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Flow of starshaped Euclidean hypersurfaces by Weingarten curvatures

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Abstract We consider the evolution of starshaped hypersurfaces in the Euclidean space by general curvature functions. Under appropriate conditions on the curvature function, we prove the global existence and convergence of the flow to a hypersurface of prescribed curvature.

1 Introduction and statement of the results

Let M_0 be a smooth closed compact hypersurface in $\mathbb{R}^{n+1}$ $(n \geq 2)$. We suppose that M_0 is starshaped with respect to a point, which we assume to be the origin of $\mathbb{R}^{n+1}$ for simplicity, and in the rest of the paper all starshaped hypersurfaces are with respect to the origin of $\mathbb{R}^{n+1}$. This means that for every point $P \in M_0$, we have $P \notin T_P M_0$, where $T_P M_0$ is the tangent space of M_0 at P. If we let $\pi : M_0 \to \mathbb{S}^n$ to be the projection on $\mathbb{S}^n$ defined by

$$\pi(P) = \frac{P}{|P|} \ , \ P \in M_0,$$

then one can prove that M_0 is starshaped if and only if π is a diffeomorphism. It follows that the inverse diffeomorphism $X_0 := \pi^{-1} : \mathbb{S}^n \to M_0$ can be used as a parametrization of M_0. The function $\rho_0 : \mathbb{S}^n \to \mathbb{R}^+$ defined by $\rho_0(x) = |X_0(x)|$ is called the radial function of M_0. Thus we have

$$X_0(x) = \rho_0(x)x, \quad x \in \mathbb{S}^n. \tag{1.1}$$

From now on, we say that a smooth embedding $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth starshaped embedding if $M := X(\mathbb{S}^n)$ is a starshaped hypersurface in $\mathbb{R}^{n+1}$, so by composing by a smooth diffeomorphism of $\mathbb{S}^n$ if necessary, we may suppose that X is of the form (1.1).

We consider the evolution problem

$$\begin{cases} \partial_t X(t,x) = \Big(K \circ \kappa(X)(t,x) - f \circ X(t,x)\Big)\nu(t,x) \\ X(0,x) = X_0(x) \end{cases} \tag{1.2}$$

where $X(t,.) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth starshaped embedding, ν is the outer unit normal vector field of the hypersurface $M_t := X(t,\mathbb{S}^n)$, K is a suitable function of the principal curvatures vector $\kappa(X) = (\kappa_1(X), ..., \kappa_n(X))$ of M_t, referred as the curvature function, and $f : \mathbb{R}^{n+1} \setminus \{ 0 \} \to \mathbb{R}$ is a given smooth function referred as the prescribed function. We suppose that the function K is expressed as an inverse function of the principal curvatures, that is

$$K \circ \kappa(X) = \frac{1}{F \circ \kappa(X)} = \frac{1}{F \circ (\kappa_1(X), .., \kappa_n(X))} \ ,$$

where $F \in C^\infty(\Gamma) \cap C^0\left(\overline{\Gamma}\right)$ is a positive, symmetric function on an open, convex symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin, which contains the positive cone

$$\Gamma^+ = \{ (\lambda_1, .., \lambda_n) \in \mathbb{R}^n : \lambda_i > 0 \ \ \forall i \in [1, .., n] \} .$$

This implies in particular that

$$\Gamma \subset \{ (\lambda_1, .., \lambda_n) \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n > 0 \} .$$

The function $F(\lambda) = F(\lambda_1, .., \lambda_n)$ is assumed to satisfy the following structure conditions

$$\frac{\partial F}{\partial \lambda_i} > 0 \text{ on } \Gamma \quad \forall i \in [1, .., n] \tag{1.3}$$

$$F \text{ is homogeneous of degree } k > 0 \text{ on } \Gamma \text{ and } F \equiv 0 \text{ on } \partial\Gamma \tag{1.4}$$

$$\log F \text{ is concave on } \Gamma. \tag{1.5}$$

By scaling, we may suppose

$$F(1, .., 1) = 1. \tag{1.6}$$

The above conditions on F are usually assumed in the study of fully nonlinear partial differential equations. Condition (1.3) ensures that the system (1.2) is parabolic, which is of great importance in proving short time existence of solutions. The other conditions will be used to control the C^1 and C^2-norms of solutions. Some examples of suitable curvature functions satisfying (1.3)-(1.6) are

$$F(\lambda_1, .., \lambda_n) = \binom{k}{n}^{-1} S_k(\lambda_1, ..., \lambda_n) = \binom{k}{n}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

the k-th elementary symmetric functions normalised so that $F(1, .., 1) = 1$. In this case we take Γ to be the component of the set where S_k is positive which contains the positive cone. Thus we obtain the mean curvature when $k = 1$ and the Gauss curvature when $k = n$. Other examples of curvature functions are

$$F(\lambda_1, .., \lambda_n) = \binom{k}{n} \left(S_k(\lambda_1^{-1}, ..., \lambda_n^{-1}) \right)^{-1} .$$

In this case, we take $\Gamma = \Gamma^+$. A particular case of interest in the previous example is the harmonic mean curvature when $k = 1$.

Finally, we notice that if a function F satisfies conditions (1.3)-(1.6) above, then for any $\alpha > 0$, the function F^α satisfies the same conditions where k is replaced by αk. This invariance property is due to the fact that the convexity condition (1.5) concerns $\log F$ but not F.

When the prescribed function $f \equiv 0$, problem (1.2) has been studied by J. Urbas [10] assuming that the curvature function F satisfies (1.3)-(1.6) with $k = 1$ and that F is concave instead of $\log F$ concave. He showed the existence of a global solution on $[0, +\infty)$, and for the convergence at infinity, he proved that if $\tilde{M}_t$ is the hypersurface parametrized by $\tilde{X}(t, .) = e^{-t} X(t, .)$, then $\tilde{M}_t$ converges to a sphere in the C^∞ topology as $t \to +\infty$. There is an extensive literature on curvature evolution equation like (1.2) and similar evolution curvature problems corresponding to other settings. We refer the reader to [1], [3], [6], [9], [11] and the references therein.

In this paper, we study the global existence and convergence for equation (1.2) assuming that F satisfies the structure conditions (1.3)-(1.6), and the prescribed function $f : \mathbb{R}^{n+1} \setminus \{ 0 \} \to \mathbb{R}^+$ is a smooth function satisfying

$$\frac{\partial}{\partial \rho} \left(\rho^{-k} f(X) \right) > 0, \; X \in \mathbb{R}^{n+1} \setminus \{ 0 \} \tag{1.7}$$

where $\rho = |X|$. We will also assume that there exist two positive real numbers $r_1 \le r_2$ such that

$$\begin{cases} f(X) \le r_1^k \text{ if } |X| = r_1 \\ f(X) \ge r_2^k \text{ if } |X| = r_2. \end{cases} \tag{1.8}$$

These assumptions were made by L. Caffarelli, L. Nirenberg and J. Spruck [4] for the existence by elliptic methods of starshaped embedding X whose $\frac{1}{F}$-curvature is equal to f, i.e, satisfying the equation :

$$\frac{1}{F(\kappa(X))} = f(X). \tag{1.9}$$

See also a related work of P. Delanoe [5] concerning the Gauss curvature.

