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NUMERICAL SOLUTION OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS USING HAAR WAVELET OPERATIONAL MATRIX

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Abstract. A fractional differential equation has a wide range of applications in engineering and science. Haar wavelet operational matrix has been widely applied in system analysis, system identification, Optimal control and numerical solution of integral and differential equations. In this paper, a numerical scheme, based on the Haar wavelet operational matrix of the fractional order integration for the solution of fractional differential equation is presented. The operational matrix is used to reduce the fractional differential equation in to a system of algebraic equations. Numerical examples are provided to demonstrate the accuracy, efficiency and simplicity of the proposed method.

1 Introduction

Fractional calculus involves integration and differentiation of arbitrary order. The application of fractional calculus just emerged in last few decades in various areas of engineering and science, namely in signal processing, control engineering, electrochemistry, electromagnetism , diffusion processes, biosciences, fluid mechanics, propagation of spherical flames, dynamics of viscoelastic materials, Continuum and statistical mechanics, quantum mechanics, quantum chemistry and damping laws and rheology. Fractional order derivatives are used in modeling and control of many dynamic systems. Fractional calculus does a remarkable job in modeling of a dynamic system as it depends on two factors such as the time instant and the prior time history. Due to the applications and advantages of Fractional differential equations (FDEs), many researchers are trying to develop more efficient and accurate methods to solve them.

But finding the exact solution of a FDE is not easy as its structure is very complicated. Hence analytical and numerical methods are good for finding the solution of FDEs.

Motivated by increasing number of applications of FDEs [11] considerable attention has been given to provide efficient methods for the exact and numerical solution of FDEs. The most commonly used methods are [1, 2, 9, 15]. All these methods have their own advantages, disadvantages, restrictions and limitations. Some of them are very complicated and tough to implement and convergence of results also not very good.

The recent years have witnessed the development of Wavelet theory, a new tool, which emerged from mathematics and quickly adopted by diverse field of engineering and science [6, 8]. Wavelets are special types of oscillatory functions with compact support. The Haar wavelet is the simplest example of orthogonal wavelets, compactly supported on the interval [0, 1). Historically Chen C. and Hsiao C., [7], first proposed a Haar operational matrix for the integration of Haar function vectors and used it for solving differential equations. Recently, there has been some significant interest in the applications of wavelet methods for the numerical solutions of FDEs [18].

In this paper, our purpose is to provide a numerical scheme, based on Haar wavelet operational matrices of integration, to solve various types of FDEs by converting them in to a system of algebraic equations. The paper is organized as follows: In section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus and Haar wavelet. In section 3, we discuss function approximation by the Haar wavelet and operational matrices of integration. In section 4, we present several examples to demonstrate the accuracy and simplicity of the numerical scheme.

2 Definitions, Mathematical Preliminaries and Notations

2.1 Fractional Calculus

In this section, some necessary definitions and mathematical preliminaries of fractional calculus theory are given, which will be used further in this paper. The Riemann-Liouville fractional integral operator I^α of order $\alpha > 0$ on the usual Lebesgue space $L_1[a,b]$ is given as [16],

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} f(z) dz \tag{2.1}$$

$$(I^0 f)(t) = f(t) \tag{2.2}$$

Where $f \in C_\mu$, $\mu \geq -1$

Its fractional derivative of order $\alpha > 0$ is given by

$$(D^\alpha f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha} f)(t) \text{ for } n-1 < \alpha \leq n \tag{2.3}$$

Where n is an integer and $f \in C_1^n$

Now by the Riemann-Liouvilles definition

$$\left.\begin{aligned} &I^\alpha t^\nu = \tfrac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1} t^{\alpha+\nu} \\ &I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t) \\ &I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) \end{aligned}\right\} \tag{2.4}$$

Where $\alpha, \beta \geq 0, t > 0$ and $\nu > -1$

There are many disadvantages of Riemann-Liouvilles derivatives when trying to model real world phenomena with fractional differential equations. Therefore we need to introduce a modified fractional differential operator D^α proposed by Caputo.

