Palestine Journal of Mathematics

Vol. 6(2)(2017) , 487–495

Generalized derivations and multilinear polynomials in prime rings

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Communicated by Ayman Badawi

MSC 2010 Classifications: 16W25, 16W80, 16N60.

Keywords and phrases: Prime ring, derivation, generalized derivation, multilinear polynomial.

First named-author is supported by a grant from Science and Engineering Research Board (SERB), DST, New Delhi, India. Grant No. SR/S4/MS:852/13.

Abstract. Let R be a prime ring, I a nonzero right ideal of R, U the two sided Utumi quotient ring of R, $C = Z(U)$ extended centroid of R, $f(x_1, \ldots, x_n)$ a nonzero multilinear polynomial over C and $m \geq 1$ a fixed integer. We prove that if F is a generalized derivation of R such that $(F(f(x_1, \ldots, x_n))^m = f(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in I$, then one of the following holds:

(i) $IC = eRC$ some idempotent $e \in Soc(RC)$ and $f(x_1, \ldots, x_n)$ is central valued on $eRCe$;

(ii) $m = 1$ and there exist $\alpha, \lambda \in C$ and $a \in U$ such that $F(x) = (a + \lambda)x$ for all $x \in R$, with $(a - \alpha)I = 0$ and $\alpha + \lambda = 1$.

1 Introduction

Throughout this paper R always denotes an associative prime ring with center $Z(R)$, U its Utumi ring of quotients and C extended centroid of R (see [2] for more details). For any pair of elements $x, y \in R$, the commutator $[x, y] = xy - yx$ and skew commutator $x \circ y = xy + yx$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

In [11], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I such that $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, then I is central ideal. In particular, if $I = R$, then R is commutative. These results of Daif and Bell was extended by Hongan in [17] to the central case. In [17], Hongan proved that if R is a 2-torsion free semiprime ring and I a nonzero ideal of R, then I is central if and only if $d([x, y]) - [x, y] \in Z(R)$ or $d([x, y]) + [x, y] \in Z(R)$ for all $x, y \in I$.

Recently in [14], De Filippis and Huang studied the situation $(F([x, y]))^n = [x, y]$ for all $x, y \in I$, where I is a nonzero ideal in a prime ring R, F a generalized derivation of R and $n \geq 1$ fixed integer. In this case they conclude that either R is commutative or $n = 1$ and $F(x) = x$ for all $x \in R$.

In [1], Argac and Inceboz studied the situation $d(x \circ y)^n = x \circ y$ for all x, y in some nonzero ideal of prime ring R. More precisely, they proved the following:

Let R be a prime ring, I a nonzero ideal of R, d a derivation of R and n a fixed positive integer. (i) If $d(x \circ y)^n = x \circ y$ for all $x, y \in I$, then R is commutative. (ii) If char $(R) \neq 2$ and $d(x \circ y)^n - x \circ y \in Z(R)$ for all $x, y \in I$, then R is commutative. Very recently, Huang [18] proved the following:

Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

In the present paper, we study the situations when (i) $(F(f(x_1, \ldots, x_n)))^m - f(x_1, \ldots, x_n) = 0$; (ii) $(F(f(x_1, \ldots, x_n)))^m - f(x_1, \ldots, x_n) \in Z(R)$; for all $x_1, \ldots, x_n$ in some subsets of R, where $f(x_1, \ldots, x_n)$ is a nonzero multilinear polynomial over C and $m \geq 1$ is an integer.

Let R be a prime ring, U be the Utumi quotient ring of R and $C = Z(U)$, the center of U. Note that U is also a prime ring with C a field. We will make use of the following notation extensively: $f(x_1,\ldots,x_n) = x_1x_2\ldots x_n + \sum_{I\neq\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(n)}$, where S_n is the permutation group over n elements and $\alpha_\sigma \in C$. We denote by $f^d(x_1,\ldots,x_n)$ the polynomial obtained from $f(x_1,\ldots,x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma.1)$. Thus we write

$$d(f(x_1,\ldots,x_n)) = f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n).$$

Denote by $U *_C C\{X_1,\ldots,X_n\}$ the free product of the C-algebra U and $C\{X_1,\ldots,X_n\}$, the free C-algebra in noncommuting indeterminates $X_1,\ldots,X_n$. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1,x_2,x_3,x_4) = \sum_{\sigma\in S_4}(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or an odd permutation in the symmetric group s_4.

Now we need the following facts to prove our theorems.