Our main result in this paper is that conditions (1.7)-(1.8) on the prescribed function f are also sufficient to study the evolution problem (1.2). Moreover the solution of such flow converges to a smooth starshaped embedding satisfying (1.9). Our first result concerns the case where the homogeneity degree k of F satisfies $0 < k \leq 1$. We have

Theorem 1.1. Let $F \in C^\infty(\Gamma) \cap C^0\left(\overline{\Gamma}\right)$ be a positive symmetric function satisfying conditions (1.3)-(1.6) such that the homogeneity degree k of F satisfies $0 < k \leq 1$, and let $f \in C^\infty\left(\mathbb{R}^{n+1} \setminus \{0\}\right)$ be a positive function satisfying (1.7)-(1.8). Let M_0 a closed compact starshaped hypersurface in $\mathbb{R}^{n+1}$, parametrized by a smooth embedding $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ of the form (1.1) such that

$$\kappa(X_0) \in \Gamma \text{ and } \frac{1}{F(\kappa(X_0))} - f(X_0) \geq 0\,. \tag{1.10}$$

Then the evolution problem (1.2) admits a unique smooth solution $X(t,.)$ defined on $[0,+\infty)$ such that, for every $t \in [0,+\infty)$, $X(t,.) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a mooth starshaped embedding satisfying $\kappa(X(t,.)) \in \Gamma$. Moreover, $X(t,.)$ converges in $C^\infty(\mathbb{S}^n, \mathbb{R}^{n+1})$ to a smooth starshaped embedding $X_\infty : \mathbb{S}^n \to \mathbb{R}^{n+1}$ as $t \to +\infty$, satisfying

$$\frac{1}{F(\kappa(X_\infty))} = f(X_\infty),$$

and for any $m \in \mathbb{N}$, $t \in [0,+\infty)$, we have

$$\|X(t,.) - X_\infty\|_{C^m(\mathbb{S}^n,\ \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t}, \tag{1.11}$$

where C_m and λ_m are positive constants depending only on m, f, F, r_1, r_2 and X_0.

Remark 1.1. There are many smooth starshaped embeddings $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying condition (1.10) in Theorem 1.1. Indeed, it suffices to take $X_0(x) = rx, x \in \mathbb{S}^n$, where r is any positive constant such that $0 < r \leq r_1$, with r_1 as in (1.8). Using conditions (1.7)-(1.8), it is easy to see that (1.10) is satisfyed.

As a consequence of Theorem 1.1, we recover the existence result for Weingarten hypersurfaces of L.Caffarelli, L.Nirenberg, and J.Spruck [4] stated above. Moreover, we prove the uniqueness of starshaped solutions of (1.9). Namely we have :

Corollary 1.1. Let $F \in C^\infty(\Gamma) \cap C^0\left(\overline{\Gamma}\right)$ be a positive symmetric function satisfying (1.3)-(1.6), and let $f \in C^\infty\left(\mathbb{R}^{n+1} \setminus \{0\}\right)$ be a positive function satisfying (1.7)-(1.8). Then there exists a smooth starshaped embedding $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $\kappa(X) \in \Gamma$, and satisfying

$$\frac{1}{F(\kappa(X))} = f(X)\,. \tag{1.12}$$

Moreover, X is the unique starshaped solution of (1.12) with $\kappa(X) \in \Gamma$.

When the homogeneity degree k of the curvature function F satisfies $k > 1$, we need additional conditions on the initial embedding X_0. More precisely, we have

Theorem 1.2. Let $F \in C^\infty(\Gamma) \cap C^0\left(\overline{\Gamma}\right)$ be a positive symmetric function satisfying conditions (1.3)-(1.6) such that the homogeneity degree k of F satisfies $k > 1$, and let $f \in C^\infty\left(\mathbb{R}^{n+1} \setminus \{0\}\right)$ be a positive function satisfying (1.7)-(1.8). Let M_0 be a closed compact starshaped hypersurface in $\mathbb{R}^{n+1}$, paramatrized by a smooth embedding $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ of the form (1.1) such that

$$\kappa(X_0) \in \Gamma \text{ and } 0 \leq -\left(\frac{1}{F(\kappa(X_0))} - f(X_0)\right)\frac{|\nabla X_0|}{|X_0|} \leq \frac{kR_1}{(k+1)R_2} \min_{R_1 \leq |Y| \leq R_2} f(Y)\,, \quad (1.13)$$

where

$$R_1 = \min\left(r_1, \min_{x\in\mathbb{S}^n} |X_0(x)|\right)\,,\ R_2 = \max\left(r_2, \max_{x\in\mathbb{S}^n} |X_0(x)|\right)$$

and r_1, r_2 are as in (1.8). Then the evolution problem (1.2) admits a unique smooth solution $X(t,.)$ defined on $[0,+\infty)$ such that, for every $t \in [0,+\infty)$, $X(t,.) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth starshaped embedding satisfying $\kappa(X(t,.)) \in \Gamma$. Moreover, $X(t,.)$ converges in $C^\infty(\mathbb{S}^n, \mathbb{R}^{n+1})$ to a smooth starshaped embedding $X_\infty : \mathbb{S}^n \to \mathbb{R}^{n+1}$ as $t \to +\infty$, satisfying

$$\frac{1}{F(\kappa(X_\infty))} = f(X_\infty),$$

and for any $m \in \mathbb{N}$, $t \in [0,+\infty)$, we have

$$\|X(t,.) - X_\infty\|_{C^m(\mathbb{S}^n,\, \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t}, \quad (1.14)$$

where C_m and λ_m are positive constants depending only on m, f, F, r_1, r_2 and X_0.

Remark 1.2. There are many smooth starshaped embeddings $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying condition (1.13) in Theorem 1.2. Indeed, by applying Corollary 1.1 to the functions $F^{1/k}, f^{1/k}$ instead of F, f (as it can easily be seen, conditions (1.3)-(1.6) and (1.7)-(1.8) are still satisfied with a new homgeneï£¡ity degree $k = 1$ for $F^{1/k}$), then we get a smooth starshaped embedding $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying :

$$\frac{1}{F(\kappa(X))} = f(X).$$

If we take $X_0 = rX$, where r is any positive constant such that $r \in [1, 1+\varepsilon)$, with $\varepsilon > 0$ small enough, then it is not difficult to see, by using condition (1.7)-(1.8), that X_0 satisfies condition (1.13) in Theorem 1.2.

2 Preliminaries

In this section, we recall some expressions for the relevant geometric quantities of smooth closed compact starshaped hypersurfaces $M \subset \mathbb{R}^{n+1}$. As we saw in the previous section, there is a smooth embedding $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$ parametrizing M, which is of the form

$$X(x) = \rho(x)x,\ x \in \mathbb{S}^n.$$

For any local orthonormal frame $\{e_1, ..., e_n\}$ on $\mathbb{S}^n$ (endowed with its standard metric), covariant differentiation with respect to e_i will be denoted by $\nabla_i, \nabla_{ij}, \nabla_{ijk}, ...$, and we let ∇ be the gradient on $\mathbb{S}^n$. Then in terms of the radial function ρ, the metric $g = [g_{ij}]$ induced by X and its inverse $g^{-1} = [g^{ij}]$ are given by

$$g_{ij} = \langle \nabla_i X, \nabla_j X\rangle = \rho^2\delta_{ij} + \nabla_i\rho\nabla_j\rho, \quad g^{ij} = \rho^{-2}\left(\delta_{ij} - \frac{\nabla_i\rho\nabla_j\rho}{\rho^2 + |\nabla\rho|^2}\right), \quad (2.1)$$

where $\langle\ ,\ \rangle$ is the standard metric on $\mathbb{R}^{n+1}$, and δ_{ij} are Kronecker symbols. The unit outer normal to M is

$$\nu = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}} \tag{2.2}$$

and the the second fundamental form of M is given by

$$h_{ij} = -\langle\nabla_{ij}X, \nu\rangle = \left(\rho^2 + |\nabla\rho|^2\right)^{-\frac{1}{2}}(\rho^2\delta_{ij} + 2\nabla_i\rho\nabla_j\rho - \rho\nabla_{ij}\rho), \tag{2.3}$$