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-z)^{n-\alpha-1} f^n(z) dz, \quad (n-1 < \alpha \leq n) \tag{2.5}$$

Where n is an integer $t > 0$ and $f \in C_1^n$

Caputo integral operator has a useful property,

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad (n-1 < \alpha \leq n) \tag{2.6}$$

Where n is an integer $t > 0$ and $f \in C_1^n$

2.2 Haar Wavelet

The Haar functions are orthogonal family of rectangular waveforms. The orthogonal basis $\{h_i(t)\}$ of Haar wavelets for the Hilbert space $L_2[0,1]$ consists of

$$h_i(t) = h_1(2^j t - k),\ i = 2^j + k,\ j \geq 0,\ 0 \leq k \leq 2^j,\ i,j,k \in Z \tag{2.7}$$

where

$$h_0(t) = 1,\ 0 \leq t < 1,\ h_1(t) = \begin{cases} 1, & 0 \leq t < 0.5 \\ -1, & 0.5 \leq t < 1 \end{cases} \tag{2.8}$$

Each Haar wavelet h_i has the support $\left(2^{-j}k, 2^{-j}(k+1)\right)$ so that it is zero elsewhere in the interval [0, 1). Note that, as n increases, the Haar wavelets become more and more localized. Therefore $\{h_i(t)\}$ forms a local basis.

The integer 2^j, $j = 0, 1, ---, J$ indicates the level of wavelets, $k = 0, 1, ---, m-1$ is the translation parameter. The integer J determines the maximal level of resolution. The index i can be calculated by formula $i = 2^j + k + 1$. The sequence $\{h_i(t)\}_{i=0}^{\infty}$ is a complete orthonormal system.

By simple calculations it can be shown that,

$$\int_0^1 h_i(t)h_j(t)dt = \delta_{ij}$$

Consequently, the Haar system $\{h_i(t)\}_{i=0}^{\infty}$ is orthonormal.

3 Function approximation and Operational matrices

3.1 Function approximation

Any function $y(t)$ which is square integrable in the interval [0, 1) can be expanded in to Haar series as

$$y(t) = \sum_{i=0}^{\infty} C_i h_i(t) \tag{3.1}$$

where the Haar coefficients $C_i,\ i = 0, 1, 2, - - -$ are given by

$$C_i =< y(t),\ h_i(t) >= 2^j \int_0^1 y(t)h_i(t)dt \tag{3.2}$$

The coefficients C_i are determined in such a way that integral square error ε given by,

$$\varepsilon = \int_0^1 [y(t) - \sum_{i=0}^{m-1} C_i h_i(t)]^2 dt,\ \ m = 2^j,\ \ j = 1, 2, 3? - --$$

is minimized. If the infinite series in equation (3.1) is truncated, then equation (3.1) can be written as,

$$y(t) \approx y_m(t) = \sum_{i=0}^{m-1} C_i h_i(t) \tag{3.3}$$

The error of numerical schemes mainly depends on series expansion for the unknown solution of differential equations. For differential function $y(t)$ with bounded first order derivative on (0, 1), we have by [4],

$$\|y(t) - y_m(t)\|_{L_2(0,1)} = O(1/2^j)$$

Thus the error of approximation decreases with increasing j.

The wavelet series in (3.3) can be written into the vector form as

$$y(t) \approx C_m^T H_m(t) \tag{3.4}$$

where T indicates transposition. The Haar coefficient vector C_m and Haar function vector H_m are defined as

$$C_m = [C_0, C_1, - - -, C_{m-1}]^T$$
$$H_m(t) = [h_0(t), h_1(t), - - -, h_{m-1}(t)]^T$$

Taking the collocation points $t_i = \frac{2i+1}{2m},\ i = 0, 1, 2, - - -, -1$, we can define the m - square Haar matrix $H_{m\times m}$ as:

$$H_{m\times m} = [H_m(t_0)\ \ H_m(t_1) - - - H_m(t_{m-1})]$$

For instance, when $m = 8$, the Haar matrix is given by

$$H_{8\times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.5}$$

Correspondingly, we have

$$y_m = [y_m(t_0)\ y_m(t_1) - - - y_m(t_{m-1})] = C_m^T H_{m\times m} \tag{3.6}$$

Because the m - square Haar matrix $H_{m\times m}$ is an invertible matrix, the Haar coefficient vector C_m^T can be obtained by

$$C_m^T = y_m H_{m\times m}^{-1} \tag{3.7}$$

3.2 Operational matrix of the fractional order integration

The integration of the Haar function vector $H_m(t)$ is given by,

$$\int_0^t H_m(s)ds = P_{m\times m} H_m(t) \tag{3.8}$$

where $P_{m\times m}$ is the Haar wavelet operational matrix of integration proposed by Chen C., Hsiao C. [7] and it is given by

$$P_{m\times m} = \frac{1}{2m}\begin{pmatrix} 2mP_{\frac{m}{2}\times\frac{m}{2}} & -H_{\frac{m}{2}\times\frac{m}{2}} \\ H^{-1}_{\frac{m}{2}\times\frac{m}{2}} & 0 \end{pmatrix}$$

