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2-ABSORBING IDEALS IN FORMAL POWER SERIES RINGS

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Abstract Let R be a commutative ring with identity. A proper ideal I of R is said to be $2-$absorbing if whenever $x_1x_2x_3 \in I$ for $x_1, x_2, x_3 \in R$, then there are 2 of the x_i's whose product is in I. In this paper, we prove that if R is a Noetherian ring, then for every proper ideal I of R, I is a $2-$absorbing ideal if and only if $I[[X]]$ is a $2-$absorbing ideal in the formal power series ring $R[[X]]$.

1 Introduction

All rings considered in this paper are commutative and unitary. Let R be a commutative and unitary ring and P a proper ideal of R. We say that P is a prime ideal if for all $a, b \in R$ such that $ab \in P$, we have $a \in P$ or $b \in P$. Prime ideals are very important for the study of commutative rings. Many generalizations of prime ideals were introduced like weakly prime ideals [8], $n-$absorbing ideals [1] and strongly prime ideals. In [2], Badawi generalized the concept of prime ideals as follows, a proper ideal I of R is a $2-$absorbing ideal if whenever $x_1x_2x_3 \in I$, for $x_1, x_2, x_3 \in R$, then there are 2 of the x_i's whose product is in I. Additionally, Badawi introduces a generalization of primary ideals in [4]. For more references about $2-$absorbing ideals see [6], [7] and [3]. In [1], D. F. Anderson and A. Badawi asked the question: If I is an $n-$absorbing ideal of R, is $I[X]$ an $n-$absorbing ideal of the polynomial ring $R[X]$?. For $n = 2$, they showed that I is a $2-$absorbing ideal if and only if $I[X]$ is a $2-$absorbing ideal of $R[X]$, see ([Theorem 4.15, [1]] or [Corollary 1.7, [9]]). It is natural to think about these results in the formal power series ring. In this paper, we show that in a Noetherian ring R, I is a $2-$absorbing ideal if and only if $I[[X]]$ is a $2-$absorbing ideal of the formal power series ring $R[[X]]$.

2 2-absorbing ideals

Definition 2.1. A proper ideal I of a ring R is said to be $n-$absorbing if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \cdots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I, $n \in \mathbb{N}^*$.

Lemma 2.2. *Let I be an ideal of a Noetherian ring R. Then*

(i) $I[[X]] = IR[[X]]$.

(ii) $\sqrt{I[[X]]} = \sqrt{I}[[X]]$.

Proof. (i) See [Corollary 2.2.3, [5]].

(ii) " $\subseteq$ " Since for all $P \in spec(R)$ with $I \subseteq P$, we have $I[[X]] \subseteq P[[X]]$ then, $\sqrt{I[[X]]} \subseteq P[[X]]$. Thus $\sqrt{I[[X]]} \subseteq \bigcap\limits_{I \subseteq P} P[[X]] = (\bigcap\limits_{I \subseteq P} P)[[X]]=\sqrt{I}[[X]]$.

" $\supseteq$ " We have $I \subseteq I[[X]]$, so $\sqrt{I} \subseteq \sqrt{I[[X]]}$. Thus $\sqrt{I}R[[X]] \subseteq \sqrt{I[[X]]}$.
Since R is Noetherian, so $\sqrt{I}R[[X]] = \sqrt{I}[[X]]$ by (1). Hence the result.

□

Lemma 2.3. *Let I be a $2-$absorbing ideal of a Noetherian ring R and $f = \sum_{i\geq 0} a_i X^i \in \sqrt{I}[[X]]$. Then,*

$$\bigcap_{n\geq 0}(I : a_n)R[[X]] = \bigcap_{n\geq 0}(I : a_n)[[X]] = (I : a_t)[[X]] \text{ for some } t \in \mathbb{N}.$$

Proof. (i) If $\sqrt{I} = I$, then $f \in \sqrt{I}[[X]] = I[[X]]$. Thus, $(I : a_n) = R\ \forall n \in \mathbb{N}$. So, $(\bigcap_{n\geq 0}(I : a_n))R[[X]] = R[[X]] = \bigcap_{n\geq 0}(I : a_n)[[X]]$.