The principal curvatures of M are the eigenvalues of the second fundamental form with respect to the induced metric g. Thus, λ is a principal curvature if

$$\det[h_{ij} - \lambda g_{ij}] = 0,$$

or equivalently

$$\det[a_{ij} - \lambda\delta_{ij}] = 0,$$

where the symmetric matrix $[a_{ij}]$ is given by

$$[a_{ij}] = [g^{ij}]^{\frac{1}{2}}[h_{ij}][g^{ij}]^{\frac{1}{2}} \tag{2.4}$$

and where $[g^{ij}]^{\frac{1}{2}}$ is the positive square root of $[g^{ij}]$ which is given by

$$[g^{ij}]^{\frac{1}{2}} = \rho^{-1}\left[\delta_{ij} - \frac{\nabla_i\rho\nabla_j\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}(\rho + \sqrt{\rho^2 + |\nabla\rho|^2})}\right]. \tag{2.5}$$

Let us now make some remarks about the curvature function F. Since F is symmetric, it is well known that F can be seen as a smooth function on the set of real symmetric $n \times n$ matrices $[a_{ij}]$. More precisely, we have

$$F \in C^\infty(M(\Gamma)) \cap C^0(\overline{M(\Gamma)})$$

where $M(\Gamma)$ is the convex cone of real symmetric $n \times n$ matrices with eigenvalue vector in the cone Γ. One can also prove that conditions (1.3)-(1.6) are equivalent to the following conditions when F is seen as function on $M(\Gamma)$:

$$[F_{ij}] \text{ is positive definite on } M(\Gamma)\ , \tag{2.6}$$

where $F_{ij} = \frac{\partial F}{\partial a_{ij}}$.

$$F \text{ is homogeneous of degree } k > 0 \text{ on } M(\Gamma) \text{ and } F \equiv 0 \text{ on } \partial M(\Gamma) \tag{2.7}$$

$$\log F \text{ is concave on } M(\Gamma). \tag{2.8}$$

$$F(\delta_{ij}) = 1. \tag{2.9}$$

We note here that a smooth function G on $M(\Gamma)$ is concave if

$$\sum_{i,j=1}^{n}\sum_{k,l=1}^{n} G_{ij,kl}\,\eta_{ij}\eta_{kl} \leq 0 \quad \text{on } M(\Gamma)$$

for all real symetric $n \times n$ matrices (η_{ij}), where

$$G_{ij,kl} = \frac{\partial^2 G}{\partial a_{kl}\partial a_{ij}}.$$

Now, we will show that equation (1.2) is equivalent to an evolution equation depending on the radial function ρ. We proceed as in [10], first suppose that $X(t,.)$ is a solution of (1.2) such that for each $t \in [0, +\infty)$, $X(t, .)$ is an embedding of a smooth closed compact hypersurface M_t in $\mathbb{R}^{n+1}$, which is starshaped with respect to the origin and such that the vector of its principal curvatures $\kappa = (\kappa_1, ..., \kappa_n)$ lies in the cone Γ. If we choose a family of suitable diffeomorphisms $\varphi(t, .) : \mathbb{S}^n \to \mathbb{S}^n$ then

$$X(t, x) = \rho(t, \varphi(t, x))\varphi(t, x),$$

where $\rho(t, .) : \mathbb{S}^n \to \mathbb{R}^+$ is the radial function of M_t. We have

$$\partial_t X = (\langle \nabla\rho, \partial_t\varphi\rangle + \partial_t\rho)\,\varphi + \rho\partial_t\varphi$$

and the unit outer normal is given by

$$\nu = \frac{\rho\varphi - \nabla\rho}{\sqrt{|\nabla\rho|^2 + \rho^2}}\,.$$

Using the fact that $\partial_t\varphi$ is tangential to S^n at φ, it follows that

$$\langle \partial_t X, \nu\rangle = \left(\rho^2 + |\nabla\rho|^2\right)^{-\frac{1}{2}} \rho\partial_t\rho$$

hence ρ satisfies the initial value problem

$$\begin{cases} \partial_t\rho = \mathcal{F}[\rho(t, .)] \\ \rho(0, x) = \rho_0(x), \ x \in \mathbb{S}^n \end{cases} \tag{2.10}$$

where the nonlinear operator $\mathcal{F}$ is defined on smooth functions $\rho : \mathbb{S}^n \to (0, +\infty)$, such that the matrix $[a_{ij}]$ given in (2.4) lies in $M(\Gamma)$, by

$$\mathcal{F}[\rho](x) = \left(\frac{1}{F(a_{ij}(x))} - f(\rho(x)x)\right)\frac{\sqrt{\rho^2(x) + |\nabla\rho(x)|^2}}{\rho(x)}\,. \tag{2.11}$$

From now on, what we mean by admissible function is a smooth function $\rho : [0, T] \times \mathbb{S}^n \to (0, +\infty)$ such that the matrix $[a_{ij}]$ defined by (2.4) lies in the cone $M(\Gamma)$ defined above. Conversely, suppose that $\rho : [0, T] \times \mathbb{S}^n \to (0, +\infty)$ is an admissible solution of (2.10) . If we set

$$X(t, x) = \rho(t, \varphi(t, x))\varphi(t, x)\ ,\ (t, x) \in [0, T] \times \mathbb{S}^n,$$

where $\varphi(t, .) : \mathbb{S}^n \to \mathbb{S}^n$ is a smooth diffeomorphism satisfying the ODE

$$\begin{cases} \partial_t\varphi(t, x) = Z(t, \varphi(t, x)) \\ \varphi(0, x) = x, \ x \in \mathbb{S}^n \end{cases} \tag{2.12}$$

with

$$Z(t, y) = -\left(\frac{1}{F(a_{ij}(t, y))} - f(\rho(t, y)y)\right)\frac{\nabla\rho(t, y)}{\rho\sqrt{|\nabla\rho(t, y)|^2 + \rho^2(t, y)}},\ (t, y) \in [0, T] \times \mathbb{S}^n, \tag{2.13}$$

then it is not difficult to see that X is a smooth starshaped embedding which is a solution of (1.2) with $X_0(x) = \rho_0(x)x$.

The condition (2.6) implies that (2.10) is parabolic on admissible functions ρ. The classical theory of parabolic equations yields the existence and uniqueness of a smooth admissible solution ρ defined on a small intervall $[0, T]$. From the classical theory of ordinary differential equations, there exists a family of diffeomorphisms $\varphi(t, .)$ defined on a small interval $[0, T]$ and satisfying (2.12). Thus by taking $X(t, x) = \rho(t, \varphi(t, x))\varphi(t, x)$ we obtain a solution of (1.2) defined on $[0, T]$.