Now we define a m - set of Block pulse functions (BPF) as:

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \le t < \frac{i+1}{m} \\ 0, & otherwise \end{cases} \tag{3.9}$$

where $i = 0, 1, 2, - - -, m-1$

The functions $b_i(t)$ are disjoint and orthogonal, that is

$$b_i(t)b_j(t) = \begin{cases} 0, & i \ne j \\ b_i(t), & i = j \end{cases} \tag{3.10}$$

$$\int_0^1 b_i(t)b_j(t)dt = \begin{cases} 0, & i \ne j \\ \frac{1}{m}, & i = j \end{cases} \tag{3.11}$$

Because the Haar functions are piecewise constant, it may be expanded into an m - term block pulse functions (BPF) as,

$$H_m(t) = H_{m\times m} B_m(t) \tag{3.12}$$

where the block - pulse function vector $B_m(t)$ is defined as,

$$B_m(t) = [b_0(t)\ \ b_1(t) - - - b_i(t) - - - b_{m-1}(t)]^T \tag{3.13}$$

Fractional integration of the block - pulse function vector is given as,

$$(I^\alpha B_m)(t) = F^\alpha B_m(t) \tag{3.14}$$

where $F^\alpha_{m\times m}$ is the block - pulse operational matrix of the fractional order integration [12].

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & - & - & - & \xi_{m-1} \\ 0 & 1 & \xi_1 & - & - & - & \xi_{m-2} \\ 0 & 0 & 1 & - & - & - & \xi_{m-3} \\ - & - & - & - & & & - \\ - & - & - & & - & & - \\ - & - & - & & & - & - \\ 0 & 0 & 0 & - & - & - & 1 \end{bmatrix} \tag{3.15}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$

Now let

$$(I^\alpha H_m)(t) = P^\alpha_{m\times m} H_m(t) \tag{3.16}$$

where the m - square matrix $P^{\alpha}_{m\times m}$ is called the Haar wavelet operational matrix of the fractional order integration. Using equations (3.12) and (3.14), we have

$$(I^{\alpha}H_m)(t) = (I^{\alpha}H_{m\times m}B_m)(t) = H_{m\times m}(I^{\alpha}B_m)(t) = H_{m\times m}F^{\alpha}B_m(t) \tag{3.17}$$

Now from equations (3.16) and (3.17), we have

$$P^{\alpha}_{m\times m}H_m(t) = P^{\alpha}_{m\times m}H_{m\times m}B_m(t) = H_{m\times m}F^{\alpha}B_m(t) \tag{3.18}$$

then the Haar wavelet operational matrix of the fractional order integration $P^{\alpha}_{m\times m}$ is given by,

$$P^{\alpha}_{m\times m} = H_{m\times m}F^{\alpha}H^{-1}_{m\times m} \tag{3.19}$$

For example, let $\alpha = 0.5, m = 8$, the operational matrix $P^{\alpha}_{m\times m}$ is given as:

$$P^{0.5}_{8\times 8} = \begin{bmatrix} 0.7532 & -0.2203 & -0.1558 & -0.0820 & -0.1102 & -0.0580 & -0.0447 & -0.0377 \\ 0.2203 & 0.3116 & -0.1558 & 0.2296 & -0.1102 & -0.0580 & 0.1756 & 0.0782 \\ 0.0410 & 0.1148 & 0.2203 & -0.0350 & -0.1102 & 0.1623 & -0.0389 & -0.0063 \\ 0.0779 & -0.0779 & 0 & 0.2203 & 0 & 0 & -0.1102 & 0.1623 \\ 0.0094 & 0.0196 & 0.0812 & -0.0032 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0112 & 0.0439 & -0.0551 & -0.0194 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0145 & -0.0145 & 0 & 0.0812 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0275 & -0.0275 & 0 & -0.0551 & 0 & 0 & 0 & 0.1558 \end{bmatrix}$$

4 Numerical Examples

In this section, the proposed Haar wavelet operational matrices of the fractional order integration is discussed to find the numerical solution of the linear and non - linear fractional differential equations. The illustrative examples show the correctness, effectiveness and accuracy of the proposed method.