Fact 1. It is well known that any derivation of R can be uniquely extended to a derivation of U (see [23, Lemma 2]).

Fact 2. Let I be a nonzero two-sided ideal of R. Then I, R, U satisfy the same generalized polynomial identities with coefficients in U (see [6]).

Fact 3. Let I be a nonzero two-sided ideal of R. Then I, R and U satisfy the same differential identities with coefficients in U (see [23, Theorem 2]).

Fact 4. Let I be a nonzero right ideal of R. If I satisfies a nontrivial polynomial identity, then RC is a primitive ring with $soc(RC) \neq 0$ and $IC = eRC$ for some idempotent $e = e^2 \in soc(RC)$ (see [22, Proposition]).

Fact 5. Let I be a nonzero right ideal of R and $a \in U$. If for every $x \in I$, ax and x are linearly C-dependent, then there exists $\alpha \in C$ such that $(a-\alpha)I = 0$.

Proof. Let $x \in I$ a fixed element. Then there exists $\alpha \in C$ such that $ax = \alpha x$. Now choose any element $y \in I$. By hypothesis, there exists $\alpha_y \in C$ such that $ay = \alpha_y y$. If x and y are linearly C-dependent, then $x = \beta y$, for $\beta \in C$. In this case, we see that $ax = a\beta y = \beta ay = \beta\alpha_y y = \alpha_y\beta y = \alpha_y x$, implying $\alpha = \alpha_y$.

Now if x and y are linearly C-independent, then we have $\alpha_{x+y}(x+y) = a(x+y) = ax+ay = \alpha_x x + \alpha_y y$, which implies $(\alpha_{x+y} - \alpha_x)x + (\alpha_{x+y} - \alpha_y)y = 0$. Since x and y are linearly C-independent, we have $\alpha_{x+y} - \alpha_x = 0 = \alpha_{x+y} - \alpha_y$ and so $\alpha = \alpha_y$. Thus for any $x \in I$, $ax = \alpha x$, where $\alpha \in C$ fixed. Hence, $(a-\alpha)I = 0$. □

Fact 6. R satisfies s_4 if and only if R is commutative or R embeds in $M_2(K)$ for K a field (see [3, Lemma 1]).

2 The case for both-sided ideals

We begin with a matrix ring case.

Lemma 2.1. *Let $R = M_k(F)$ be the ring of $k \times k$ matrices over the field F with $k \geq 2$. Let $f(x_1,\ldots,x_n)$ be a multilinear polynomial over F which is not central valued on R, $a, b \in R$ and $m \geq 1$ a fixed positive integer.*

(I) If $(af(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)b)^m = f(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in R$, then $m = 1$ and $a, b \in F \cdot I_k$ with $a + b = 1$.

(II) If $(af(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)b)^m - f(x_1,\ldots,x_n) \in Z(R)$ for all $x_1,\ldots,x_n \in R$, then $m = 1$ and $a, b \in F \cdot I_k$ with $a + b = 1$ or $k = 2$.

Proof. Let e_{ij} be the usual matrix unit with 1 in (i,j) entry and zero elsewhere. By our assumption $(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b)^m - f(r_1,\ldots,r_n) \in Z(R)$ for all $r_1,\ldots,r_n \in R$.

Since $f(r_1,\ldots,r_n)$ is not central valued on R, by [25, Lemma 2, proof of Lemma 3] there exist $r_1,\ldots,r_n \in R$ such that $f(r_1,\ldots,r_n) = \alpha e_{ij}$, with $0 \neq \alpha \in F$ and $i \neq j$. Since the subset $\{f(r_1,\ldots,r_n) : r_1,\ldots,r_n \in R\}$ is invariant under any F-automorphism, then for any $i \neq j$ there exist $t_1,\ldots,t_n \in R$ such that $f(t_1,\ldots,t_n) = \alpha e_{ij}$. Thus for any $i \neq j$, $(a\alpha e_{ij} + \alpha e_{ij} b)^m - \alpha e_{ij} \in Z(R)$. If $k \geq 3$, then since rank of $(a\alpha e_{ij} + \alpha e_{ij} b)^m - \alpha e_{ij}$ is ≤ 2, we have