Usually in order to get high order estimates it is useful to represent the hypersurface locally as graph over an open set $\Omega \subset \mathbb{R}^n$. Locally, after rotating the coordinates axes , we may suppose that M is the graph of a smooth function $u : \Omega \to \mathbb{R}$. Hence the metric of M, the outer normal vector and the second fundamental form can be written respectively

$$g_{ij} = \delta_{ij} + D_iuD_ju\ ,\ \ g^{ij} = \delta_{ij} - \frac{D_iuD_ju}{1+|Du|^2} \tag{2.14}$$

$$\nu = \frac{1}{\sqrt{1+|Du|^2}}(Du,-1)\ , \tag{2.15}$$

$$h_{ij} = \frac{D_{ij}u}{\sqrt{1+|Du|^2}} \tag{2.16}$$

where D_k, D_{ij} are the usual first and second order derivatives in $\mathbb{R}^n$, and $Du = (D_1u, ..., D_nu)$. The principal curvatures of M are the eigenvalues of the symmetric matrix $[a_{ij}]$ given by

$$[a_{ij}] = [g^{ij}]^{\frac{1}{2}}[h_{ij}][g^{ij}]^{\frac{1}{2}} \tag{2.17}$$

where $[g^{ij}]^{\frac{1}{2}}$ is the positive square root of $[g^{ij}]$. On ca compute

$$a_{ij} = \frac{1}{v}\left(D_{ij}u - \frac{D_iuD_luD_{jl}u}{v(1+v)} - \frac{D_juD_luD_{il}u}{v(1+v)} + \frac{D_iuD_juD_kuD_luD_{kl}u}{v^2(1+v)^2}\right) \tag{2.18}$$

with $v = \sqrt{1+|Du|^2}$.

In this case equation (1.2) takes the form

$$\partial_t u = -\left(\frac{1}{F(a_{ij})} - f(x,u)\right)\sqrt{1+|Du|^2}\ . \tag{2.19}$$

In what follows, what we mean by an admissible solution of (2.19) is a smooth function $u : [0,T]\times\Omega \to \mathbb{R}$ such that the matrix $[a_{ij}]$ defined by (2.18) lies in the cone $M(\Gamma)$ defined above, and satisfying (2.19).

3 C^1-estimates and exponential decay

In this section we prove C^1-estimates on solutions ρ of (2.10) and exponential decay of its derivatives $\partial_t\rho$. First we prove C^0-estimates.

Proposition 3.1. Suppose that F satisfies conditions (1.3)-(1.6) and that f satisfies conditions (1.7)-(1.8). Let $\rho : [0,T]\times\mathbb{S}^n \to (0,+\infty)$ be an admissible solution of (2.10). Then we have, for all $(t,x)\in[0,T]\times\mathbb{S}^n$,

$$R_1 \le \rho(t,x) \le R_2 \tag{3.1}$$

where

$$R_1 = \min\left(r_1, \min_{x\in\mathbb{S}^n}\rho_0(x)\right) \quad \text{and} \quad R_2 = \max\left(r_2, \max_{x\in\mathbb{S}^n}\rho_0(x)\right)$$

and where r_1, r_2 are as in (1.8).

Proof. Let $\rho : [0,T]\times\mathbb{S}^n \to (0,+\infty)$ be an admissible solution of (2.10). Let $(t_0,x_0)\in[0,T]\times\mathbb{S}^n$ such that

$$\rho(t_0,x_0) = \max_{(t,x)\in[0,T]\times\in\mathbb{S}^n}\rho(t,x). \tag{3.2}$$

We want to prove

$$\rho(t_0,x_0) \le R_2. \tag{3.3}$$

If $t_0 = 0$, then

$$\rho(t_0,x_0) = \rho_0(x_0) \le R_2,$$

so (3.3) is proved in this case. Suppose now that $t_0 > 0$. Then we have

$$\partial_t \rho(t_0, x_0) \geq 0 \tag{3.4}$$

$$\nabla \rho(t_0, x_0) = 0 \tag{3.5}$$

and the matrix

$$[\nabla_{ij} \rho(t_0, x_0)] \text{ is negative semi-definite.} \tag{3.6}$$

It follows from (3.5) and (3.6) that the matrix $[a_{ij}]$ defined by (2.4) satisfies in the sense of operators

$$a_{ij}(t_0, x_0) \geq \rho^{-1}(t_0, x_0)\delta_{ij}. \tag{3.7}$$

Since by (1.3) F is monotone, then by using (3.7) we have at (t_0, x_0)

$$F(a_{ij}) \geq F(\rho^{-1}\delta_{ij}) = \rho^{-k} F(\delta_{ij}) = \rho^{-k}, \tag{3.8}$$

where we have used the fact that F is homogenous of degree k and $F(\delta_{ij}) = 1$. Using equation (2.10) and (3.8), we obtain

$$\partial_t \rho(t_0, x_0) \leq \rho^k(t_0, x_0) - f(\rho(t_0, x_0)x_0). \tag{3.9}$$

Combining (3.4) and (3.9) gives

$$f(\rho(t_0, x_0)x_0) \leq \rho^k(t_0, x_0). \tag{3.10}$$

But from (1.7) and (1.8) we have that if $X \in \mathbb{R}^{n+1}$ satisfies $|X| > r_2$, then $f(X) > |X|^k$. So it follows from (3.10) that $\rho(t_0, x_0) \leq r_2$. This proves (3.3) since $r_2 \leq R_2$.

It remains now to prove that $\rho(t, x) \geq R_1$. As before, if we let $(t_0, x_0) \in [0, T] \times \mathbb{S}^n$ such that

$$\rho(t_0, x_0) = \min_{(t,x) \in [0,T] \times \in \mathbb{S}^n} \rho(t, x),$$

then in the same way as before, we prove that $\rho(t_0, x_0) \geq R_1$. This achieves the proof of Proposition 3.1.

□

We prove now the exponential decay of $\partial_t \rho$.

Proposition 3.2. Assume that F satisfies conditions (1.3)-(1.6) and that f satisfies conditions (1.7)-(1.8). Let $\rho : [0, T] \times \mathbb{S}^n \to (0, +\infty)$ be an admissible solution of (2.10). We suppose that $\mathcal{F}[\rho_0] \geq 0$ if $k \leq 1$ and $\mathcal{F}[\rho_0] \leq 0$ if $k > 1$, where the operator $\mathcal{F}$ is given by (2.11), and k is the homogeneity degree of F. Then we have, for any $(t, x) \in [0, T] \times \mathbb{S}^n$,

$$\partial_t \rho(t, x) \geq 0 \text{ if } k \leq 1$$

and

$$\partial_t \rho(t, x) \leq 0 \text{ if } k > 1.$$

Moreover, there exists a positive constant λ depending only on f, r_1, r_2 and ρ_0 such that, for any $t \in [0, T]$, we have

$$\max_{x \in \mathbb{S}^n} |\partial_t \rho(t, x)| \leq \frac{R_2}{R_1} \max_{x \in \mathbb{S}^n} \left| \mathcal{F}[\rho_0](x) \right| e^{-\lambda t}, \tag{3.11}$$

where

$$R_1 = \min\left(r_1, \min_{x \in \mathbb{S}^n} \rho_0(x)\right) \text{ and } R_2 = \max\left(r_2, \max_{x \in \mathbb{S}^n} \rho_0(x)\right)$$

and where r_1, r_2 are as in (1.8).

The proof of the above proposition is based on the following lemma which asserts that the function $\rho^{-1}\partial_t \rho$ satisfies a second order parabolic equation.

Lemma 3.1. Suppose that F satisfies conditions (1.3)-(1.6). Let $\rho : [0,T] \times \mathbb{S}^n \to (0,+\infty)$ be an admissible solution of (2.10) and set $G = \rho^{-1}\partial_t \rho$. Then we have for some smooth functions $A_l, l = 1, ..., n$ (depending on ρ and its derivatives) ,

$$\partial_t G = \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} G + \sum_{l=1}^{n} A_l \nabla_l G - \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left(\rho \partial_\rho f - f - \frac{k-1}{F} \right) G$$

where

$$A_{ij} = \frac{1}{\rho^2 F^2} \sum_{l,m=1}^{n} \gamma_{il} F_{lm} \gamma_{mj} \tag{3.12}$$

and

$$\gamma_{ij} = \delta_{ij} - \frac{\nabla_i \rho \nabla_j \rho}{\sqrt{\rho^2 + |\nabla \rho|^2} \left(\rho + \sqrt{\rho^2 + |\nabla \rho|^2} \right)} . \tag{3.13}$$

Proof. We recall that by (2.10), ρ satifies

$$\partial_t \rho = \mathcal{F}[\rho] \tag{3.14}$$

where

$$\mathcal{F}[\rho] = \left(\frac{1}{F(a_{ij})} - f(\rho x) \right) \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho} \tag{3.15}$$

and where a_{ij} is given by (2.4).

