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Finitely generated powers of prime ideals

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Abstract. Let R be a commutative ring. If P is a maximal ideal of R with a finitely generated power then we prove that P is finitely generated if R is either locally coherent or arithmetical or a polynomial ring over a ring of global dimension ≤ 2. And, if P is a prime ideal of R with a finitely generated power then we show that P is finitely generated if R is either a reduced coherent ring or a polynomial ring over a reduced arithmetical ring. These results extend a theorem of Roitman, published in 2001, on prime ideals of coherent integral domains.

1 Introduction

All rings are commutative and unitary. In this paper the following question is studied:

question A: Suppose that some power P^n of the prime ideal P of a ring R is finitely generated. Does it follow that P is finitely generated?

When P is maximal it is the *question 0.1* of [7], a paper by Gilmer, Heinzer and Roitman. The first author posed this question in [6, p.74]. In [7] some positive answers are given to the *question 0.1* (see [7, for instance, Theorem 1.24]), but also some negative answers (see [7, Example 3.2]). The authors proved a very interesting result ([7, Theorem 1.17]): a reduced ring R is Noetherian if each of its prime ideals has a finitely generated power. This *question 0.1* was recently studied in [12] by Mahdou and Zennayi, where some examples of rings with positive answers are given, but also some examples with negative responses. In [13] Roitman investigated the **question A**. In particular, he proved that P is finitely generated if R is a coherent integral domain ([13, Theorem 1.8]).

We first study *question 0.1* in Section 2. It is proven that P is finitely generated if R is either locally coherent or arithmetical. In Section 3 we investigate **question A** and extend the Roitman's result. We get a positive answer when R is a reduced ring which is either coherent or arithmetical. If R is not reduced, we obtain a positive answer for all prime ideals P, except if P is minimal and not maximal. In Section 4, by using Greenberg and Vasconcelos's results, we deduce that **question A** has also a positive response if R is a polynomial ring over either a reduced arithmetical ring or a ring of global dimension ≤ 2. In Section 5, we consider rings of constant functions defined over a totally disconnected compact space X with values in a ring O for which a quotient space of Spec O has a unique point, and we examine when these rings give a positive answer to our questions. This allows us to provide some examples and counterexamples.

We denote respectively Spec R, Max R and Min R, the space of prime ideals, maximal ideals and minimal prime ideals of R, with the Zariski topology. If A is a subset of R, then we denote $(0 : A)$ its annihilator and

$$V(A) = \{P \in \text{Spec } R \mid A \subseteq P\} \text{ and } D(A) = \text{Spec } R \setminus V(A).$$

2 Powers of maximal ideals

Recall that a ring R is **coherent** if each finitely generated ideal is finitely presented. It is well known that R is coherent if and only if $(0 : r)$ and $A \cap B$ are finitely generated for each $r \in R$ and any two finitely generated ideals A and B.

Theorem 2.1. *Let R be a coherent ring. If P is a maximal ideal such that P^n is finitely generated for some integer $n > 0$ then P is finitely generated too.*

Proof. First, suppose there exists an integer $n > 0$ such that $P^n = 0$. So, R is local of maximal ideal P. We can choose n minimal. If $n = 1$ then P is clearly finitely generated. Suppose $n > 1$. It follows that $P^{n-1} \neq 0$. So, $P = (0 : r)$ for each $0 \neq r \in P^{n-1}$. Since R is coherent, P is finitely generated. Now, suppose that P^n is finitely generated for some integer $n \geq 1$. If $R' = R/P^n$ and $P' = P/P^n$ then R' is coherent and $P'^n = 0$. From above we deduce that P' is finitely generated. Hence P is finitely generated too. □

The following theorem can be proven by using [7, Lemma 1.8].