$$(a\alpha e_{ij} + \alpha e_{ij} b)^m - \alpha e_{ij} = 0 \tag{2.1}$$

in R. Right multiply by e_{ij} we get $0 = ((a\alpha e_{ij} + \alpha e_{ij} b)^m - \alpha e_{ij})e_{ij} = (\alpha e_{ij} b)^m e_{ij}$. It follows that the (j,i)-th entry of the matrix b is zero, for all $i \neq j$ and this means that b is diagonal, that is $b = \sum_t \alpha_t e_{tt}$, with $\alpha_t \in F$. For any F-automorphism θ of R, b^θ enjoys the same property as b does, namely, $(a^\theta f(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b^\theta)^m - f(r_1,\ldots,r_n) \in Z(R)$ for all $r_1,\ldots,r_n \in R$. Hence, b^θ must be diagonal. Write $b = \sum_{i=1}^{k} b_{ii} e_{ii}$; then for each $j \neq 1$, we have

$$(1 + e_{1j})b(1 - e_{1j}) = \sum_{i=1}^{k} b_{ii} e_{ii} + (b_{jj} - b_{11})e_{1j}$$

diagonal. Therefore, $b_{jj} = b_{11}$ and so b is a scalar matrix. Similarly, left multiplying by e_{ij} in (2.1) and then by same argument as above we have that a is a scalar matrix. Therefore $a, b \in F.I_k$.

Then (2.1) becomes

$$(a + b)^m (\alpha e_{ij})^m = \alpha e_{ij}. \tag{2.2}$$

If $m \geq 2$, then $0 = e_{ij}$, a contradiction. Hence $m = 1$ and so $(a + b - 1)\alpha e_{ij} = 0$, implying $a + b - 1 = 0$. □

Lemma 2.2. *Let R be a prime ring, I a nonzero ideal of R and $f(x_1,\ldots,x_n)$ a multilinear polynomial over C which is not central valued on R. Suppose that $F(x) = ax + xb$ is an inner generalized derivation of R such that $(F(f(x_1,\ldots,x_n)))^m = f(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in I$, where $m \geq 1$ is a fixed integer. Then $m = 1$ and $a, b \in C$ with $a + b = 1$.*

Proof. Since I, R and U satisfy the same generalized polynomial identities (see *Fact-2*), without loss of generality, we may assume that $(af(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)b)^m = f(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in U$.

First we assume that U does not satisfy any nontrivial GPI. Then $(af(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)b)^m = f(x_1,\ldots,x_n)$ is a trivial GPI for U. This implies that $b \in C$. Then U satisfies $((a+b)f(x_1,\ldots,x_n))^m - f(x_1,\ldots,x_n) = 0$. Again this implies that $a + b \in C$. Therefore, we have in this case that $a, b \in C$.

Next we assume that U satisfies nontrivial GPI $(af(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)b)^m = f(x_1,\ldots,x_n)$. Let $g(x_1,\ldots,x_n) = (af(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)b)^m - f(x_1,\ldots,x_n)$. In case C is infinite, we have $g(r_1,\ldots,r_n) = 0$ for all $r_1,\ldots,r_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are centrally closed [15, Theorem 2.5 and 3.5] we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $g(r_1,\ldots,r_n) = 0$ for all $r_1,\ldots,r_n \in R$. By Martindale's theorem [26], R is a primitive ring having nonzero socle H with C as the associated division ring. In light of Jacobson's theorem [19, p. 75], R is isomorphic to a dense ring of linear transformations on a vector space V over C. Now, if V is finite dimensional over C, then the density of R on V implies that $R \cong M_k(C)$ with $k = \dim_C V$. Since $f(x_1,\ldots,x_n)$ is not central valued on R, R must be noncommutative. Hence $k \geq 2$. Then by Lemma 2.1(I), we conclude that $a, b \in C$.

If V is infinite dimensional over C, then as in [27, Lemma 2] the set $f(R)$ is dense on R and so from $(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b)^m = f(r_1,\ldots,r_n)$ for all $r_1,\ldots,r_n \in R$, we have $(ar+rb)^m - r = 0$ for all $r \in R$. Let v and bv are C-independent for some $v \in R$. By the density of R, there exist $r \in R$ such that $rv = 0, rbv = v$. Therefore we have $0 = ((ar + rb)^m - r)v = v \neq 0$, which is a contradiction. Thus v and bv must be C-dependent, for any $v \in V$. By standard argument, there exists $\alpha \in C$ such that $bv = v\alpha$, for all $v \in V$. Let now $r \in R$ and $v \in V$. As we have just seen, there exist $\alpha \in C$ such that $bv = v\alpha$, $r(bv) = r(v\alpha)$, and also $b(rv) = (rv)\alpha$. Thus $[b, r]v = 0$ for any $v \in V$, that is $[b, r]V = 0$. Since V is left faithful irreducible R-module, $[b, r] = 0$ for all $r \in R$, i.e. $b \in C$. Similarly, we can prove that $a \in C$.