(ii) If $\sqrt{I} \neq I$, then set $H := \{(I : a_n)/\ \ n \in \mathbb{N}\}$. If $a_n \in I$, then $(I : a_n) = R$. Otherwise, for all $a_n, a_m \in \sqrt{I} \setminus I$, either $(I : a_n) \subseteq (I : a_m)$ or $(I : a_m) \subseteq (I : a_n)\ \forall n, m \in \mathbb{N}$ by $[Theorem.2.5$, [2]] and $[Theorem.2.6$, [2]]. Thus H is a nonempty totally ordered set of ideals of R. Since R is Noetherian, then H has a minimal element, and since H is totally ordered, this element is the smallest element. Hence $\bigcap_{n\geq 0}(I : a_n) = (I : a_t)$ for some $t \in \mathbb{N}$.

□

Lemma 2.4. *Let I be a $2-$absorbing ideal of R and p, q two prime ideals of R.*

(i) If $\sqrt{I} = p$, then $(I :_R x)$ is a $2-$absorbing ideal of R for all $x \in R \setminus p$ with $\sqrt{(I :_R x)} = p$ and $S = \{(I :_R x)/\ x \in R\}$ is a totally ordered set.

(ii) If $\sqrt{I} = p \cap q$, then $(I :_R x)$ is a $2-$absorbing ideal of R, for all $x \in R \setminus p \cup q$ with $\sqrt{(I :_R x)} = p \cap q$ and $S = \{(I :_R x)\,/\, x \in R \setminus p \cup q\}$ is a totally ordered set.

Proof. See $[Theorem.1.4$, [9]]. □

Theorem 2.5. *Let I be a $2-$absorbing ideal of a Noetherian ring R and $f(X) = \sum_{i\geq 0} a_i X^i \in R[[X]]$.*

(i) If $f(X) \in \sqrt{I[[X]]} \setminus I[[X]]$, then $(I[[X]] :_{R[[X]]} f(X)) = (I :_R a_t)R[[X]]$ for some $t \geq 0$ and is a prime ideal of $R[[X]]$.

(ii) If $f(X) \notin \sqrt{I[[X]]}$, then either $(I[[X]] :_{R[[X]]} f(X)) = (I :_R a_t)R[[X]]$ for some $t \geq 0$ or $(I[[X]] :_{R[[X]]} f(X)) = P[[X]] \cap Q[[X]]$, where P and Q are two prime ideals of R.