Theorem 2.2. *Let R be a ring. Suppose that R_L is coherent for each maximal ideal L. If P is a maximal ideal such that P^n is finitely generated for some integer $n > 0$ then P is finitely generated too.*

Proof. Suppose that P^n is generated by $\{x_1, \dots, x_k\}$. Let $L \neq P$ be a maximal ideal. Let $s \in P \setminus L$. Then $s^n \in P^n \setminus L$. It follows that $s^n R_L = P^n R_L = PR_L = R_L$. So, there exists i, $1 \leq i \leq k$ such that $PR_L = x_i R_L$. Since R_P is coherent, PR_P is finitely generated by Theorem 2.1. So, there exist $y_1, \dots, y_m$ in P such that $PR_P = y_1 R_P + \cdots + y_m R_P$. Let Q be the ideal generated by $\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_m\}$. Then $Q \subseteq P$ and it is easy to check that $QR_L = PR_L$ for each maximal ideal L. Hence $P = Q$ and P is finitely generated. □

A ring R is a **chain ring** if its lattice of ideals is totally ordered by inclusion, and R is **arithmetical** if R_P is a chain ring for each maximal ideal P.

Theorem 2.3. *Let R be an arithmetical ring. If P is a maximal ideal such that P^n is finitely generated for some integer $n > 0$ then P is finitely generated too.*

Proof. First, assume that R is local. Let P be its maximal ideal. Suppose that P is not finitely generated and let $r \in P$. Since $P \neq Rr$ there exists $a \in P \setminus Rr$. So, $r = ab$ with $b \in P$. It follows that $P^2 = P$ and $P^n = P$ for each integer $n > 0$. So, P^n is not finitely generated for each integer $n > 0$. Now, we do as in the proof of Theorem 2.2 to complete the demonstration. □

Remark 2.4. There exist arithmetical rings which are not coherent. In [12] several other examples of non-coherent rings which satisfy the conclusion of the previous theorem are given.

Let R be a ring. For a polynomial $f \in R[X]$, denote by $c(f)$ (the content of f) the ideal of R generated by the coefficients of f. We say that R is **Gaussian** if $c(fg) = c(f)c(g)$ for any two polynomials f and g in $R[X]$ (see [14]). A ring R is said to be a **fqp-ring** if each finitely generated ideal I is projective over $R/(0 : I)$ (see [1, Definition 2.1 and Lemma 2.2]).

By [1, Theorem 2.3] each arithmetical ring is a fqp-ring and each fqp-ring is Gaussian, but the converses do not hold. The following examples show that Theorem 2.3 cannot be extented to the class of fqp-rings and the one of Gaussian rings.

Example 2.5. Let R be a local ring and P its maximal ideal. Assume that $P^2 = 0$. Then it is easy to see that R is a fqp-ring. But P is possibly not finitely generated.

Example 2.6. Let A be a valuation domain (a chain domain), M its maximal ideal generated by m and E a vector space over A/M. Let $R = \{\left(\begin{smallmatrix} a & e \\ 0 & a \end{smallmatrix}\right) \mid a \in A,\ e \in E\}$ be the trivial ring extension of A by E. By [5, Corollary 2.2 and Theorem 4.2] R is a local Gaussian ring which is not a fqp-ring. Let P be its maximal ideal. Then P^2 is generated by $\left(\begin{smallmatrix} m^2 & 0 \\ 0 & m^2 \end{smallmatrix}\right)$. But, if E is of infinite dimension over A/M then P is not finitely generated over R (see also [12, Theorem 2.3(*iv*)**a**)]).

3 Powers of prime ideals

By [13, Theorem 1.8], if R is a coherent integral domain then each prime ideal with a finitely generated power is finitely generated too. The following example shows that this result does not extend to any coherent ring.

Example 3.1. Let D be a valuation domain. Suppose there exists a non-zero prime ideal L' which is not maximal. Moreover assume that $L' \neq L'^2$ and let $d \in L' \setminus L'^2$. If $R = D/Dd$ and $L = L'/Dd$, then R is a coherent ring, L is not finitely generated and $L^2 = 0$.

Remark 3.2. Let R be an arithmetical ring. In the previous example we use the fact that each non-zero prime ideal L which is not maximal is not finitely generated. In Theorem 3.9 we shall prove that L^n is not finitely generated for each integer $n > 0$ if L is not minimal.