In view of the definition of G and (3.15) it will be usefull to work with the function $r = \log \rho$ instead of ρ. Equation (3.14) becomes then

$$\partial_t r = \left(\frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \sqrt{1 + |\nabla r|^2} \tag{3.16}$$

where a_{ij} takes the form

$$a_{ij} = \frac{e^{-r} b_{ij}}{\sqrt{1 + |\nabla r|^2}} \tag{3.17}$$

with

$$\begin{cases} b_{ij} = \gamma_{il} (\delta_{lm} + \nabla_l r \nabla_m r - \nabla_{lm} r) \gamma_{mj} \\ \gamma_{ij} = \delta_{ij} - \dfrac{\nabla_i r \nabla_j r}{\sqrt{1 + |\nabla r|^2} \left(1 + \sqrt{1 + |\nabla r|^2} \right)}. \end{cases} \tag{3.18}$$

Now, we have

$$G = \rho^{-1} \partial_t \rho = \partial_t r = \left(\frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \sqrt{1 + |\nabla r|^2}, \tag{3.19}$$

so

$$\begin{aligned} \partial_t G = & -e^{-r} \sqrt{1 + |\nabla r|^2} \sum_{i,j=1}^{n} \frac{F_{ij}}{F^2} \partial_t a_{ij} - \sqrt{1 + |\nabla r|^2} \partial_\rho f(e^r x) \partial_t r \\ & + \left(\frac{1}{F(a_{ij})} - f(e^r x) \right) e^{-r} \left(-\sqrt{1 + |\nabla r|^2} \partial_t r + \frac{\langle \nabla \partial_t r, \nabla r \rangle}{\sqrt{1 + |\nabla r|^2}} \right). \end{aligned} \tag{3.20}$$

Using (3.17) and (3.18), one can check that for some smooth functions $B_{ij}^l(t,x)$ $(l = 1, ..., n)$, we have

$$\partial_t a_{ij} = -a_{ij} \partial_t r - \frac{e^{-r}}{\sqrt{1 + |\nabla r|^2}} \sum_{l,m=1}^{n} \gamma_{il} \gamma_{mj} \nabla_{lm} \partial_t r + \sum_{l=1}^{n} B_{ij}^l \nabla_l \partial_t r, \tag{3.21}$$

and since $\partial_t r = G$, it follows from (3.20) and (3.21) that

$$\partial_t G = \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}G + \sum_{l=1}^{n} A_l \nabla_l G - \partial_\rho f(e^r x)\sqrt{1+|\nabla r|^2}G - G^2$$
$$+ e^{-r}\sqrt{1+|\nabla r|^2}\sum_{i,j=1}^{n}\frac{F_{ij}}{F^2}a_{ij}G, \tag{3.22}$$

where

$$A_{ij} = \frac{e^{-2r}}{F^2}\sum_{l,m=1}^{n}\gamma_{il}\gamma_{mj}F_{lm}$$

and $A_l(t,x)$ ($l = 1, ..., n$) are smooth functions. Since F is homogeneous of degree k, then

$$\sum_{i,j=1}^{n}\frac{F_{ij}}{F^2}a_{ij} = \frac{k}{F},$$

so it follows from (3.22) by using (3.19) that

$$\partial_t G = \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}G + \sum_{l=1}^{n} A_l\nabla_l G - \sqrt{1+|\nabla r|^2}e^{-r}\left(e^r\partial_\rho f - \frac{k}{F}\right)G - G^2$$
$$= \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}G + \sum_{l=1}^{n} A_l\nabla_l G - \frac{\sqrt{\rho^2+|\nabla\rho|^2}}{\rho^2}\left(\rho\partial_\rho f - f - \frac{k-1}{F}\right)G.$$

This achieves the proof Lemma 3.1.

□

We need also the following lemma which is a well known version of the maximum principle for parabolic equations.

Lemma 3.2. Let $G : [0,T]\times\mathbb{S}^n \to \mathbb{R}$ be a smooth function satisfying

$$\partial_t G \geq \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}G + \sum_{l=1}^{n} A_l\nabla_l G + AG \tag{3.23}$$

for some smooth functions $A, A_l, A_{ij}, (l,i,j = 1,...,n)$, such that the matrix $[A_{ij}]$ is positive semi-definite. Suppose

$$\min_{x\in\mathbb{S}^n} G(0,x) \geq 0,$$

then

$$\min_{(t,x)\in[0,T]\times\mathbb{S}^n} G(t,x) \geq 0.$$

Proof. Let $\lambda \in \mathbb{R}$ such that

$$\lambda < -\max_{(t,x)\in[0,T]\times\mathbb{S}^n} |A(t,x)|, \tag{3.24}$$

and consider the function $\widetilde{G}$ defined by $\widetilde{G}(t,x) = e^{\lambda t}G(t,x)$. To prove the lemma it is equivalent to prove that

$$\min_{(t,x)\in[0,T]\times\mathbb{S}^n} \widetilde{G}(t,x) \geq 0. \tag{3.25}$$

By using (3.23), $\widetilde{G}$ satisfies

$$\partial_t\widetilde{G} \geq \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}\widetilde{G} + \sum_{l=1}^{n} A_l\nabla_l\widetilde{G} + (\lambda + A)\widetilde{G}. \tag{3.26}$$

Let $(t_0, x_0) \in [0, T] \times \mathbb{S}^n$ such that

$$\widetilde{G}(t_0, x_0) = \min_{(t,x) \in [0,T] \times \mathbb{S}^n} \widetilde{G}(t, x).$$

We want to prove

$$\widetilde{G}(t_0, x_0) \geq 0. \tag{3.27}$$

If $t_0 = 0$, then

$$\widetilde{G}(t_0, x_0) = \widetilde{G}(0, x_0) = G(0, x_0) \geq 0$$

and (3.27) is proved in this case. If $t_0 > 0$, then

$$\partial_t \widetilde{G}(t_0, x_0) \leq 0 \tag{3.28}$$

$$\nabla \widetilde{G}(t_0, x_0) = 0 \tag{3.29}$$

and the matrix

$$\left[\nabla_{ij} \widetilde{G}(t_0, x_0)\right] \text{ is positive semi-definite.} \tag{3.30}$$

It follows from (3.26), (3.28), (3.29) and (3.30) that

$$(\lambda + A(t_0, x_0)) \widetilde{G}(t_0, x_0) \leq 0$$

which implies that $\widetilde{G}(t_0, x_0) \geq 0$ since $\lambda + A(t_0, x_0) < 0$ by (3.24). Thus (3.27) is proved and the lemma follows.