Proof. (i) Suppose that $f(X) \in \sqrt{I[[X]]} \setminus I[[X]]$. First, we show that $\bigcap_{i\geq 0}(I : a_i)[[X]] = (I[[X]] :_{R[[X]]} f(X))$. Let $g(X) = \sum_{j\geq 0} b_j X^j \in \bigcap_{i\geq 0}(I : a_i)[[X]]$. Then for all $i, j \in \mathbb{N}$, $b_j a_i \in I$. Thus $f(X)g(X) = \sum_{n\geq 0}(\sum_{k=0}^{n} a_k b_{n-k})X^n \in I[[X]]$. So $\bigcap_{i\geq 0}(I : a_i)[[X]] \subseteq (I[[X]] :_{R[[X]]} f(X))$. Conversely, let $g(X) \in (I[[X]] :_{R[[X]]} f(X))$. We have $g(X)f(X) \in I[[X]]$. So it is clear that $b_0 \in (I : a_0)$. Let $n \geq 1$ and suppose that $b_0 \in \bigcap_{k=0}^{n-1}(I : a_k)$. We show that $b_0 \in \bigcap_{k=0}^{n}(I : a_k)$. We have $c_n := \sum_{k=0}^{n} b_k a_{n-k} \in I$. Then $b_0 c_n = b_0^2 a_n + b_0 b_1 a_{n-1} + \cdots + b_0 a_0 b_n \in I$. Thus $b_0^2 a_n \in I$ and hence $b_0^2 \in (I : a_n)$. We have $f \in \sqrt{I[[X]]} = \sqrt{I}[[X]]$ by Lemma 2.2. So $a_n \in \sqrt{I}$. If $a_n \in I$, then $(I : a_n) = R$. Otherwise, $a_n \in \sqrt{I} \setminus I$ and then $(I : a_n)$ is prime by $[Theorem.2.5$, [2]] or $[Theorem.2.6$, [2]]. Hence $b_0 \in (I : a_n)$. So $b_0 \in (I : a_n)\ \forall n \in \mathbb{N}$. Now, let $k \geq 1$ and suppose that $b_0, \ldots, b_{k-1} \in (I : a_n)\ \forall n \in \mathbb{N}$. We prove that $b_k \in (I : a_n)\ \forall n \in \mathbb{N}$. For $n = 0$, $b_k c_k = b_k b_0 a_k + b_k b_1 a_{k-1} + \cdots + b_k^2 a_0 \in I$. Thus $b_k^2 a_0 \in I$. This means that $b_k^2 \in (I : a_0)$, so $b_k \in (I : a_0)$ since $(I : a_0)$ prime for $a_0 \in \sqrt{I} \setminus I$. Let $n \geq 1$. Suppose that $b_k \in (I : a_i), \forall\, i \in \{0, \ldots, n-1\}$. We prove that $b_k \in (I : a_n)$. We have $c_{k+n} = a_0 b_{k+n} + a_1 b_{k+n-1} + \cdots + a_n b_k + a_{n+1} b_{k-1} + \cdots + a_{k+n} b_0$. So $b_k c_{k+n} = b_k a_0 b_{k+n} + b_k a_1 b_{k+n-1} + \cdots + a_n b_k^2 + b_k a_{n+1} b_{k-1} + \cdots + b_k a_{k+n} b_0$. Then $a_n b_k^2 \in I$

and thus $b_k^2 \in (I : a_n)$. So $b_k \in (I : a_n)$ since $(I : a_n)$ is prime for $a_n \in \sqrt{I} \setminus I$. Hence $b_k \in (I : a_n)\ \forall k\ \forall n \in \mathbb{N} \Rightarrow b_k \in \bigcap_{n\geq 0}(I : a_n)\ \forall k \in \mathbb{N}$. Therefore $g(X) \in \bigcap_{n\geq 0}(I : a_n)[[X]]$. Thus,

$$(I[[X]] :_{R[[X]]} f(X)) = \bigcap_{n\geq 0}(I : a_n)R[[X]] = \bigcap_{n\geq 0}(I : a_n)[[X]].$$

Now we show that $(I[[X]] :_{R[[X]]} f(X)) = (I :_R a_t)R[[X]]$ for some $t \geq 0$ and is a prime ideal of $R[[X]]$. By Lemma 2.3, $\bigcap_{n\geq 0}(I : a_n) = (I : a_t)$ for some $t \in \mathbb{N}$. So $(I[[X]] :_{R[[X]]} f(X)) = (I : a_t)R[[X]] = (I : a_t)[[X]]$ is prime because $(I : a_t)$ is prime for $a_t \in \sqrt{I} \setminus I$ by $[Theorem.2.8, [2]]$ and $[Theorem.2.9, [2]]$.

(ii) Suppose that $f(X) \notin \sqrt{I[[X]]}$. First, we show that $\bigcap_{i\geq 0}(I : a_i)[[X]] = (I[[X]] :_{R[[X]]} f(X))$. Let $g(X) = \sum_{j\geq 0} b_j X^j \in \bigcap_{i\geq 0}(I : a_i)[[X]]$. Then for all $i, j \in \mathbb{N}$, $b_j a_i \in I$. Thus $f(X)g(X) = \sum_{n\geq 0}(\sum_{k=0}^{n} a_k b_{n-k})X^n \in I[[X]]$. So $\bigcap_{i\geq 0}(I : a_i)[[X]] \subseteq (I[[X]] :_{R[[X]]} f(X))$. Conversely, let $g(X) \in (I[[X]] :_{R[[X]]} f(X))$. We have $g(X)f(X) \in I[[X]]$. We show that $b_k \in (I : a_0)\ \forall k \in \mathbb{N}$.