Thus in any case, we have proved that $a, b \in C$. By our hypothesis, we have $(a+b)^m f(x_1, \ldots, x_n)^m - f(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$. If $m = 1$, then $(a + b - 1)f(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$, implying $a + b - 1 = 0$, since $f(x_1, \ldots, x_n)$ is not an identity for R. If $m \geq 2$, then since $(a + b)^m f(x_1, \ldots, x_n)^m - f(x_1, \ldots, x_n) = 0$ is a polynomial identity for R, there exists a field F such that $R \subseteq M_k(F)$ and R and $M_k(F)$ satisfy the same polynomial identity $(a+b)^m f(x_1, \ldots, x_n)^m - f(x_1, \ldots, x_n) = 0$ [21, Lemma 1]. Since $f(x_1, \ldots, x_n)$ is noncentral valued on R, R must be noncommutative and hence $k \geq 2$. By [25, Lemma 2, proof of Lemma 3] there exist $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n) = \alpha e_{ij}$, with $0 \neq \alpha \in F$ and $i \neq j$. Thus $0 = (a+b)^m f(r_1, \ldots, r_n)^m - f(r_1, \ldots, r_n) = (a+b)^m (\alpha e_{ij})^m - \alpha e_{ij} = -\alpha e_{ij}$, a contradiction. □

Theorem 2.3. *Let R be a prime ring, I a nonzero ideal of R and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R. Suppose that F is a generalized derivation of R such that $(F(f(x_1, \ldots, x_n))^m = f(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in I$, where $m \geq 1$ is a fixed integer. Then $m = 1$ and $F(x) = x$ for all $x \in R$.*

Proof. If F is an inner generalized derivation of R, then the result follows by Lemma 2.2. Since I, R and U satisfy the same generalized polynomial identities (see *Fact-2*) as well as same differential identities (see *Fact-3*), by Lee [24] $F(x) = ax + d(x)$ for all $x \in R$, and hence U satisfies U satisfies $(a(f(x_1, \ldots, x_n)) + d(f(x_1, \ldots, x_n)))^m = f(x_1, \ldots, x_n)$, where $a \in U$ and d is a derivation of U. Since F is not inner, d cannot be inner derivation of U. In this case U satisfies the differential identity $(af(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots x_n))^m = f(x_1, \ldots, x_n)$. Then by Kharchenko's Theorem in [20], U satisfies the generalized polynomial identity $(af(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots x_n))^m = f(x_1, \ldots, x_n)$. In particular, by assuming $x_1 = 0$, we have $f(y_1, \ldots, x_n)^m = 0$. This is a polynomial identity for U, hence there exists a field E such that $U \subseteq M_k(E)$, moreover U and $M_k(E)$ satisfies the same polynomial identities [21, Lemma 1]. Thus $M_k(E)$ satisfies $f(y_1, \ldots, x_n)^m = 0$. Then by [25, Corollary 5] $f(x_1, \ldots, x_n)$ is an identity for $M_k(F)$ and so for R, a contradiction. □

Corollary 2.4. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that F is a generalized derivation with associated nonzero derivation d of R such that $(F(x \circ y))^m = x \circ y$ for all $x, y \in I$, where $m \geq 1$ is a fixed integer. Then R is commutative or $m = 1$ and $F(x) = x$ for all $x \in R$.*

Proof. By Theorem 2.3, we conclude that $x \circ y \in Z(R)$ for all $x, y \in R$ or $m = 1$ and $F(x) = x$ for all $x \in R$. Now we are only to consider the case $x \circ y \in Z(R)$, that is $[xy + yx, z] = 0$ for all $x, y \in R$. Then replacing y with yz we have $[xy + yx, z]z + [y[z, x], z] = 0$, implying $[y[z, x], z] = 0$ for all $x, y, z \in R$. Again, replacing y with xy, we have $0 = [xy[z, x], z] = x[y[z, x], z] + [x, z]y[z, x] = [x, z]y[z, x]$ for all $x, y, z \in R$. Since R is prime ring, we have $[x, z] = 0$ for all $x, z \in R$, implying R to be commutative. □