Example 4.1. Following [10] and [13], we consider the following linear fractional differential equation

$$D^{\alpha}y(t) + y(t) = 0, \quad 0 < \alpha \leq 2 \tag{4.1}$$

such that,

$$y(0) = 1,\ y^{'}(0) = 0 \tag{4.2}$$

The condition $y^{'}(0) = 0$ is only for $1 < \alpha \leq 2$. The exact solution of equation (4.1), (4.2) is given by

$$y(t) = E_{\alpha}(-t^{\alpha})$$

where

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

is the Mittage - Leffler function of order α.
We apply proposed method for $1 < \alpha \leq 2$. Let

$$D^{\alpha}y(t) = K^T_m H_m(t) \tag{4.3}$$

then

$$Dy(t) = (I^{\alpha-1}D^{\alpha}y)(t) + y^{'}(0) = K^T_m P^{\alpha-1}_{m\times m}H_m(t) \tag{4.4}$$

and

$$y(t) = K^T_m P^{\alpha}_{m\times m}H_m(t) + y(0) \tag{4.5}$$

On substituting equations (4.3) and (4.5) in the equation (4.1), we have

$$K^T_m H_m(t) + K^T_m P^{\alpha}_{m\times m}H_m(t) + 1 = 0 \tag{4.6}$$

Thus equation (4.6) has been transformed into a system of algebraic equations. Solving the equation (4.6), we can obtain the coefficients K^T_m. Then on using equation (4.5) we can get the value of required function $y(t)$. The absolute error for $\alpha = 1.5$ and for different values of $m = 8, 16, 32, 64, 128$ is shown in table 1.

Example 4.2. Next, we consider a non-homogeneous multi-term fractional differential equation

$$AD^{\alpha}y(t) + BD^{\beta}y(t) + Cy(t) = g(t) \tag{4.7}$$

With $y(0) = y_0,\ y^{'}(0) = y_1,\ 0 \leq t \leq 1,\ 0 < \beta < \alpha \leq 2$ where $A \neq 0$ and $B, C \in R$

For $\alpha = 2,\ \beta = \frac{3}{2}$ equation (4.7) reduces to the Bargely-Torvik equation originally proposed in [5]. This equation arises in the modeling of a rigid plate immersed in a Newtonian fluid.

Now let

$$D^{\alpha}y(t) = K_m^T H_m(t) \tag{4.8}$$

together with the initial states, then we have

$$D^{\alpha}y(t) = (I^{\alpha-\beta}D^{\alpha}y)(t) = K_m^T P_{m\times m}^{\alpha-\beta} H_m(t) = K_m^T H_{m\times m} F^{\alpha-\beta} H_{m\times m}^{-1} H_m(t) \tag{4.9}$$

and

$$Dy(t) = K_m^T P_{m\times m}^{\alpha-1} H_m(t) + y^{'}(0) = K_m^T P_{m\times m}^{\alpha-1} H_m(t) + y_1 \tag{4.10}$$

and

$$y(t) = K_m^T P_{m\times m}^{\alpha} H_m(t) + y_1[1, 1, ---, 1] H_{m\times m}^{-1} P_{m\times m}^{\alpha-1} H_m(t) + y_0 \tag{4.11}$$

Similarly the input signal $g(t)$ may be expanded by the Haar functions as follows,

$$g(t) = g_m^T H_m(t) \tag{4.12}$$

where g_m^T is a known constant vector. On substituting equations (4.8), (4.9), (4.11) and (4.12) in the equation (4.7), we have

$$AK_m^T H_m(t) + BK_m^T P_{m\times m}^{\alpha-\beta} H_m(t) + C(K_m^T P_{m\times m}^{\alpha} H_m(t) + y_1[1, 1, ---, 1] H_{m\times m}^{-1} P_{m\times m}^{\alpha-1} H_m(t) + y_0[1, 1, ---, 1]) = g_m^T H_m(t) \tag{4.13}$$

Thus equation (4.13) has been transformed into a system of algebraic equations. Substituting the values of A,B and C into the equation (4.13) and solving the system of algebraic equations, we can obtain the coefficients K_m^T. Then using the equation (4.11) we can get the required output y(t).

