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ON A DIOPHANTINE EQUATION OF M. J. KARAMA

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Abstract For every positive integer n, the infinite family of positive integral solutions of the diophantine equation $x^n - y^n = z^{n+1}$ is constructed.

1 The equation $x^n - y^n = z^{n+1}$

In a recent paper, M. J. Karama [1] studied the diophantine equation $x^2 - y^2 = z^3$, and conjectured that the diophantine equation $x^3 - y^3 = z^4$ has no solution in positive integers. A standard reference for diophantine equations is the book by Mordell [3], but this very interesting equation is not discussed there.

We shall prove that, for every positive integer n, the diophantine equation $x^n - y^n = z^{n+1}$ has infinitely many positive integral solutions.

2 Powerful triples

The triple (a, b, c) of positive integers is called *n-powerful* if $a > b$ and c^{n+1} divides $a^n - b^n$. Define the function

$$t_n(a, b, c) = \frac{a^n - b^n}{c^{n+1}}. \tag{2.1}$$

The triple (a, b, c) of positive integers is n-powerful if and only if $t_n(a, b, c)$ is a positive integer. The triple (a, b, c) is *relatively prime* if $\gcd(a, b, c) = 1$, where gcd is the greatest common divisor.

Theorem 2.1. *Let n be a positive integer. If (a, b, c) is an n-powerful triple with $t = t_n(a, b, c)$, then the triple of positive integers*

$$(x, y, z) = (at, bt, ct) \tag{2.2}$$

is a solution of the diophantine equation

$$x^n - y^n = z^{n+1}. \tag{2.3}$$

Moreover, there is a one-to-one correspondence between positive integral solutions of (2.3) and relatively prime n-powerful triples.

For example, if a and b are positive integers with $a > b$, then the triple $(a, b, 1)$ is n-powerful with $t = t_n(a, b, 1) = a^n - b^n$, and so

$$(x, y, z) = (at, bt, t) = (a(a^n - b^n), b(a^n - b^n), a^n - b^n) \tag{2.4}$$

is a positive integral solution of (2.3). Moreover,

$$(a(a^n - b^n), b(a^n - b^n), a^n - b^n) = (a_1(a_1^n - b_1^n), b_1(a_1^n - b_1^n), a_1^n - b_1^n)$$

if and only if $a = a_1$ and $b = b_1$. It follows that, for every n, the diophantine equation (2.3) has infinitely many solutions.

Different n-powerful triples (a, b, c) can generate identical solutions to (2.3). For example, for every positive integer n, the triple $(8, 4, 2)$ is n-powerful with $t = 2^{2n-1} - 2^{n-1}$, and produces the solution

$$(x, y, z) = \left(2^{2n+2} - 2^{n+2}, 2^{2n+1} - 2^{n+1}, 2^{2n} - 2^n\right)$$

of the diophantine equation (2.3). The triple $(4, 2, 1)$ is also n-powerful with $t = 2^{2n} - 2^n$, and produces exactly the same solution of (2.3).

Proof. Let (a, b, c) be an n-powerful triple with $t = t_n(a, b, c)$. Defining (x, y, z) by (2.2), we obtain

$$\begin{aligned}
x^n - y^n &= (at)^n - (bt)^n \\
&= a^n \left(\frac{a^n - b^n}{c^{n+1}}\right)^n - b^n \left(\frac{a^n - b^n}{c^{n+1}}\right)^n \\
&= (a^n - b^n) \left(\frac{a^n - b^n}{c^{n+1}}\right)^n \\
&= \left(\frac{a^n - b^n}{c^n}\right)^{n+1} \\
&= \left(c\left(\frac{a^n - b^n}{c^{n+1}}\right)\right)^{n+1} \\
&= (ct)^{n+1} \\
&= z^{n+1}.
\end{aligned}$$

Thus, (x, y, z) solves (2.3).

Let (a, b, c) be an n-powerful triple with $t = t_n(a, b, c)$, and let d be a common divisor of a, b, and c. The relatively prime triple $(a/d, b/d, c/d)$ is n-powerful because

$$\begin{aligned}
t' = t_n(a/d, b/d, c/d) &= \frac{(a/d)^n - (b/d)^n}{(c/d)^{n+1}} \\
&= d\left(\frac{a^n - b^n}{c^{n+1}}\right) = d\, t_n(a, b, c) \\
&= dt
\end{aligned}$$

is a positive integer. The solution of equation (2.3) constructed from $(a/d, b/d, c/d)$ is

$$(x, y, z) = ((a/d)t', (b/d)t', (c/d)t') = (at, bt, ct)$$

which is also the solution constructed from (a, b, c).

If (x, y, z) is a positive integral solution of the diophantine equation (2.3), then (x, y, z) is an n-powerful triple with $t_n(x, y, z) = 1$. Let $d = \gcd(x, y, z)$, and define $(a, b, c) = (x/d, y/d, z/d)$. It follows that (a, b, c) is an n-powerful triple with $t_n(a, b, c) = dt_n(x, y, z) = d$, and that (x, y, z) is the solution of (2.3) produced by (a, b, c). Thus, every positive integral solution of (2.3) can be constructed from a relatively prime n-powerful triple.