In the sequel let $\Phi = \text{Max } R \cup (\text{Spec } R \setminus \text{Min } R)$ for any ring R.
The proof of the following theorem is similar to that of [13, Theorem 1.8].

Theorem 3.3. *Let R be a coherent ring. Then, for any $P \in \Phi$, P is finitely generated if P^n is finitely generated for some integer $n > 0$.*

Proof. Let $P \in \Phi$ such that P^k is finitely generated for some integer $k > 0$. By Theorem 2.1 we may assume that P is not maximal. So, there exists a minimal prime ideal P' such that $P' \subset P$. It follows that $P^n \neq 0$ for each integer $n > 0$. By [13, Lemma 1.7] there exist an integer $n > 1$ such that P^n is finitely generated and $a \in P^{n-1} \setminus P^{(n)}$ where $P^{(n)}$ is the inverse image of $P^n R_P$ by the natural map $R \to R_P$. This implies that $aP = aR \cap P^n$. We may assume that $a \notin P'$, else, we replace a with $a + b$ where $b \in P^n \setminus P'$. Since R is coherent, aP and $(0 : a)$ are finitely generated. From $a \notin P'$ we deduce $(0 : a) \subseteq P' \subset P$, whence $P \cap (0 : a) = (0 : a)$. Hence P is finitely generated. □

Corollary 3.4. *Let R be a reduced coherent ring. Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.*

Proof. Let P be a prime ideal of R such that P^n is finitely generated for some integer $n > 1$. We may assume that $P \neq 0$ and by Theorem 3.3 that P is minimal. So, $P^n \neq 0$. It is easy to check that $(0 : P) = (0 : P^n)$ because R is reduced. Since R is coherent, it follows that $(0 : P)$ is finitely generated. On the other hand, since P^n is finitely generated, there exists $t \in (0 : P^n) \setminus P$. This implies that $P = (0 : (0 : P))$. We conclude that P is finitely generated. □

An exact sequence of R-modules $0 \to F \to E \to G \to 0$ is **pure** if it remains exact when tensoring it with any R-module. Then, we say that F is a **pure** submodule of E. The following proposition is well known.

Proposition 3.5. *[4, Proposition 2.4] Let A be an ideal of a ring R. The following conditions are equivalent:*

(i) A is a pure ideal of R;

(ii) for each finite family $(a_i)_{1 \leq i \leq n}$ of elements of A there exists $t \in A$ such that $a_i = a_i t$, $\forall i$, $1 \leq i \leq n$;

(iii) for all $a \in A$ there exists $b \in A$ such that $a = ab$ (so, $A = A^2$);

(iv) R/A is a flat R-module.

Moreover:

- *if A is finitely generated, then A is pure if and only if it is generated by an idempotent;*
- *if A is pure, then $R/A = S^{-1}R$ where $S = 1 + A$.*

If R is a ring, we consider on Spec R the equivalence relation $\mathcal{R}$ defined by $L\mathcal{R}L'$ if there exists a finite sequence of prime ideals $(L_k)_{1 \leq k \leq n}$ such that $L = L_1$, $L' = L_n$ and $\forall k$, $1 \leq k \leq (n-1)$, either $L_k \subseteq L_{k+1}$ or $L_k \supseteq L_{k+1}$. We denote by pSpec R the quotient space of Spec R modulo $\mathcal{R}$ and by λ : Spec $R \to$ pSpec R the natural map. The quasi-compactness of Spec R implies the one of pSpec R, but generally pSpec R is not T_1: see [10, Propositions 6.2 and 6.3].

Lemma 3.6. *[4, Lemma 2.5]. Let R be a ring and let C a closed subset of* Spec R*. Then C is the inverse image of a closed subset of* pSpec R *by λ if and only if $C = V(A)$ where A is a pure ideal. Moreover, in this case, $A = \cap_{P \in C} \ker(R \to R_P)$.*

In the sequel, for each $x \in \mathrm{pSpec}\, R$ we denote by $A(x)$ the unique pure ideal which verifies $\overline{\{x\}} = \lambda(V(A(x)))$, where $\overline{\{x\}}$ is the closure of $\{x\}$ in pSpec R.

