n' \in V_n \cap D_{n+1}$ for each $n \in \omega$. i.e. $x_n' \in D_{n+1}$ for each $n \in \omega$.

$$\text{Therefore, } U_0 \subseteq St(x_0, \mathcal{U}),$$

$$y_0 \in V_0 \subseteq St(x_0', \mathcal{U}),$$

$$y_1 \in V_1 \subseteq St(x_1', \mathcal{U}),$$

$$.$$

$$.$$

$$.$$

$$y_n \in V_n \subseteq St(x_n', \mathcal{U}),$$

$$.$$

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.

$\therefore U_0 \cup \{y_1, y_2, y_3, ..., y_n, ...\} \subseteq U_0 \cup V_0 \cup V_1 \cup ... \cup V_n \cup \subseteq St(\{x_0, x_0^{'}, x_1^{'}, ..., x_n^{'}, ...\}, \mathcal{U})$.
i.e. $X = St(\{x_0, x_0^{'}, x_1^{'}, ..., x_n^{'}, ...\}, \mathcal{U})$.
$\therefore X$ is R-star-Lindelöf space.

Theorem 3.13. *Let $f : X \to Y$ be an open continuous surjection. If X is a $\mathbb{R}$-star-Lindelöf space, then so is also Y.*

Proof. Let $\{E_n : n \in \omega\}$ be a sequence of dense subsets of Y and let $\mathcal{V}$ be an open cover of Y. Then $\{D_n$=$f^{-1}(E_n)$: $n \in \omega\}$ is a sequence of dense subsets of X and $\mathcal{U}$ =$\{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of X. Now by the property of R-star-Lindelöfness of X there are points $x_n \in D_n$, $(n \in \omega)$ such that St$(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}) = X$.

Let y_n=$f(x_n)$, $n \in \omega$. Clearly, $y_n \in E_n$ for each $n \in \omega$.

Let $y \in Y$. So there exists $x \in X$ such that $f(x) = y$. Also there exists an $n \in \omega$ such that $x \in$St$(\{x_n\},\mathcal{U})$, i.e. there exists $U = f^{-1}(V)$ for some $V \in \mathcal{V}$ such that $x \in U$ and $\{x_n\} \cap U \neq \phi, \therefore x_n \in U$.

Thus, $y \in V$ and $y_n \in V$, i.e., $y \in$St$(\{y_n\}, \mathcal{V})$.

Then, St$(\bigcup_{n\in\omega}\{y_n\}, \mathcal{V}) = Y$.

Hence Y is R-star-Lindelöf. □

Theorem 3.14. *Every R-star Lindelöf space is a selectively star-Lindelöf space.*

Remark 3.15. Every R-separable space is M-separable. Soukup et. al [12] have shown that the space $Fn(\omega, \omega; \omega)$ of all finite partial functions from ω to ω is a countable M-separable non-R-separable space.

Bhowmik et. al [3] had shown that every M-separable space is M-star-Lindelöf space and there exists a space (the Tychonoff cube $\mathbb{I}^c$) which is M-star-Lindelöf but not M-separable.

In this paper, we have shown that every R-separable space is R-star-Lindelöf but not every R-star-Lindelöf space is R-separable.

Also every R-star-Lindelöf space is selectively star-Lindelöf space.

Thus we have,

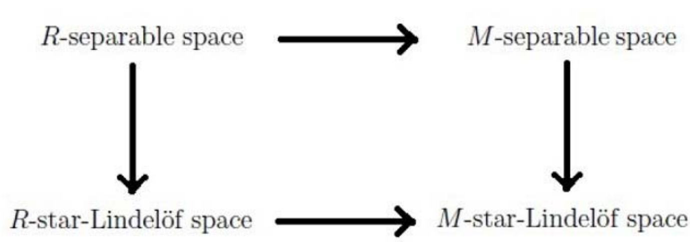

Figure 1. Relation chart

Problem 3.16. Does there exists a space which is M-star-Lindelöf but not $\mathbb{R}$-star-Lindelöf?

4 R-k-Star-Lindelöf Space

In this section we assume that $\mathbb{N} = \omega \setminus \{0\}$. Here we study the iterative star version of R-star-Lindelöf spaces.

Definition 4.1. For each $k \in \mathbb{N}$, a topological space X is said to be R-k-star-Lindelöf if for every sequence $\{D_n : n \in \omega\}$ of dense subsets of X and for every open cover $\mathcal{U}$ of X there are points $x_n \in D_n$ such that $\mathrm{St}^k(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}) = X$.