Let (x, y, z) be a positive integral solution of (2.3), and let (a, b, c) and (a_1, b_1, c_1) be relatively prime n-powerful triples that produce (x, y, z). We must prove that $(a, b, c) = (a_1, b_1, c_1)$.

If $t = t_n(a, b, c)$ and $t' = t_n(a_1, b_1, c_1)$, then

$$(x, y, z) = (at, bt, ct) = (a_1t', b_1t', c_1t').$$

If $d = \gcd(t, t')$, then t/d and t'/d are positive integers. The equation $x = at = a_1t'$ implies that $a(t/d) = a_1(t'/d)$, and so t/d divides $a_1(t'/d)$. Because t/d and t'/d are relatively prime, it follows that t/d divides a_1, and $a_1 = A(t/d)$ for some positive integer A. Therefore,

$$a\left(\frac{t}{d}\right) = a_1\left(\frac{t'}{d}\right) = A\left(\frac{t}{d}\right)\left(\frac{t'}{d}\right)$$

and $a = A(t'/d)$. Similarly, there exist positive integers B and C such that $b = B(t'/d)$, $b_1 = B(t/d)$, $c = C(t'/d)$, and $c_1 = C(t/d)$. Because t'/d is a common divisor of a, b, and c, and because $\gcd(a, b, c) = 1$, it follows that $t'/d = 1$ and so $a = A$, $b = B$, and $c = C$. Because $\gcd(a_1, b_1, c_1) = 1$, we also have $a_1 = A$, $b_1 = B$, and $c_1 = C$. Therefore, $(a, b, c) = (A, B, C) = (a_1, b_1, c_1)$. This completes the proof. □

3 Open problems

A Maple computation produces 39 positive integral solutions of $x^3 - y^3 = z^4$ with $x \leq 5000$ (see page 527). There are 35 relatively prime 3-powerful triples of the form $(a, b, 1)$, and the following four relatively prime 3-powerful triples (a, b, c) with $c > 1$:

$$\begin{aligned} t_3(71, 23, 14) &= 9 \\ t_3(39, 16, 7) &= 23 \\ t_3(190, 163, 21) &= 13 \\ t_3(103, 101, 7) &= 26. \end{aligned}$$

How often is a difference of cubes divisible by a nontrivial fourth power? More generally, how often is a difference of nth powers divisible by a nontrivial $(n+1)$st power?

It would also be interesting to know, for positive integers n and $k \geq 2$, the positive integral solutions of the diophantine equation

$$x^n - y^n = z^{n+k}.$$

The Beal conjecture [2] states that if k, ℓ, m are integers with $\min(k, \ell, m) > 2$ and if x, y, z are positive integers such that

$$x^k - y^\ell = z^m$$

then $\gcd(x, y, z) > 1$. Does the Beal conjecture hold for the diophantine equation $x^n - y^n = z^{n+1}$?

References

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Table 1. Solutions of $x^3 - y^3 = z^4$ for $x \leq 5000$ with the associated relatively prime 3-powerful triples (a, b, c) and $t_3 = t_3(a, b, c)$. An asterisk (*) indicates a solution with $c > 1$.

x	y	z	a	b	c	t_3
14	7	7	2	1	1	7
57	38	19	3	2	1	19
78	26	26	3	1	1	26
148	111	37	4	3	1	37
224	112	56	4	2	1	56
252	63	63	4	1	1	63
305	244	61	5	4	1	61
490	294	98	5	3	1	98
546	455	91	6	5	1	91
585	234	117	5	2	1	117
620	124	124	5	1	1	124
*639	207	126	71	23	14	9
889	762	127	7	6	1	127
*897	368	161	39	16	7	23
912	608	152	6	4	1	152
1134	567	189	6	3	1	189
1248	416	208	6	2	1	208
1290	215	215	6	1	1	215
1352	1183	169	8	7	1	169
1526	1090	218	7	5	1	218
1953	1116	279	7	4	1	279
1953	1736	217	9	8	1	217
2212	948	316	7	3	1	316
2345	670	335	7	2	1	335
2368	1776	296	8	6	1	296
2394	342	342	7	1	1	342
*2470	2119	273	190	163	21	13
*2678	2626	182	103	101	7	26
2710	2439	271	10	9	1	271
3096	1935	387	8	5	1	387
3474	2702	386	9	7	1	386
3584	1792	448	8	4	1	448
3641	3310	331	11	10	1	331
3880	1455	485	8	3	1	485
. 4032	1008	504	8	2	1	504
4088	511	511	8	1	1	511
4617	3078	513	9	6	1	513
4764	4367	397	12	11	1	397
4880	3904	488	10	8	1	488