Corollary 2.5. *Let R be a prime ring I be a nonzero ideal of R. Suppose that F is a generalized derivation with associated nonzero derivation d of R such that $(F([x, y]))^m = [x, y]$ for all $x, y \in I$, where $m \geq 1$ is a fixed integer. Then R is commutative or $m = 1$ and $F(x) = x$ for all $x \in R$.*

Proof. By Theorem 2.3, we conclude that $[x, y] \in Z(R)$ for all $x, y \in R$ or $m = 1$ and $F(x) = x$ for all $x \in R$. Now we are only to consider the case $[x, y] \in Z(R)$, that is $[[x, y], z] = 0$ for all $x, y \in R$. Then replacing y with yz we have $[[x, y], z]z + [y[x, z], z] = 0$, implying $[y[x, z], z] = 0$ for all $x, y, z \in R$. Again, replacing y with xy, we have $0 = [xy[x, z], z] = x[y[x, z], z] + [x, z]y[x, z] = [x, z]y[x, z]$ for all $x, y, z \in R$. Since R is prime ring, we have

$[x,z]=0$ for all $x,z\in R$, implying R to be commutative. □

Theorem 2.6. *Let R be a prime ring, I a nonzero ideal of R and $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. Suppose that F is a generalized derivation of R such that $(F(f(x_1,\dots,x_n)))^m - f(x_1,\dots,x_n)\in Z(R)$ for all $x_1,\dots,x_n\in I$, where $m\geq 1$ is a fixed integer. Then one of the following holds:*
(1) $m=1$ and $F(x)=x$ for all $x\in R$;
(2) R satisfies s_4;
(3) $f(x_1,\dots,x_n)^m\in C$ for all $x_1,\dots,x_n\in R$.

Proof. By the hypothesis

$$[F(f(x_1,\dots,x_n))^m - f(x_1,\dots,x_n), x_{n+1}]=0 \qquad (2.3)$$

for all $x_1,\dots,x_{n+1}\in I$. Since I, R and U satisfy the same generalized polynomial identities (see *Fact-2*) as well as same differential identities (see *Fact-3*), by Lee [24] $F(x)=ax+d(x)$ for all $x\in R$, and hence U satisfies

$$[(af(x_1,\dots,x_n)+d(f(x_1,\dots,x_n)))^m - f(x_1,\dots,x_n), x_{n+1}]=0, \qquad (2.4)$$

where $a\in U$ and d is a derivation of U. Now we consider the following two cases:
Case-I: Let d be inner derivation of U, say $d(x)=[b,x]$ for all $x\in U$ and for some $b\in U$. Then by (2.4), U satisfies

$$((a+b)f(x_1,\dots,x_n)-f(x_1,\dots,x_n)b)^m - f(x_1,\dots,x_n)\in C. \qquad (2.5)$$

If $((a+b)f(x_1,\dots,x_n)-f(x_1,\dots,x_n)b)^m - f(x_1,\dots,x_n)=0$ for all $x_1,\dots,x_n\in U$, then by Lemma 2.2, $m=1$ and $a,b\in C$, with $a+b=1$. In this case $F(x)=x$ for all $x\in U$ and so for all $x\in R$, as desired.
If for some $r_1,\dots,r_n\in U$ $((a+b)f(r_1,\dots,r_n)-f(r_1,\dots,r_n)b)^m - f(r_1,\dots,r_n)\neq 0$, then $((a+b)f(x_1,\dots,x_n)-f(x_1,\dots,x_n)b)^m - f(x_1,\dots,x_n)\in C$ is a nonzero central generalized identity for U, by [9, Theorem 1] U is a PI-ring and hence U is a nontrivial GPI-ring simple with 1. By lemma 2 in [21] and Theorem 2.3.29 in [28], there exists a field E such that $U\subseteq M_k(E)$ and U and $M_k(E)$ satisfy the same generalized identities. Thus $((a+b)f(x_1,\dots,x_n)-f(x_1,\dots,x_n)b)^m - f(x_1,\dots,x_n)\in Z(M_k(E))$ for all $x_1,\dots,x_n\in M_k(E)$. Then by Lemma 2.1 (II), we conclude that either $m=1$, $a=1$ and $b\in C$ or $k=2$. In the first case $F(x)=x$ for all $x\in R$, as desired. In the second case, U and so R satisfies s_4.
Case-II: Let d be not inner derivation of U. Then from (2.4), U satisfies

$$\left[\left(af(x_1,\dots,x_n)+f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,d(x_i),\dots,x_n)\right)^m - f(x_1,\dots,x_n), x_{n+1}\right]=0.$$

By Kharchenko's Theorem [20], U satisfies the generalized polynomial identity

$$\left[\left(af(x_1,\dots,x_n)+f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)\right)^m - f(x_1,\dots,x_n), x_{n+1}\right]=0.$$

In particular, for $x_1=0$, we have $f(y_1,\dots,x_n)^m\in C$ for all $y_1,x_2,\dots,x_n\in U$ and so for all $y_1,x_2,\dots,x_n\in R$. □

3 The case for one sided ideals

In this section we will prove our next Theorem for a one sided ideal of R. To prove this theorem, we need the following Lemmas.