□

Proof of Proposition 3.2. Let $G = \rho^{-1} \partial_t \rho$. Then by Lemma 3.1 we have

$$\partial_t G = \sum_{i,j} A_{ij} \nabla_{ij} G + \sum_{l=1}^{n} A_l \nabla_l G$$
$$- \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left(\rho \partial_\rho f - f - \frac{k-1}{F} \right) G. \tag{3.31}$$

By (1.3) (or equivalently (2.6)) the matrix $[F_{ij}]$ is positive definite. So it follows from (3.12) that $[A_{ij}]$ is positive semi-definite. We distinguish two cases :

<u>First case</u> : $0 < k \leq 1$. Since G satisfies (3.31) and $G(0, x) = \rho_0^{-1}(x) \partial_t \rho(0, x) = \rho_0^{-1}(x) \mathcal{F}[\rho_0](x) \geq 0$ by hypothesis, then by Lemma 3.2 we have for any $t \in [0, T]$,

$$\min_{x \in \mathbb{S}^n} G(t, x) \geq 0. \tag{3.32}$$

In particular, (3.32) implies that $\partial_t \rho \geq 0$ since $\partial_t \rho = \rho G$. Now we have, since ρ satisfies (2.10),

$$G = \rho^{-1} \partial_t \rho = \left(\frac{1}{F(a_{ij})} - f(\rho x) \right) \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} ,$$

so it follows from (3.32) that

$$\frac{1}{F(a_{ij})} \geq f(\rho x)$$

which implies that the last term in (3.31) is bounded from below as

$$\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left(\rho \partial_\rho f - f - \frac{k-1}{F} \right) \geq \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2} \left(\rho \partial_\rho f - kf \right). \tag{3.33}$$

Since f satisfies (1.7), then $\rho \partial_\rho f - kf > 0$, and since $R_1 \leq \rho(t, x) \leq R_2$ by Proposition 3.1, we deduce that

$$\rho \partial_\rho f - kf \geq \delta_0 \tag{3.34}$$

for some constant $\delta_0 > 0$ depending only on f, R_1 and R_2. It follows from (3.33) and (3.34) by using Proposition 3.1 that

$$\frac{\sqrt{\rho^2 + |\nabla\rho|^2}}{\rho^2}\left(\rho\partial_\rho f - f - \frac{k-1}{F}\right) \geq \frac{\delta_0}{R_2}\,. \tag{3.35}$$

By setting $\lambda = \frac{\delta_0}{R_2}$ and $\widetilde{G}(t,x) = e^{\lambda t}G(t,x)$, it follows from (3.31) that $\widetilde{G}$ satisfies

$$\begin{aligned}\partial_t\widetilde{G} &= \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}\widetilde{G} + \sum_{l=1}^{n} A_l\nabla_l\widetilde{G} \\ &- \frac{\sqrt{\rho^2 + |\nabla\rho|^2}}{\rho^2}\left(\rho\partial_\rho f - f - \frac{k-1}{F}\right)\widetilde{G} + \lambda\widetilde{G}\end{aligned}$$

which gives by using (3.35) and the fact that $\widetilde{G} \geq 0$,

$$\partial_t\widetilde{G} \leq \sum_{i,j=1}^{n} A_{ij}\nabla_{ij}\widetilde{G} + \sum_{l=1}^{n} A_l\nabla_l\widetilde{G}. \tag{3.36}$$

It follows from (3.36) by applying Lemma 3.2 to the function $-\widetilde{G} + \max_{x\in\mathbb{S}^n}\widetilde{G}(0,x)$ that

$$-\widetilde{G} + \max_{x\in\mathbb{S}^n}\widetilde{G}(0,x) \geq 0$$

which implies

$$\max_{x\in\mathbb{S}^n} G(t,x) \leq e^{-\lambda t}\max_{x\in\mathbb{S}^n} G(0,x). \tag{3.37}$$

But from the definition of G we have

$$\partial_t\rho = \rho G, \tag{3.38}$$

so it follows from (3.37) and (3.38) since $\partial_t\rho \geq 0$ and $R_1 \leq \rho \leq R_2$ by Proposition 3.1, that

$$|\partial_t\rho| \leq R_2 e^{-\lambda t}\max_{x\in\mathbb{S}^n} G(0,x) = R_2 e^{-\lambda t}\max_{x\in\mathbb{S}^n}\left(\frac{\mathcal{F}[\rho_0](x)}{\rho_0(x)}\right) \leq \frac{R_2}{R_1}e^{-\lambda t}\max_{x\in\mathbb{S}^n}\mathcal{F}[\rho_0](x).$$

This proves Proposition 3.2 in the case $0 < k \leq 1$.

Second case : $k > 1$. Since G satisfies (3.31) and $G(0,x) = \rho_0^{-1}(x)\partial_t\rho(0,x) = \rho_0^{-1}(x)\mathcal{F}[\rho_0](x) \leq 0$ by hypothesis, then by Lemma 3.2 we have for any $t \in [0,T]$,

$$\max_{x\in\mathbb{S}^n} G(t,x) \leq 0. \tag{3.39}$$

In particular, (3.39) implies that $\partial_t\rho \leq 0$ since $\partial_t\rho = \rho G$. Now we have, since ρ satisfies (2.10),

$$G = \rho^{-1}\partial_t\rho = \left(\frac{1}{F(a_{ij})} - f(\rho x)\right)\frac{\sqrt{\rho^2 + |\nabla\rho|^2}}{\rho^2}\,,$$

so it follows from (3.39) that

$$\frac{1}{F(a_{ij})} \leq f(\rho x)$$

which implies that the last term in (3.31) is bounded from below as

$$\frac{\sqrt{\rho^2 + |\nabla\rho|^2}}{\rho^2}\left(\rho\partial_\rho f - f - \frac{k-1}{F}\right) \geq \frac{\sqrt{\rho^2 + |\nabla\rho|^2}}{\rho^2}\left(\rho\partial_\rho f - kf\right). \tag{3.40}$$

Since f satisfies (1.7), then $\rho\partial_\rho f - kf > 0$, and since $R_1 \leq \rho(t,x) \leq R_2$ by Proposition 3.1, we deduce that

$$\rho\partial_\rho f - kf \geq \delta_0 \tag{3.41}$$

for some constant $\delta_0 > 0$ depending only on f, R_1 and R_2. It follows from (3.40) and (3.41) by using Proposition 3.1 that

$$\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2}\left(\rho \partial_\rho f - f - \frac{k-1}{F}\right) \geq \frac{\delta_0}{R_1}\,. \tag{3.42}$$

By setting $\lambda = \frac{\delta_0}{R_1}$ and $\widetilde{G}(t,x) = e^{\lambda t} G(t,x)$, it follows from (3.31) that

$$\partial_t \widetilde{G} = \sum_{i,j=1}^{n} A_{ij} \nabla_{ij} \widetilde{G} + \sum_{l=1}^{n} A_l \nabla_l \widetilde{G}$$

$$- \frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho^2}\left(\rho \partial_\rho f - f - \frac{k-1}{F}\right)\widetilde{G} + \lambda \widetilde{G}$$

which gives by using (3.42) and the fact that $\tilde{G} \leq 0$,

$$\partial_t \widetilde{G} \geq \sum_{i,j=1}^{n} A_{ij}(t,x) \nabla_{ij} \widetilde{G} + \sum_{l=1}^{n} A_l(t,x) \nabla_l \widetilde{G}. \tag{3.43}$$

It follows from (3.43) by applying Lemma 3.2 to the function $\widetilde{G} - \min\limits_{x \in \mathbb{S}^n} \widetilde{G}(0,x)$ that

$$\widetilde{G} - \min_{x \in \mathbb{S}^n} \widetilde{G}(0,x) \geq 0$$

which implies

$$\min_{x \in \mathbb{S}^n} G(t,x) \geq e^{-\lambda t} \min_{x \in \mathbb{S}^n} G(0,x). \tag{3.44}$$

But from the definition of G we have

$$\partial_t \rho = \rho G, \tag{3.45}$$

so it follows from (3.44) and (3.45) since $\partial_t \rho \leq 0$ and $\rho \leq R_2$ by Proposition 3.1, that

$$|\partial_t \rho| \leq -R_2 e^{-\lambda t} \min_{x \in \mathbb{S}^n} G(0,x) = R_2 e^{-\lambda t} \max_{x \in \mathbb{S}^n} |G(0,x)| = R_2 e^{-\lambda t} \max_{x \in \mathbb{S}^n} \left(\frac{|\mathcal{F}[\rho_0](x)|}{\rho_0(x)}\right)$$

$$\leq \frac{R_2}{R_1} e^{-\lambda t} \max_{x \in \mathbb{S}^n} |\mathcal{F}[\rho_0](x)|.$$

The proof of Proposition 3.2 is then complete. □

Now we are in position to prove C^1-estimates on the function ρ.