Theorem 3.7. *Let R be a ring. Assume that $R/A(x)$ is coherent for each $x \in \mathrm{pSpec}\, R$. Then, for any $P \in \Phi$, P is finitely generated if P^n is finitely generated for some integer $n > 0$.*

Proof. Let $P \in \Phi$ and $I = A(\lambda(P))$. Suppose that P^n is generated by $\{x_1, \dots, x_k\}$. Let L be a maximal ideal such that $I \nsubseteq L$. As in the proof of Theorem 2.2 we show that $PR_L = x_i R_L$ for some integer i, $1 \le i \le k$. By Theorem 3.3 P/I is finitely generated over R/I. So, there exist $y_1, \dots, y_m$ in P such that $(y_1 + I, \dots, y_m + I)$ generate P/I. Let Q be the ideal generated by $\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_m\}$. Then $Q \subseteq P$ and it is easy to check that $QR_L = PR_L$ for each maximal ideal L. Hence $P = Q$ and P is finitely generated. □

From Corollary 3.4 and Theorem 3.7 we deduce the following.

Corollary 3.8. *Let R be a reduced ring. Assume that $R/A(x)$ is coherent for each $x \in \mathrm{pSpec}\, R$. Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.*

Theorem 3.9. *Let R be an arithmetical ring. Then, for any $P \in \Phi$, P is finitely generated if P^n is finitely generated for some integer $n > 0$.*

Proof. Let P be a prime ideal. By Theorem 2.3 we may assume that P is not maximal. Let M be a maximal ideal containing P. If P is not minimal then $P^n R_M$ contains strictly the minimal prime ideal of R_M for each integer $n > 0$. So, $P^n R_M \neq 0$ for each integer $n > 0$. On the other hand, since R_M is a chain ring it is easy to check that $PR_M = MPR_M$. It follows that $P^n R_M = MP^n R_M$ for each integer $n > 0$. By Nakayama Lemma we deduce that $P^n R_M$ is not finitely generated over R_M. Hence, P^n is not finitely generated for each integer $n > 0$. □

Remark 3.10. Example 3.1 shows that the assumption "$P \in \Phi$" cannot be omitted in some previous results. However, if each minimal prime ideal which is not maximal is idempotent then the conclusions hold for each prime ideal P.

Proposition 3.11. *Let R be a ring. Let P be a minimal prime ideal such that P^n is finitely generated for some integer $n > 0$. Then P is an isolated point of* Min R.

Proof. Let N be the nilradical of R. For any finitely generated ideal I we easily check that $V(I) \cap \mathrm{Min}\, R = D((N : I)) \cap \mathrm{Min}\, R$. Hence it is a clopen (closed and open) subset of Min R. Since $V(P^n) \cap \mathrm{Min}\, R = \{P\}$, P is an isolated point of Min R if P^n is finitely generated. □

From Theorems 3.7 and 3.9 and Proposition 3.11 we deduce the following corollary.

Corollary 3.12. *Let R be a ring. Assume that* Min R *contains no isolated point and R satisfies one of the following conditions:*

- *$R/A(x)$ is coherent for each $x \in \mathrm{pSpec}\, R$;*
- *R is arithmetical.*

Then, each prime ideal with a finitely generated power is finitely generated too.

Proposition 3.13. *Let R be a ring for which each prime ideal contains only one minimal prime ideal. Let P be a minimal prime ideal such that P^n is finitely generated for some integer $n > 0$. Then $\lambda(P)$ is an isolated point of* pSpec R.