Lemma 3.1. *([5, Lemma 2]) Let R be a prime ring, I a nonzero right ideal of R, $f(x_1,\dots,x_n)$ a multilinear polynomial over C, $a\in R$ and m a fixed positive integer.*
(I) If $af(x_1,\dots,x_n)^m=0$ for all $x_1,\dots,x_n\in I$, then either $aI=0$ or $f(I)I=0$.
(II) If $f(x_1,\dots,x_n)^m a=0$ for all $x_1,\dots,x_n\in I$, then either $a=0$ or $f(I)I=0$.

Lemma 3.2. *Let R be a prime ring with extended centroid C, I a nonzero right ideal of R and $f(x_1,\ldots,x_n)$ a nonzero multilinear polynomial over C. If for some $a,b\in R$, $(a(f(x_1,\ldots,x_n)+(f(x_1,\ldots,x_n)b)^m-f(x_1,\ldots,x_n)=0$ for all $x_1,\ldots,x_n\in I$, then R satisfy a nontrivial generalized polynomial identity or $m=1$ and there exists $\alpha\in C$ such that $(a-\alpha)I=0$, $b\in C$ with $b+\alpha=1$.*

Proof. By our hypothesis, for any $u\in I$, R satisfies the following generalized identity

$$(a(f(ux_1,\ldots,ux_n)+(f(ux_1,\ldots,ux_n)b)^m-f(ux_1,\ldots,ux_n)=0. \tag{3.1}$$

We assume that this is a trivial GPI for R, for otherwise we are done. If there exists $u\in I$ such that $\{u,au\}$ is linearly C-independent, then from above R satisfies

$$af(ux_1,\ldots,ux_n)(af(ux_1,\ldots,ux_n)+f(ux_1,\ldots,ux_n)b)^{m-1}=0. \tag{3.2}$$

Again, since $\{u,au\}$ is linearly C-independent, we have from above relation that R satisfies

$$(af(ux_1,\ldots,ux_n))^2(af(ux_1,\ldots,ux_n)+f(ux_1,\ldots,ux_n)b)^{m-2}=0 \tag{3.3}$$

and hence $(af(ux_1,\ldots,ux_n))^m=0$, which is nontrivial, a contradiction. Thus $\{u,au\}$ is linearly dependent over C for all $u\in I$. Then by *Fact-5* $(a-\alpha)I=0$ for some $\alpha\in C$. Then (3.1) becomes

$$(f(ux_1,\ldots,ux_n)(b+\alpha))^m-f(ux_1,\ldots,ux_n)=0. \tag{3.4}$$

Since this is trivial identity for R, we have that $b+\alpha\in C$, that is $b\in C$. Thus identity reduces to

$$(b+\alpha)^m f(ux_1,\ldots,ux_n)^m-f(ux_1,\ldots,ux_n)=0. \tag{3.5}$$

Since this is trivial identity for R, we conclude that $m=1$ and $b+\alpha-1=0$. □

Lemma 3.3. *Let R be a prime ring with extended centroid C, I a nonzero right ideal of R, $f(x_1,\ldots,x_n)$ a nonzero multilinear polynomial over C and $m\geq 1$ a fixed integer. If F is an inner generalized derivation of R such that $(F(f(x_1,\ldots,x_n))^m=f(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n\in I$, then one of the following holds:*

(i) $IC=eRC$ some idempotent $e\in Soc(RC)$ and $f(x_1,\ldots,x_n)$ is central valued on $eRCe$;

(ii) $m=1$ and there exist $\alpha,\lambda\in C$ and $a\in U$ such that $F(x)=(a+\lambda)x$ for all $x\in R$, with $(a-\alpha)I=0$ and $\alpha+\lambda=1$.