Proposition 3.3. Supoose that F satisfies conditions (1.3)-(1.6) and that f satisfies conditions (1.7)-(1.8). Let $\rho : [0,T] \times \mathbb{S}^n \to \mathbb{R}^+$ be an admissible solution of (2.10). We suppose that $\mathcal{F}[\rho_0] \geq 0$ if $k \leq 1$ and $\mathcal{F}[\rho_0] \leq 0$ if $k > 1$, where the operator $\mathcal{F}$ is given by (2.11), and k is the homogeneity degree of F. Then there exists a positive constant C depending only on f, r_1, r_2 and ρ_0 such that

$$\max_{(t,x) \in [0,T] \times \mathbb{S}^n} |\nabla \rho(t,x)| \leq C,$$

where r_1 and r_2 are as in (1.8).

Proof. As in the proof of Lemma 3.1, we introduce the function $r = \log \rho$. We have then

$$\partial_t r = \left(\frac{1}{F(a_{ij})} - f(e^r x)\right) e^{-r} \sqrt{1 + |\nabla r|^2} \tag{3.46}$$

where we recall that a_{ij} takes the form

$$a_{ij} = \frac{e^{-r} b_{ij}}{\sqrt{1+|\nabla r|^2}} \tag{3.47}$$

with

$$\begin{cases} b_{ij} = \gamma_{il}(\delta_{lm} + \nabla_l r \nabla_m r - \nabla_{lm} r)\gamma_{mj} \\ \gamma_{ij} = \delta_{ij} - \dfrac{\nabla_i r \nabla_j r}{\sqrt{1+|\nabla r|^2}\left(1+\sqrt{1+|\nabla r|^2}\right)}. \end{cases} \tag{3.48}$$

Set $H = \frac{1}{2}|\nabla r|^2$, and let $(t_0, x_0) \in [0,T] \times \mathbb{S}^n$ such that

$$H(t_0, x_0) = \max_{(t,x)\in[0,T]\times\mathbb{S}^n} H(t,x).$$

Let $\{e_1, ..., e_n\}$ be an orthonormal frame in a neighborhood of x_0 such that $\nabla_i(e_j) = 0$ at x_0, for $i, j = 1, ..., n$.

If $t_0 = 0$, then

$$H(t_0, x_0) = H(0, x_0) = \max_{x\in\mathbb{S}^n} H(0,x). \tag{3.49}$$

If $t_0 > 0$, then

$$\partial_t H(t_0, x_0) \geq 0 \tag{3.50}$$

$$\nabla_i H(t_0, x_0) = 0, \ i = 1, ..., n \tag{3.51}$$

and the matrix

$$[\nabla_{ij} H(t_0, x_0)] \text{ is negative semi-definite.} \tag{3.52}$$

In what follows, to simplify the notation we shall write F instead of $F(a_{ij})$, and f instead $f(e^r x)$. We have at (t_0, x_0), by using (3.51),

$$\begin{aligned} \partial_t H &= \langle \nabla \partial_t r, \nabla r \rangle = \left\langle \nabla \left(\left(\frac{1}{F} - f \right) e^{-r}\sqrt{1+|\nabla r|^2} \right), \nabla r \right\rangle \\ &= -e^{-r}\sqrt{1+|\nabla r|^2} \sum_{i,j=1}^{n} \frac{F_{ij}}{F^2}\langle \nabla a_{ij}, \nabla r\rangle - 2\sqrt{1+|\nabla r|^2}\partial_\rho f H \\ &\quad - \sqrt{1+|\nabla r|^2}\langle \nabla f, \nabla r\rangle - 2\left(\frac{1}{F} - f\right) e^{-r}\sqrt{1+|\nabla r|^2} H \ . \end{aligned} \tag{3.53}$$

Using (3.47) and (3.48), one can check that for some smooth functions $B^l_{ij}(t,x)$ $(l = 1, ..., n)$, we have, for any $\alpha = 1, ..., n$, at (t_0, x_0),

$$\nabla_\alpha a_{ij} = -\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{l,m=1}^{n} \gamma_{il}\gamma_{mj}\nabla_{\alpha lm} r + \sum_{l=1}^{n} B^l_{ij} \nabla_{\alpha l} r - a_{ij}\nabla_\alpha r.$$

It follows that, at (t_0, x_0),

$$\begin{aligned} \langle \nabla a_{ij}, \nabla r\rangle &= \sum_{\alpha=1}^{n} \nabla_\alpha a_{ij} \nabla_\alpha r \\ &= -\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}} \sum_{\alpha,l,m=1}^{n} \gamma_{il}\gamma_{mj}\nabla_{\alpha lm} r \nabla_\alpha r - 2a_{ij} H. \end{aligned} \tag{3.54}$$

The formula for commuting the order of covariant differentiation gives at (t_0, x_0)

$$\nabla_{\alpha lm} r = \nabla_{lm\alpha} r + \delta_{\alpha m}\nabla_l r - \delta_{lm}\nabla_\alpha r. \tag{3.55}$$

Combining (3.54) and (3.55) we get at (t_0, x_0)

$$
\begin{aligned}
\langle \nabla a_{ij}, \nabla r\rangle = &-\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}}\sum_{\alpha,l,m=1}^{n}\gamma_{il}\gamma_{mj}\nabla_{lm\alpha}r\nabla_\alpha r\\
&-\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}}\sum_{l,m=1}^{n}\gamma_{il}\gamma_{mj}\nabla_l r\nabla_m r\\
&+2\frac{e^{-r}}{\sqrt{1+|\nabla r|^2}}\sum_{l=1}^{n}\gamma_{il}\gamma_{lj}H-2a_{ij}H. \qquad (3.56)
\end{aligned}
$$

But we have at (t_0, x_0)

$$
\nabla_{lm}H=\frac{1}{2}\nabla_{lm}\left(|\nabla r|^2\right)=\sum_{\alpha=1}^{n}\nabla_{lm\alpha}r\nabla_\alpha r+\sum_{\alpha=1}^{n}\nabla_{l\alpha}r\nabla_{m\alpha}r. \qquad (3.57)
$$