Proof. Let P be a minimal prime ideal and $A = A(\lambda(P))$. Clearly $\lambda(P) = V(P) = V(A)$. We have $A^2 = A$. From $A \subseteq P$ we deduce that $A \subseteq P^2$. It follows that $A \subseteq P^n$ for each integer $n > 0$. Suppose that P^n is finitely generated for some integer $n > 0$. Since P/A is the nilradical of R/A, $P^m = A$ for some integer $m \ge n$. We deduce that $P^m = Re$ for some idempotent e of R by Proposition 3.5. It follows that $\lambda(P) = V(P^m) = D(1 - e)$. Hence $\lambda(P)$ is an isolated point of pSpec R. □

4 pf-rings

Now, we consider the rings R for which each prime ideal contains a unique minimal prime ideal. So, the restriction λ' of λ to Min R is bijective. In this case, for each minimal prime ideal L we put $A(L) = A(\lambda(L))$. By [3, Proposition IV.1] pSpec R is Hausdorff and λ' is a homeomorphism if and only if Min R is compact. We deduce the following from Lemma 3.6.

Proposition 4.1. *Let R be a ring. Assume that each prime ideal contains a unique minimal prime ideal. Then, for each minimal prime ideal L, $V(L) = V(A(L))$. Moreover, if R is reduced then $A(L) = L$.*

Proof. If R is reduced, then, for each $P \in V(L)$, $LR_P = 0$, whence $L = \ker(R \to R_P)$. □

As in [15, p.14] we say that a ring R is a **pf-ring** if one of the following equivalent conditions holds:

(i) R_P is an integral domain for each maximal ideal P;

(ii) each principal ideal of R is flat;

(iii) each cyclic submodule of a flat R-module is flat.

Moreover, if R is a pf-ring then each prime ideal P contains a unique minimal prime ideal P' and $A(P') = P'$ by Proposition 4.1.

So, from the previous section and the fact that each minimal prime ideal of a pf-ring is idempotent, we deduce the following three results. Let us observe that each prime ideal of an arithmetical ring R contains a unique minimal prime ideal because R_P is a chain ring for each maximal ideal P.

Corollary 4.2. *Let R be a coherent pf-ring. Then each prime ideal with a finitely generated power is finitely generated too.*

Corollary 4.3. *Let R be a pf-ring. Assume that R/L is coherent for each minimal prime ideal L. Then each prime ideal with a finitely generated power is finitely generated too.*

Corollary 4.4. *Let R be a reduced arithmetical ring. Then each prime ideal with a finitely generated power is finitely generated too.*

The following three corollaries allows us to give some examples of pf-ring satisfying the conclusion of Corollary 4.3. Let n be an integer ≥ 0 and G a module over a ring R. We say that pd $G \leq n$ if $\mathrm{Ext}_R^{n+1}(G, H) = 0$ for each R-module H.

Corollary 4.5. *Let R be a coherent ring. Assume that each finitely generated ideal I satisfies pd $I < \infty$. Then each prime ideal with a finitely generated power is finitely generated too.*

Proof. By, either [2, Théorème A] or [8, Corollary 6.2.4], R_P is an integral domain for each maximal ideal P. So, R is a pf-ring. □

Corollary 4.6. *Let A be a ring and $X = \{X_\lambda\}_{\lambda \in \Lambda}$ a set of indeterminates. Consider the polynomial ring $R = A[X]$. Assume that A is reduced and arithmetical. Then each prime ideal of R with a finitely generated power is finitely generated too.*

Proof. Let P be a maximal ideal of R and $P' = P \cap A$. Thus R_P is a localization of $A_{P'}[X]$. Since $A_{P'}$ is a valuation domain, R_P is an integral domain. So, R is a pf-ring. Now, let P be a minimal prime ideal of R and L be a minimal prime ideal of A contained in $P \cap A$. We put $A' = A/L$ and $R' = A'[X]$. So, A' is an arithmetical domain (a Prüfer domain). By [9, 3.(b)] R' is coherent. Since R/P is flat over R and R', R/P is a localization of R'. Hence R/P is coherent. We conclude by Corollary 4.3. □

Let n be an integer ≥ 0. We say that a ring R is of global dimension $\leq n$ if pd $G \leq n$ for each R-module G.