a. If $a_0 \in \sqrt{I}$.

i. If $\sqrt{I} = P$, then $f(X)g(X) \in I[[X]] \subseteq \sqrt{I}[[X]] = P[[X]]$. Since $f(X) \notin P[[X]]$ then $g(X) \in P[[X]]$. Thus $a_0 b_k \in P^2 \subseteq I\ \forall k \in \mathbb{N}$ by $[Theorem.2.4, [2]]$.

ii. If $\sqrt{I} = P \cap Q$, then $f(X) \notin P[[X]]$ or $f(X) \notin Q[[X]]$ since $f(X) \notin \sqrt{I}[[X]] = (P \cap Q)[[X]] = P[[X]] \cap Q[[X]]$. Note first that if $f(X) \notin P[[X]] \cup Q[[X]]$, then $g(X) \in P[[X]] \cap Q[[X]] = (P \cap Q)[[X]]$ since $f(X)g(X) \in P[[X]] \cap Q[[X]]$. Thus $b_k \in P \cap Q = \sqrt{I}\ \forall k \in \mathbb{N}$. Hence $b_k a_0 \in PQ \subseteq I$ by $[Theorem.2.4, [2]]$. So $b_k \in (I : a_0)\ \forall k \in \mathbb{N}$. On the other hand, if $f(X) \in P[[X]]$ and $f(X) \notin Q[[X]]$, then $g(X) \in Q[[X]]$ since $f(X)g(X) \in Q[[X]]$. Thus $b_k \in (I : a_0)\ \forall k \in \mathbb{N}$ since $b_k a_0 \in QP \subseteq I$ by $[Theorem.2.4, [2]]$.

b. If $a_0 \notin \sqrt{I}$. We have $f(X)g(X) \in I[[X]]$ then $b_0 a_0 \in I$. Let $k \geq 1$, suppose that $b_0, \ldots, b_{k-1} \in (I : a_0)$. We have $c_k = a_0 b_k + \cdots + a_k b_0$. Thus, $a_0 c_k = a_0^2 b_k + a_0 a_1 b_{k-1} + \cdots + a_0 a_k b_0 \in I$. Then $a_0^2 b_k \in I$. Hence $a_0^2 \in I$ or $a_0 b_k \in I$ since I is a $2-$absorbing ideal. If $a_0^2 \in I$, then $a_0 \in \sqrt{I}$ absurd. So $a_0 b_k \in I$.

Now we prove that $b_k \in (I : a_n)\ \forall k, n \in \mathbb{N}$. We have already shown that $b_k \in (I : a_0)$ $\forall k \in \mathbb{N}$. Let $n \in \mathbb{N}^*$. We suppose that $b_k \in (I : a_m)\ \forall 0 \leq m \leq n-1, \forall k \in \mathbb{N}$ and we prove that $b_k \in (I : a_n)\ \forall k \in \mathbb{N}$. Indeed, for $k = 0$, we have $c_n = a_0 b_n + \cdots + a_n b_0 \in I$. Thus $b_0 a_n \in I$. Let $k \geq 1$. Suppose that $b_r \in (I : a_n)\ \forall 1 \leq r \leq k-1$. We show that $b_k \in (I : a_n)$. we do the same proof of a_0 to a_n.

a. If $a_n \in \sqrt{I}$.

i. If $\sqrt{I} = P$, then $a_n b_k \in P^2 \subseteq I$ (as for a_0).

ii. If $\sqrt{I} = P \cap Q$, then $a_n b_k \in PQ \subseteq I$.

b. If $a_n \notin \sqrt{I}$. We have $a_n c_{n+k} = a_n a_0 b_{n+k} + a_n a_1 b_{n+k-1} + \cdots + a_n a_{n-1} b_{k+1} + a_n^2 b_k + a_n a_{n+1} b_{k-1} + \cdots + a_n a_{n+k} b_0 \in I$. Thus $a_n^2 b_k \in I$. Hence $b_k \in (I : a_n)$ since I is a $2-$absorbing ideal and $a_n \notin \sqrt{I}$.