Theorem 4.2. *Every R-k-star-Lindelöf space is a R-(k+1)-star-Lindelöf space.*

Proof. Directly follows from the definition, hence omitted. □

We recall the definition of star-separability. Given a class (or a property) $\mathcal{P}$ of topological spaces. We say that a space X is star-$\mathcal{P}$ if, for any open cover $\mathcal{U}$ of the space X, there is a subspace $Y \subseteq X$ such that $Y \in \mathcal{P}$ and $St(Y, \mathcal{U}) = X$ [10]. So, a topological space X is said to be star-separable, if for every open cover $\mathcal{U}$ of X, there exists a separable subspace Y of X such that St($Y, \mathcal{U}$)=X.

Theorem 4.3. *If X is star-separable, then X is R-2-star-Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X and $\{D_n : n \in \omega\}$ be a sequence of dense subsets of X. Since X is star-separable, there exists a separable subspace Y of X such that St(Y, $\mathcal{U}$)=X. Now there exits a countable subset $B = \{x_n : n \in \omega\}$ of Y such that $\overline{B}^{\tau(Y)} = Y$, hence $Y \subseteq \overline{B}^{\tau(Y)} \subseteq \overline{B}^{\tau(X)} = \overline{B}$. Now for each n there exists $U_n \in \mathcal{U}$ such that $U_n \cap D_n \neq \emptyset$. We take $x_n \in U_n \cap D_n$.

Let $x \in X$. Since St(Y, $\mathcal{U}$)=X, there exists $U \in \mathcal{U}$ such that $x \in U$ and $U \cap Y \neq \emptyset$, and so $U \cap Y \cap B \neq \emptyset$. Let $k \in \omega$ be such that $x_k \in U \cap Y \cap B$. Then $U \cap U_k \neq \emptyset$, hence $U \cap$St($\bigcup_{n\in\omega}\{x_n\}$, $\mathcal{U}$)$\neq \emptyset$ and so $x \in \mathrm{St}^2(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$. Therefore $\mathrm{St}^2(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$=$X$ and thus X is R-2-star-Lindelöf. □

Corollary 4.4. *A separable space is a R-2-star-Lindelöf space.*

Theorem 4.5. *If X is a R-star-Lindelöf space and Y is a compact space, then $X \times Y$ is a R-2-star-Lindelöf space.*

Proof. Let, $\mathcal{U}$ be an open cover of $X \times Y$ by basic open sets of $X \times Y$.

Now by Remark 1.4 of [1], for each $x \in X$ there exists a open neighborhood W_x of x in X such that $W_x \times Y$ is covered by finite number of elements of $\mathcal{U}$, say $W_x \times Y \subseteq \bigcup\{U_k(x) \times V_k(x) : 1 \leqslant k \leqslant n_x\}$, where $W_x \subseteq \bigcap_{1\leqslant k\leqslant n_x} U_k(x)$.

Now, $\mathcal{U}_X = \{W_x : x \in X\}$ is an open cover of X. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets of $X \times Y$, $\pi_X : X \times Y \to X$ be the natural projection from $X \times Y$ to X. Then $\{\pi_X(D_n) : n \in \omega\}$ is a sequence of dense subsets of X. By the R-star-Lindelöfness there are points $x_n \in \pi_X(D_n)$ for each $n \in \omega$ and $St(\cup_{n\in\omega}\{x_n\}, \mathcal{U}_X) = X$.

For each x_n,we choose $(x^k, y^k) \in D_n \cap (U_k(x) \times V_k(x))$, $1 \leq k \leq n_x$.

Let, $(x, y) \in X \times Y$.

So, there exists W_{x_0} such that $x \in W_{x_0}$ and $W_{x_0} \cap (\cup_{n\in\omega}\{x_n\}) \neq \emptyset$.

Let, $x_p \in W_{x_0} \cap (\bigcup_{n\in\omega}\{x_n\})$, for some $p \in \omega$.

So, there exists, $(x^{k'}, y^{k'}) \in D_n \cap (U_{k'}(x_p) \times V_{k'}(x_p)), 1 \leqslant k' \leqslant n_x$.

$\Rightarrow (x^{k'}, y^{k'}) \in (U_{k'}(x_p) \times V_{k'}(x_p)), 1 \leqslant k' \leqslant n_x$

$\Rightarrow W_{x_p} \times Y \subseteq \mathrm{St}(\bigcup_{k\in\omega}\{(x^k, y^k)\}, \cup_{k=1}^{n_x}(U_k(x_p) \times V_k(x_p))) \subseteq \mathrm{St}(\bigcup_{k\in\omega}\{(x^k, y^k)\}, \mathcal{U})$.