Proof. Since F is inner, there exist $a,b\in U$ such that $F(x)=ax+xb$ for all $x\in R$. If R does not satisfy any nontrivial GPI, then by Lemma 3.2, we conclude that $m=1$ and there exists $\alpha\in C$ such that $(a-\alpha)I=0$, $b\in C$, $b+\alpha=1$. In this case $F(x)=ax+xb=(a+b)x$ for all $x\in R$, where $0=(a-\alpha)I=(a+b-1)I$. This gives particular case of conclusion (2), when $\lambda=0$.
So we assume that R satisfies a nontrivial GPI. If $I=R$, then by Lemma 2.2, $m=1$ and $a,b\in C$ with $a+b=1$. In this case we have $F(x)=x$ for all $x\in R$. This is also a particular case of conclusion (2).
So let $I\neq R$. We assume first that $[f(I),I]I=0$, that is $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}=0$ for all $x_1,x_2,\ldots,x_{n+2}\in I$. Then by *Fact-4*, $IC=eRC$ for some idempotent $e\in soc(RC)$. Since $[f(I),I]I=0$, we have $[f(IR),IR]IR=0$ and hence $[f(IU),IU]IU=0$ by [6, Theorem 2]. In particular, $[f(IC),IC]IC=0$, or equivalently, $[f(eRC),eRC]eRC=0$. Then $[f(eRCe),eRCe]=0$, that is, $f(x_1,\ldots,x_n)$ is central-valued on $eRCe$ and hence conclusion (1) is obtained.
So we assume that $[f(I),I]I\neq 0$, that is $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$ is not an identity for I. In this case R is a prime GPI-ring and so is U (see *Fact-2*). Since U is centrally closed over C, it follows that [26] U is a primitive ring with $H=Soc(U)\neq 0$. Then $[f(IH),IH]IH\neq 0$. For otherwise $[f(IU),IU]IU=0$ by [6], a contradiction. Choose $u_n,\ldots,u_{n+2}\in IH$ such that $[f(u_1,\ldots u_n),u_{n+1}]u_{n+2}\neq 0$. Let $u\in IH$. Since H is a regular ring, there exists $e^2=e\in H$ such that $eH=uH+u_1H+\cdots+u_{n+2}H$. Then $e\in IH$ and $u=eu$, $u_i=eu_i$ for $i=1,\ldots,n+2$.

Thus, we have $0 \neq [f(eH), eH]eH = [f(eHe), eHe]H$ i.e., $f(r_1, \ldots, r_n)$ is not central-valued in eHe.
By our assumption and by *Fact-2* we may also assume that

$$(a(f(x_1, \ldots, x_n) + (f(x_1, \ldots, x_n)b)^m = f(x_1, \ldots, x_n)$$

is an identity for IU. In particular,

$$(a(f(x_1, \ldots, x_n) + (f(x_1, \ldots, x_n)b)^m = f(x_1, \ldots, x_n)$$

is an identity for IH and so for eH. It follows that for all $r_1, \ldots, r_n \in H$

$$(a(f(er_1, \ldots, er_n) + (f(er_1, \ldots, er_n)b)^m = f(er_1, \ldots, er_n) \tag{3.6}$$

we may write

$$f(x_1, \ldots, x_n) = \sum_i t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i,$$

where t_i is a suitable multilinear polynomial in $n-1$ variables and x_i never appears in any monomials of t_i. Since $f(eHe) \neq 0$, there exists some t_i which does not vanish in eHe. Without loss of generality $t_n(eHe) \neq 0$. Let $r \in R$. Then replacing r_n with $r(1-e)$ in (3.6), we have

$$\Big(at_n(er_1, \ldots, er_{n-1})er(1-e) + t_n(er_1, \ldots, er_{n-1})er(1-e)b\Big)^m$$
$$= t_n(er_1, \ldots, er_{n-1})er(1-e). \tag{3.7}$$

Left multiplying by $(1-e)$ in (3.7), we obtain $(1-e)(at_n(er_1, \ldots, er_{n-1})er(1-e))^m = 0$, that is $((1-e)at_n(er_1, \ldots, er_{n-1})er)^{m+1} = 0$ for all $r \in H$. By [16], we have $(1-e)at_n(er_1, \ldots, er_{n-1})eH = 0$ implying $(1-e)aet_n(er_1e, \ldots, er_{n-1}e) = 0$ for all $r_1, \ldots, r_{n-1} \in H$. Since eHe is a simple Artinian ring and $t_n(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [7, Lemma 2], $(1-e)ae = 0$ that is, $eae = ae$. Analogously right multiplying by e in (3.7) and then by above argument we conclude that $(1-e)be = 0$. Moreover, since in particular from (3.6) we can write that H satisfies

$$e\{(af(er_1e, \ldots, er_ne) + f(er_1e, \ldots, er_ne)b)^m - f(er_1e, \ldots, er_ne)\}e = 0,$$

and so using the facts $ae = eae$ and $be = ebe$, we have eHe satisfies

$$(eaef(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)ebe)^m - f(r_1, \ldots, r_n) = 0.$$