In particular, if we choose $\alpha = 2,\ \beta = \frac{3}{2},\ A = B = C = 1$ and $g(t) = 8$ for $t \in [0, 1]$ and $y_0 = y_1 = 0$. The numerical solutions obtained by the proposed method and some other numerical methods such as fractional finite difference method, the Adomian decomposition method, fractional differential transform method and the variational iteration method given in [3, 14] are given in the table 2. The exact solution refers to the closed form series solution given by [14]. Clearly, the approximations obtained by the proposed Haar wavelet operational matrix of integration method are in good agreement with those obtained with above mentioned numerical methods.

Example 4.3. In example 4.2, for $A = B = 1,\ C = 0,\ \alpha = 2,\ 0 \leq \beta \leq 1,\ y_0 = y_1 = 0$ and $g(t) = 6t^3\left(\frac{t^{-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{-\beta}}{\Gamma(4-\beta)}\right)$, one can easily verify that the exact solution in this case is $y(t) = t^3$. Computer simulations are carried out for $t \in [0, 1]$ and the maximum absolute errors by our proposed method are given in table 3. This confirms that the proposed method gives very accurate results for the solution of FDEs.

Example 4.4. Consider the following non-linear fractional order differential equation

$$D^{\alpha}y(t) + ay^n(t) = g(t) \tag{4.14}$$

where $y(0) = 0, 1 < \alpha \leq 2,\ n \in N$ and $g(t)$ is the given function.

Now let,

$$D^{\alpha}y(t) = K_m^T H_m(t) \tag{4.15}$$

together with the initial states, then we have

$$y(t) = K_m^T P_{m\times m}^{\alpha} H_m(t) = K_m^T P_{m\times m}^{\alpha} H_{m\times m} B_m(t) \tag{4.16}$$

where $B_m(t)$ is the block-pulse function vector.

Now let,

$$K_m^T P_{m\times m}^{\alpha} H_{m\times m} = [a_1, a_2, ---, a_m]$$

and using equation (3.10), we have

$$[y(t)]^n = [a_1b_1(t) + a_2b_2(t) + ---- + a_mb_m(t)]^n$$
$$= a_1^n b_1(t) + a_2^n b_2(t) + ---- + a_m^n b_m(t)$$
$$[y(t)]^n = [a_1^n, a_2^n, ---, a_m^n]B_m(t) \tag{4.17}$$

Similarly, the input signal g(t) may be expanded by the Haar functions as follows,

$$g(t) = g_m^T H_m(t) = g_m^T H_{m\times m} B_m(t) \tag{4.18}$$

where g_m^T is a known constant vector. On substituting (4.15), (4.17) and (4.18) in (4.14), we have

$$K_m^T H_{m\times m} B_m(t) + a[a_1^n, a_2^n, ---, a_m^n]B_m(t) - g_m^T H_{m\times m} B_m(t) = 0 \tag{4.19}$$

This is a non-linear system of algebraic equations. Here, we use MATLAB to solve equation (4.19). In particular, for $\alpha = \frac{3}{2},\ a = exp(-2\pi),\ n = 2$ and $g(t) = \frac{105}{32}t^2 + e^{2\pi}t^7$, it can be easily verified that the exact solution is $y(t) = t^{\frac{7}{2}}$. The absolute error is given for $m = 64$ and different values of α in the table 4.

Table 1: Absolute error for $\alpha = 1.5$ and different values of m.

t	m = 8	m = 16	m = 32	m = 64	m = 128
0.1	8.291×10^{-4}	1.233×10^{-4}	1.677×10^{-5}	6.001×10^{-6}	8.554×10^{-7}
0.2	7.925×10^{-4}	2.558×10^{-4}	2.636×10^{-5}	3.223×10^{-6}	9.223×10^{-7}
0.3	5.568×10^{-4}	2.525×10^{-5}	2.489×10^{-6}	7.288×10^{-6}	6.365×10^{-8}
0.4	8.329×10^{-5}	1.246×10^{-5}	5.347×10^{-6}	8.834×10^{-7}	4.945×10^{-8}
0.5	9.564×10^{-5}	1.028×10^{-5}	6.854×10^{-6}	5.364×10^{-7}	9.564×10^{-8}
0.6	6.885×10^{-4}	3.258×10^{-5}	7.228×10^{-6}	3.384×10^{-7}	5.658×10^{-8}
0.7	2.645×10^{-4}	7.562×10^{-5}	5.118×10^{-6}	4.735×10^{-7}	3.374×10^{-8}
0.8	3.646×10^{-4}	8.244×10^{-5}	4.487×10^{-6}	2.527×10^{-7}	5.338×10^{-8}
0.9	9.875×10^{-4}	1.223×10^{-4}	3.751×10^{-5}	1.854×10^{-6}	2.426×10^{-7}