Thus $(I[[X]] :_{R[[X]]} f(X)) = \bigcap_{i\geq 0}(I : a_i)[[X]]$. Now we are ready to show that either $(I[[X]] :_{R[[X]]} f(X)) = (I :_R a_t)R[[X]]$ for some $t \geq 0$ or $(I[[X]] :_{R[[X]]} f(X)) = P[[X]] \cap Q[[X]]$, where P and Q are two prime ideals of R.

(i) If $\sqrt{I} = P$, then the set $\{(I : a_n)/\ a_n \in R\}$ is totally ordered by Lemma 2.4. Since R is Noetherian we deduce that $\bigcap_{n\geq 0}(I : a_n) = (I : a_t)$ for some $t \in \mathbb{N}$. So $(I[[X]] :_{R[[X]]} f(X)) = (I : a_t)R[[X]]$ for some $t \in \mathbb{N}$.

(ii) If $\sqrt{I} = P \cap Q$.

a. If there exists $t \in \mathbb{N}$ such that $a_t \notin P \cup Q$, then $\{(I : a_t)/\ a_t \notin P \cup Q\}$ is totally ordered and $\sqrt{(I : a_t)} = P \cap Q$ by Lemma 2.4. Thus $(I : a_t) \subseteq \sqrt{(I : a_t)} = P \cap Q$ forall $t \in \mathbb{N}$ with $a_t \notin P \cup Q$. If there exists $t \in \mathbb{N}$ with $a_t \in P$ or Q, for example if $a_t \in P$, then $Q \subseteq (I : a_t)$ because $\forall x \in Q$, $xa_t \in QP \subseteq I$ by $[Theorem.2.4$, [2]]. By the same way if $a_t \in Q$, then $P \subseteq (I : a_t)$. Hence $\forall t_0 \in \mathbb{N}$ with $a_{t_0} \notin P \cup Q$ we have $(I : a_{t_0}) \subseteq \sqrt{(I : a_{t_0})} = P \cap Q \subseteq Q \subseteq (I : a_t)$ $\forall a_t \in P$ and $(I : a_{t_0}) \subseteq \sqrt{(I : a_{t_0})} = P \cap Q \subseteq P \subseteq (I : a_t)$ $\forall a_t \in Q$. So if there exists $t \in \mathbb{N}$ with $a_t \notin P \cup Q$ we have $\bigcap_{n\geq 0}(I : a_n) = \bigcap_{n\geq 0}(I : a_n)$ where $a_n \notin P \cup Q$. So $\bigcap_{n\geq 0}(I : a_n)R[[X]] = (I : a_t)R[[X]]$ for some $t \in \mathbb{N}$ since these ideals are comparable by Lemma 2.4 (2) and R is Noetherian.

b. If $a_t \in P \cup Q$ $\forall t \in \mathbb{N}$. Remark that $(I : a_t) = Q$ (resp. $(I : a_t) = P$) for all $t \in \mathbb{N}$ with $a_t \in P \setminus Q$ (resp. $a_t \in Q \setminus P$). Indeed, $xa_t \in PQ \subseteq I$ $\forall x \in Q$. So $Q \subseteq (I : a_t)$. On the other hand, if $xa_t \in I \subseteq \sqrt{I} = P \cap Q$, then $xa_t \in Q$. Thus $x \in Q$. The same way for $(I : a_t) = P$. We have $f(X) \notin \sqrt{I}[[X]] = (P \cap Q)[[X]]$. Thus there exists $t \in \mathbb{N}$ such that $a_t \in P \setminus Q$ or $a_t \in Q \setminus P$. So for all $i \in \mathbb{N}$ with $a_i \in P \cap Q = \sqrt{I}$ we have $(I : a_t) = Q \subseteq (I : a_i)$ with $a_t \in P \setminus Q$ and in the same way we have $(I : a_t) = P \subseteq (I : a_i)$ with $a_t \in Q \setminus P$ by $[Theorem.2.4$, [2]]. Thus $(I[[X]] :_{R[[X]]} f(X)) = \bigcap_{i\geq 0}(I : a_i)[[X]] = \bigcap_{i\geq 0}(I : a_i)[[X]]$ with $a_i \in P \setminus Q$ or $a_i \in Q \setminus P$.