Also, $W_{x_0} \times Y \cap W_{x_p} \times Y \neq \emptyset$,

$\therefore W_{x_0} \times Y \subseteq \mathrm{St}^2(\bigcup_{k\in\omega}\{(x^k, y^k\}), \mathcal{U})$

$\Rightarrow (x, y) \in \mathrm{St}^2(\bigcup_{k\in\omega}\{(x^k, y^k\}), \mathcal{U})$

So, $X \times Y = \mathrm{St}^2(\bigcup_{k\in\omega}\{(x^k, y^k)\}\, \mathcal{U})$.

Hence the theorem.

□

Applying Theorem 4.5, by mathematical induction we get the following corollary :

Corollary 4.6. *If X is a R-star-Lindelöf space and $Y_1, Y_2, ..., Y_n$ are compact spaces, then $X \times Y_1 \times Y_2 \times ... \times Y_n$ is a R-$(n+1)$-star-Lindelöf space.*

Theorem 4.7. *If X is star-Lindelöf and has the property that for any $x \in X$, for any sequence $\{U_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap_{n\in\omega} \overline{U_n}$ and for any open cover $\mathcal{U}$ of X we can choose points $x_n \in U_n$ with $x \in$St$(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$, then X is R-2-star-Lindelöf.*

Proof. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets of X and $\mathcal{U}$ be an open cover of X. Since X is star-Lindelöf, there exists a countable subset $F = \{x_n : n \in \omega\}$ of X such that St(F, $\mathcal{U}$)=X.

Let $L = \{L_n : n \in \omega\}$ be a sequence of disjoint infinite subsets of ω, such that ω=$\bigcup_{n\in\omega} L_n$.

Now $x_n \in \bigcap\{\overline{D_k} : k \in L_n\}$. So there are points $x_k \in D_k$ for each $k \in L_n$ such that $x_n \in St(\bigcup_{k\in L_n}\{x_k\}, \mathcal{U})$, $n \in \omega$.

Hence we have points $x_n \in D_n$ for each $n \in \omega$.

Let $x \in X$. There exists $U \in \mathcal{U}$ such that $x \in U$. Since $F \cap U \neq \emptyset$, there exists $x_n \in U$ for some $n \in \omega$. Since $x_n \in$St$(x_k, \mathcal{U})$ for some $k \in L_n$, we can choose $V \in \mathcal{U}$ such that $x_n \in V$ and $x_k \in V$. Then $V \subseteq$St$(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$ and $U \cap V \neq \emptyset$, so St$(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})\cap U \neq \emptyset$, therefore $x \in$St$^2(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$, i.e. St$^2(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U}) = X$. Hence X is R-2-star-Lindelöf. □

Definition 4.8. Let $k \in \mathbb{N}$. A subset Y of topological space X is said to be R-k-star-Lindelöf with respect to X (or Y is a R-k-star-Lindelöf subset of X) if for every sequence $\{D_n : n \in \omega\}$ of subsets of X such that $Y \subseteq \overline{D_n}$ for each $n \in \omega$ and for every open cover $\mathcal{U}$ of Y by the open sets in X there are $x_n \in D_n$ for each $n \in \omega$ such that $Y \subseteq$St$^k(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$.

Theorem 4.9. *If A is a R-k-star-Lindelöf subset of a topological space X, then A is also a R-$(k+1)$-star-Lindelöf subset of X.*

Theorem 4.10. *If A is a R-star-Lindelöf subset of a topological space X and $A \subseteq B \subseteq \overline{A}$, then B is a R-2-star-Lindelöf subset of X.*

Proof. Let $\{D_n : n \in \omega\}$ be a sequence of subsets of X such that $B \subseteq \overline{D_n}$ for each $n \in \omega$, so $A \subseteq \overline{D_n}$ for each $n \in \omega$. Let $\mathcal{U}$ be an open cover of B, so also an open cover of A. Then there are points $x_n \in D_n$ for each $n \in \omega$ such that $A \subseteq$St$(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$.

Now let $x \in B$. So $x \in \overline{A}$ so that there exists a $U \in \mathcal{U}$ such that $x \in U$. Let $y \in U\cap A$. Then there exists $V \in \mathcal{U}$ and $n \in \omega$ with $y \in V$ and $x_n \in V$. So $U \cap V \neq \emptyset$, thus $U\cap$St$(\cup_{n\in\omega}\{x_n\}$, $\mathcal{U}) \neq \emptyset$. Therefore, $x \in$St$^2(\bigcup_{n\in\omega}\{x_n\}$, $\mathcal{U})$. So $B \subseteq$St$^2(\bigcup_{n\in\omega}\{x_n\}, \mathcal{U})$, i.e. B is R-2-star-Lindelöf subset. □

Corollary 4.11. *If a topological space X has a R-star-Lindelöf dense subset, then X is R-2-star-Lindelöf.*

Problem 4.12. If A and B are R-star-Lindelöf subspaces of a topological space X such that $X = A \cup B$, is X a R-2-star-Lindelöf space?

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