Corollary 4.7. *Let A be a ring and $X = \{X_\lambda\}_{\lambda\in\Lambda}$ a set of indeterminates. Consider the polynomial ring $R = A[X]$. Assume that A is of global dimension ≤ 2. Then each prime ideal of R with a finitely generated power is finitely generated too.*

Proof. Let P be a maximal ideal of R and $P' = P \cap A$. Thus R_P is a localization of $A_{P'}[X]$. Since $A_{P'}$ is an integral domain by [11, Lemme 2], R_P is an integral domain. So, A and R are pf-rings. By [11, Proposition 2] A/L is coherent for each minimal prime ideal L. Now, we conclude as in the proof of the previous corollary, by using [9, (4.4) Corollary]. □

5 Rings of locally constant functions

A topological space is called **totally disconnected** if each of its connected components contains only one point. Every Hausdorff topological space X with a base of clopen (closed and open) neighbourhoods is totally disconnected and the converse holds if X is compact (see [16, Lemma 29.6]).

Proposition 5.1. *Let X be a totally disconnected compact space, let O be a ring with a unique point in* pSpec O. *Let R be the ring of all locally constant maps from X into O. Then,* pSpec R *is homeomorphic to X and $R/A(z) \cong O$ for each $z \in$* pSpec R.

Proof. If U is a clopen subset of X then there exists an idempotent e_U defined by $e_U(x) = 1$ if $x \in U$ and $e_U(x) = 0$ else. Let $x \in X$ and $\phi_x : R \to O$ be the map defined by $\phi_x(r) = r(x)$ for every $r \in R$. Clearly ϕ_x is a ring homomorphism, and since R contains all the constant maps, ϕ_x is surjective. Let $x \in X$, $r \in \ker(\phi_x)$ and $U = \{y \in X \mid r(y) \neq 0\}$. Then U is a clopen subset. It is easy to check that $e_U \in \ker(\phi_x)$ and $r = re_U$. Since $\ker(\phi_x)$ is generated by idempotents, $R/\ker(\phi_x)$ is flat over R. For each $x \in X$, let $\Pi(x)$ be the image of Spec O by $\lambda \circ \phi_x^a$ where ϕ_x^a : Spec $O \to$ Spec R is the continuous map induced by ϕ_x. We shall prove that $\Pi : X \to$ pSpec R is a homeomorphism. Clearly, $V(\ker(\phi_x)) \subseteq \Pi(x)$. Conversely, let $P \in \Pi(x)$. Then there exists $L \in V(\ker(\phi_x))$ such that $P\mathcal{R}L$. We may assume that $L \subseteq P$ or $P \subseteq L$. The first case is obvious. For the second case let e an idempotent of $\ker(\phi_x)$. Then, $e \in L$, $(1-e) \notin L$, $(1-e) \notin P$ and $e \in P$. We conclude that $V(\ker(\phi_x)) = \Pi(x)$ because $\ker(\phi_x)$ is generated by its idempotents. Let $x, y \in X$, $x \neq y$. By using the fact there exists a clopen subset U of X such that $x \in U$ and $y \notin U$ then $e_U \in \ker(\phi_y)$ and $(1-e_U) \in \ker(\phi_x)$. So, $\ker(\phi_x) + \ker(\phi_y) = R$, whence Π is injective. By way of contradiction suppose there exists a prime ideal P of R such that $\ker(\phi_x) \nsubseteq P$ for each $x \in X$. There exists an idempotent $e'_x \in \ker(\phi_x) \setminus P$ whence $e_x = (1-e'_x) \in P \setminus \ker(\phi_x)$. Let V_x be the clopen subset associated with e_x. Clearly $X = \cup_{x\in X} V_x$. Since X is compact, a finite subfamily $(V_{x_i})_{1\leq i\leq n}$ covers X. We put $U_1 = W_1 = V_{x_1}$, and for $k = 2, \dots, n$, $W_k = \cup_{i=1}^k V_{x_i}$ and $U_k = W_k \setminus W_{k-1}$. Then U_k is clopen for each $k = 1, \dots, n$. For $i = 1, \dots, n$ let $\epsilon_i \in R$ be the idempotent associated with U_i. Since $U_i \subseteq V_{x_i}$, we have $\epsilon_i = e_{x_i}\epsilon_i$. So, $\epsilon_i \in P$ for $i = 1, \dots, n$. It is easy to see that $1 = \Sigma_{i=1}^n \epsilon_i$. We get $1 \in P$. This is false. Hence Π is bijective. We easily check that $x \in U$, where U is a clopen subset of X, if and only if $\Pi(x) \subseteq D(e_U)$. Since $A(\Pi(x)) = \ker(\phi_x)$ is generated by its idempotents, pSpec R has a base of clopen neighbourhoods. We conclude that Π is a homeomorphism. □