i. If there exists $t_1, t_2 \in \mathbb{N}$ such that $a_{t_1} \in P \setminus Q$ and $a_{t_2} \in Q \setminus P$ then $(I[[X]] :_{R[[X]]} f(X)) = (P \cap Q)R[[X]] = (P \cap Q)[[X]] = P[[X]] \cap Q[[X]]$.

ii. If for all $t \in \mathbb{N}$ $a_t \in P \setminus Q$ (resp. $Q \setminus P$), then $(I[[X]] :_{R[[X]]} f(X)) = (I : a_t)R[[X]] = QR[[X]] = Q[[X]]$ (resp. $PR[[X]] = P[[X]]$).

□

Corollary 2.6. *Let I be a proper ideal of a Noetherian ring R. Then, I is a 2-absorbing ideal of R if and only if $I[[X]]$ is a 2-absorbing ideal of $R[[X]]$.*

Proof. Suppose that $I[[X]]$ is a $2-$absorbing ideal of $R[[X]]$. Since $I = I[[X]] \cap R$ hence I is a $2-$absorbing ideal of R. Conversely, suppose that I is a $2-$absorbing ideal of R. We show that $I[[X]]$ is a $2-$absorbing ideal of $R[[X]]$.

(i) If $\sqrt{I} = I$, then $\sqrt{I[[X]]} = \sqrt{I}[[X]] = I[[X]]$ but I is $2-$absorbing so $\sqrt{I} = P$ or $\sqrt{I} = P \cap Q$ hence $I[[X]] = P[[X]]$ or $I[[X]] = (P \cap Q)[[X]] = P[[X]] \cap Q[[X]]$ therefore $I[[X]]$ is a $2-$absorbing ideal of $R[[X]]$.

(ii) If $\sqrt{I} \neq I$, then $\sqrt{I[[X]]} \neq I[[X]]$. For all $f(x) \in \sqrt{I[[X]]} \setminus I[[X]]$ we have $(I[[X]] :_{R[[X]]} f(X)) = (I :_R a_t)R[[X]]$ for some $t \geq 0$ is a prime ideal of $R[[X]]$ by Theorem 2.5 (1). Thus, $I[[X]]$ is a $2-$absorbing ideal of $R[[X]]$ by $[Theorem.2.8$, [2]] and $[Theorem.2.9$, [2]].

□

References

[1] D. F. Anderson and A. Badawi, On n-absorbing ideals of commutative rings, *Comm. Algebra*, **39**, 1646–1672 (2011).

[2] A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.* **75**, 417–429 (2007).

[3] A. Badawi and A. Y. Darani, On weakly 2-absorbing ideals of commutative rings, *Houston J. Math.* **39**, 441–452 (2013).

[4] A. Badawi, U. Tekir and E.Yetkin, On 2-absorbing primary ideals in commutative rings, *Bull. Korean Math. Soc.* **51**, 1163–1173 (2014).

[5] A. Benhissi, Séries formelles, *Queen's Papers in Pure and Applied Mathematics* **124** Kingston, Ontario, Canada, (2003).

[6] A. Y. Darani and E.R Puczylowski, On 2-absorbing commutative semigroups and their applications to rings, *Semigroup Forum* **86**, 83–91 (2013).

[7] A. Y. Darani and F. Soheilnia, On 2-absorbing and weakly 2-absorbing submodules, *Thai J. Math.* **9**, 577–584 (2011).

[8] S. Galovich, Unique factorization rings with zero divisors, *Math. Mag.* **51**, 276–283 (1978).

[9] Sh. Payrovi and S. Babaei, On the 2-absorbing ideals, *International mathematical forum* **7**, 265–271 (2012).

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