Then by Lemma 2.2, since $f(r_1, \ldots, r_n)$ is not central valued in eHe, we conclude that $m = 1$ and $eae, ebe \in Ce$. Therefore $ae = eae \in Ce$ and $be = ebe \in Ce$. Thus $au = aeu = eaeu \in Cu$ and hence au, u are linearly C-dependent for each $u \in I$. So by *Fact-5* $(a-\alpha)I = 0$ for some $\alpha \in C$. Similarly $(b-\beta)I = 0$ for some $\beta \in C$.
Then our hypothesis

$$(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)^m - f(r_1, \ldots, r_n) = 0 \tag{3.8}$$

for all $r_1, \ldots, r_n \in I$ gives

$$f(r_1, \ldots, r_n)(b+\alpha) - f(r_1, \ldots, r_n) = 0 \tag{3.9}$$

for all $r_1, \ldots, r_n \in I$, since $m = 1$. Thus

$$f(r_1, \ldots, r_n)(b+\alpha-1) = 0 \tag{3.10}$$

for all $r_1, \ldots, r_n \in I$. Then by Lemma 3.1(II), either $b+\alpha-1 = 0$ or $f(I)I = 0$. Since $f(I)I = 0$ implies $[f(I), I]I = 0$, a contradiction, we have $b = 1-\alpha \in C$. Thus $F(x) = ax+xb = (a+b)x$ for all $x \in R$, which gives our conclusion (2). □

Now we are in a position to prove our main theorem for a one sided ideal of R.

Theorem 3.4. *Let R be a prime ring with extended centroid C, I a nonzero right ideal of R, $f(x_1,\ldots,x_n)$ a nonzero multilinear polynomial over C and $m \geq 1$ a fixed integer. If F is a generalized derivation of R such that $(F(f(x_1,\ldots,x_n)))^m = f(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in I$, then one of the following holds:*

(i) $IC = eRC$ some idempotent $e \in Soc(RC)$ and $f(x_1,\ldots,x_n)$ is central valued on $eRCe$;

(ii) $m = 1$ and there exist $\alpha, \lambda \in C$ and $a \in U$ such that $F(x) = (a+\lambda)x$ for all $x \in R$, with $(a-\alpha)I = 0$ and $\alpha + \lambda = 1$.

Proof. If F is inner generalized derivation of R, then by Lemma 3.3, we are done. Now let F be not inner. By [24], we have $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Let $u_1,\ldots,u_n \in I$. Then by [21], U satisfies

$$\Big(af(u_1x_1,\ldots,u_1x_n) + d(f(u_1x_1,\ldots,u_1x_n))\Big)^m = f(u_1x_1,\ldots,u_1x_n),$$

that is

$$\Big(af(u_1x_1,\ldots,u_1x_n) + f^d(u_1x_1,\ldots,u_nx_n)$$
$$+\sum_j f(u_1x_1,\ldots,d(u_j)x_j + u_jd(x_j),\ldots,x_n)\Big)^m = f(u_1x_1,\ldots,u_1x_n).$$

Since F is not inner, d is also not inner derivation. Then by Kharchenko's theorem [20], U satisfies

$$\Big(af(u_1x_1,\ldots,u_1x_n) + f^d(u_1x_1,\ldots,u_nx_n)$$
$$+\sum_j f(u_1x_1,\ldots,d(u_j)x_j + u_jy_j,\ldots,x_n)\Big)^m = f(u_1x_1,\ldots,u_1x_n).$$

In particular, putting $x_1 = 0$, U satisfies

$$f(u_1y_1,\ldots,u_nx_n)^m = 0.$$

Since I and IU satisfies the same polynomial identities, we have that I satisfies $f(x_1,\ldots,x_n)^m = 0$. By Lemma 3.1, $f(I)I = 0$ and hence $[f(I), I]I = 0$. Then conclusion (1) is obtained by *Fact-4*. □

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Received: March 18, 2016.
Accepted: September 13, 2016.