From Corollary 3.8 we deduce the following proposition.

Proposition 5.2. *Let R be the ring defined in Proposition 5.1. Assume that O is a reduced coherent ring. Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.*

Proposition 5.3. *Let R be the ring defined in Proposition 5.1. Assume that O has a unique minimal prime ideal M. Then, every prime ideal of R contains only one minimal prime ideal and* Min R *is compact. If $M = 0$ then R is a pp-ring, i.e. each principal ideal is projective.*

Proof. If P is a prime ideal of R then there exists a unique $x \in X$ such that $P \in \Pi(x)$. So, $\phi_x^a(M)$ is the only minimal prime ideal contained in P.

Assume that $M = 0$. Let $r \in R$, $e = e_U$ where U is the clopen subset of X defined by $U = \{x \in X \mid r(x) \neq 0\}$. We easily check that the map $Re \to Rr$ induced by the multiplication by r is an isomorphism. This proves that R is a pp-ring.

Let R' be the ring obtained like R by replacing O with O/M. It is easy to see that $R' \cong R/N$ where N is the nilradical of R. So, Min R and Min R' are homeomorphic. Since R' is a pp-ring, Min R is compact by [15, Proposition 1.13]. □

From Theorems 3.7 and 3.9 and Propositions 3.13 and 5.3 we deduce the following corollary.

Corollary 5.4. *Let R be the ring defined in Proposition 5.1. Suppose that O has a unique minimal prime ideal M. Assume that O is either coherent or arithmetical and that one of the following conditions holds:*

(i) M is either idempotent or finitely generated;

(ii) X contains no isolated point.

Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

Example 5.5. Let R be the ring defined in Proposition 5.1. Assume that:

- O is either coherent or arithmetical, with a unique minimal prime ideal M;
- M is not finitely generated and $M^k = 0$ for some integer $k > 1$ (for example, O is the ring R defined in Example 3.1);
- X contains no isolated points (for example the Cantor set, see [16, Section 30]).

Then the property "for each prime ideal P, P^n is finitely generated for some integer $n > 0$ implies P is finitely generated" is satisfied by R, but not by $R/A(L)$ for each minimal prime ideal L.

From Theorems 2.2 and 2.3 and Proposition 3.13 we deduce the following corollary.

Corollary 5.6. *Let R be the ring defined in Proposition 5.1. Assume that O is local with maximal ideal M. Then each prime ideal of R is contained in a unique maximal ideal, and for each maximal ideal P, $R_P \cong O$. Moreover, if one of the following conditions holds:*

(i) O is coherent;

(ii) O is a chain ring;

(iii) X contains no isolated point and M is the sole prime ideal of O.

then, for each maximal ideal P, P^n finitely generated for some integer $n > 0$ implies P is finitely generated.

Example 5.7. Let R be the ring defined in Proposition 5.1. Assume that M is the sole prime ideal of O, M is not finitely generated, $M^k = 0$ for some integer $k > 1$ and X contains no isolated points. Then the property "for each maximal ideal P, P^n is finitely generated for some integer $n > 0$ implies P is finitely generated" is satisfied by R, but not by R_L for each maximal ideal L.

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