Table 1

Table 2: Numerical results obtained by the proposed method of Haar wavelet operational matrix of integration (HWM) for $m = 64$ with comparison to solutions given by [3, 14].

t	y_{FDM}	y_{ADM}	y_{FDTM}	y_{VIM}	y_{HWM}
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.039473	0.039874	0.039750	0.039874	0.039754
0.2	0.157703	0.158512	0.157036	0.158512	0.157032
0.3	0.352403	0.353625	0.347370	0.353625	0.347362
0.4	0.620435	0.622083	0.604695	0.622083	0.604670
0.5	0.957963	0.960047	0.921768	0.960047	0.921752
0.6	1.360551	1.363093	1.290457	1.363093	1.290442
0.7	1.823267	1.826257	1.702008	1.826257	1.702008
0.8	2.340749	2.344224	2.147287	2.344224	2.147286
0.9	2.907324	2.911278	2.617001	2.617000	2.617002
1.0	3.517013	3.521462	3.101906	3.521462	3.101905

Table 2

Table 3: Maximum absolute error by the proposed method for $\alpha = 2$ and different values of β.

β	m = 8	m = 32	m = 128	m = 512
0.25	0.0074	4.606×10^{-4}	3.000×10^{-5}	1.872×10^{-6}
0.50	0.0070	4.475×10^{-4}	2.800×10^{-5}	1.740×10^{-6}
0.75	0.0064	4.160×10^{-4}	2.602×10^{-5}	1.624×10^{-6}
1.00	0.0061	3.902×10^{-4}	2.440×10^{-5}	1.522×10^{-6}

Table 3

Table 4: Absolute error for m = 64 and different values of α.

t	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	$\alpha = 2.0$
0.1	2.6121×10^{-4}	1.8352×10^{-4}	8.5462×10^{-5}	3.2457×10^{-5}	2.3447×10^{-5}	1.2239×10^{-5}
0.2	4.8424×10^{-4}	2.2284×10^{-4}	9.5872×10^{-4}	5.8146×10^{-4}	1.3654×10^{-4}	1.1438×10^{-5}
0.3	6.2754×10^{-4}	5.6584×10^{-3}	7.2548×10^{-3}	5.2647×10^{-3}	3.6657×10^{-4}	2.4854×10^{-5}
0.4	3.5472×10^{-4}	3.5812×10^{-3}	8.5573×10^{-3}	6.2173×10^{-3}	2.4572×10^{-4}	5.4769×10^{-5}
0.5	7.2843×10^{-4}	6.3487×10^{-3}	9.0547×10^{-3}	5.0118×10^{-3}	6.5047×10^{-4}	3.9073×10^{-5}
0.6	6.5804×10^{-4}	5.2409×10^{-3}	8.5706×10^{-3}	6.8740×10^{-3}	9.2074×10^{-4}	8.8460×10^{-5}
0.7	7.5406×10^{-4}	8.5429×10^{-3}	5.8706×10^{-3}	4.0764×10^{-3}	7.4706×10^{-4}	6.0161×10^{-5}
0.8	5.2476×10^{-4}	4.9057×10^{-3}	6.5409×10^{-3}	2.8706×10^{-3}	6.8735×10^{-4}	4.7421×10^{-5}
0.9	3.2780×10^{-4}	6.023×10^{-4}	4.6058×10^{-3}	3.0487×10^{-3}	3.5190×10^{-4}	1.7608×10^{-5}

Table 4

5 Conclusion

In this work a numerical scheme, based on operational matrices of integration for Haar wavelet, is proposed and is used to solve fractional differential equations numerically. A general procedure of forming this matrix $P^{\alpha}_{m\times m}$ is summarized. The proposed method is used to solve the linear and nonlinear multi - term fractional orders differential equations. The obtained results by the proposed method are compared with exact solutions and with the solutions obtained by some other numerical methods. The numerical examples show that the proposed method is very convenient, efficient and accurate for solving FDEs.

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