Hence it follows from (3.50), (3.53), (3.56) and (3.57) that, at (t_0, x_0),

$$
\begin{aligned}
0\le\ & e^{-2r}\sum_{i,j=1}^{n}A_{ij}\nabla_{ij}H-e^{-2r}\sum_{\alpha,l,m=1}^{n}A_{lm}\nabla_{l\alpha}r\nabla_{m\alpha}r\\
&+2e^{-r}\sqrt{1+|\nabla r|^2}\sum_{i,j=1}^{n}\frac{F_{ij}}{F^2}a_{ij}H+e^{-2r}\sum_{i,j=1}^{n}A_{ij}\nabla_i r\nabla_j r-2e^{-2r}\text{Trace}\left[A_{ij}\right]H\\
&-2\sqrt{1+|\nabla r|^2}\partial_\rho fH-\sqrt{1+|\nabla r|^2}\langle\nabla f,\nabla r\rangle-\ 2\left(\frac{1}{F}-f\right)e^{-r}\sqrt{1+|\nabla r|^2}H, \qquad (3.58)
\end{aligned}
$$

where

$$
A_{ij}=\sum_{l,m=1}^{n}\frac{F_{lm}}{F^2}\gamma_{il}\gamma_{mj}.
$$

Since $[F_{ij}]$ is positive definite, then $[A_{ij}]$ is positive semi-definite. So we have at (t_0, x_0), by using (3.52),

$$
\sum_{i,j=1}^{n}A_{ij}\nabla_{ij}H\le 0, \qquad (3.59)
$$

$$
\sum_{\alpha,l,m=1}^{n}A_{lm}\nabla_{l\alpha}r\nabla_{m\alpha}r\ge 0 \qquad (3.60)
$$

and

$$
\sum_{i,j=1}^{n}A_{ij}\nabla_i r\nabla_j r-2\ \text{Trace}\left[A_{ij}\right]H\le 0. \qquad (3.61)
$$

Since F is homogenous of degree k, we have also

$$
\sum_{i,j=1}^{n}\frac{F_{ij}}{F^2}a_{ij}=\frac{k}{F}. \qquad (3.62)
$$

Thus we get from (3.58), (3.59), (3.60), (3.61), (3.61) and (3.62), at (t_0, x_0)

$$
\begin{aligned}
0\le\ & 2e^{-r}\sqrt{1+|\nabla r|^2}\frac{k}{F}H-2\sqrt{1+|\nabla r|^2}\partial_\rho fH\\
&-2\left(\frac{1}{F}-f\right)e^{-r}\sqrt{1+|\nabla r|^2}H-\sqrt{1+|\nabla r|^2}\langle\nabla f,\nabla r\rangle. \qquad (3.63)
\end{aligned}
$$

But by Proposition 3.2 we have $\partial_t\rho \geq 0$ if $k \leq 1$, and $\partial_t\rho \leq 0$ if $k > 1$. This implies, since ρ satifies (2.10), that $\frac{1}{F(a_{ij})} - f(\rho x) \geq 0$ if $k \leq 1$, and $\frac{1}{F(a_{ij})} - f(\rho x) \leq 0$ if $k > 1$. That is,

$$\frac{k-1}{F(a_{ij})} \leq (k-1)f(\rho x)$$

Hence it follows from (3.63) that at (t_0, x_0)

$$2\left(e^r\partial_\rho f - kf\right)H \leq e^r\langle\nabla f, \nabla r\rangle. \tag{3.64}$$

By (1.7) we have $\rho\partial_\rho f(\rho x) - kf(\rho x) > 0$, which implies that

$$\delta_0 = \min_{(\rho,x)\in[R_1,R_2]\times\mathbb{S}^n}\left(\rho\partial_\rho f(\rho x) - kf(\rho x)\right) > 0,$$

where R_1 and R_2 are defined in Proposition 3.1. Since $R_1 \leq \rho(t,x) \leq R_2$ by Proposition 3.1, then $e^r\partial_\rho f - kf \geq \delta_0$. Thus it follows from (3.64) at (t_0, x_0)

$$2\delta_0 H \leq e^r\langle\nabla f, \nabla r\rangle \leq R_2|\nabla f||\nabla r| = R_2|\nabla f|\sqrt{2}\sqrt{H}$$

that is

$$H(t_0, x_0) \leq \frac{C_0^2 R_2^2}{2\delta_0^2}\,, \tag{3.66}$$

where

$$C_0 = \sup_{R_1\leq|y|\leq R_2}|\nabla f(y)|.$$

It follows from (3.49) and (3.66) that

$$H(t_0, x_0) \leq \max\left(\max_{x\in\mathbb{S}^n} H(0,x), \frac{C_0^2 R_2^2}{2\delta_0^2}\right).$$

This ends the proof of Proposition 3.3.

□

4 C^2-estimates and proof of the main results

To get C^2-estimates we need to controll the principal curvatures.

Proposition 4.1. Suppose that F satisfies conditions (1.3)-(1.6) and that f satisfies conditions (1.7)-(1.8). Let $\rho : [0,T]\times\mathbb{S}^n \to (0,+\infty)$ be an admissible solution of (2.10). We suppose that

$$\begin{cases} \mathcal{F}[\rho_0] \geq 0 \;\text{ if }\; k \leq 1 \\ \\ 0 \leq -\mathcal{F}[\rho_0] \leq \dfrac{kR_1}{(k+1)R_2}\displaystyle\min_{R_1\leq|Y|\leq R_2} f(Y) \;\text{ if }\; k > 1, \end{cases} \tag{4.1}$$

where the operator $\mathcal{F}$ is given by (2.11), k is the homogeneity degree of F, and

$$R_1 = \min\left(r_1, \min_{x\in\mathbb{S}^n}\rho_0(x)\right) \quad , \quad R_2 = \max\left(r_2, \max_{x\in\mathbb{S}^n}\rho_0(x)\right)$$

with r_1, r_2 as in (1.8). Then there exists a positive constant C depending only on f, r_1, r_2 and ρ_0 sucht that

$$\max_{(t,x)\in[0,T]\times\mathbb{S}^n}\max_{1\leq i\leq n}|\kappa_i(t,x)| \leq C,$$

where $\kappa_1, ..., \kappa_n$ are the principal curvatures of the hypersurface M_t parametrized by $X(t,x) = \rho(t,x)x$.

Proof. Define the function $h : [0, T] \times \mathbb{S}^n \to \mathbb{R}$ by

$$h(t,x) = \log \frac{\max\limits_{1\le i\le n} \kappa_i(t,x)}{\langle X(t,x), \nu(t,x)\rangle} \tag{4.2}$$

where $\kappa_1, ..., \kappa_n$ are the principal curvatures of the hypersurface M_t parametrized by $X(t,x) = \rho(t,x)x$, and $\nu(t,.)$ is its outer normal vector. First we shall give an upper bound on the function h. Let $(t_0, x_0) \in [0,T] \times \mathbb{S}^n$ the point where h achieves its maximum on $[0,T] \times \mathbb{S}^n$, that is,

$$h(t_0,x_0) = \max_{(t,x)\in[0,T]\times\mathbb{S}^n} h(t,x) = \max_{(t,x)\in[0,T]\times\mathbb{S}^n} \log \frac{\max\limits_{1\le i\le n} \kappa_i(t,x)}{\langle X(t,x), \nu(t,x)\rangle}.$$

We want to prove that

$$h(t_0,x_0) \le C_0, \tag{4.3}$$

where the constant C_0 depends only on f, r_1, r_2 and ρ_0. If $t_0 = 0$, then $h(t_0,x_0) = h(0,x_0)$, and (4.3) is trivially satisfied in this case. From now on, we suppose that $t_0 > 0$. Without loss of generality, we may suppose that x_0 is the south pole of $\mathbb{S}^n$. Let Σ the tangent hyperplane to M_{t_0} at the point $Z_0 = X(t_0,x_0)$. Then near (t_0, Z_0), the family of hypersurfaces M_t can be represented as the graph of a smooth function u defined on a neighborhood of (t_0, Z_0) in $[0,T] \times \Sigma$. Since ρ is an admissible solution of (2.10), then u is an admissible solution of (2.19).

By choosing a new coordinate system in the hyperplane Σ, with origin at the point Z_0, then in the coordinate parallel to the new ones with centre at the original origin, denoted by $x_1, ..., x_n$, we have

$$Z_0 = (a_1, ..., a_n, -a) \text{ , for some constants } a_1, ..., a_n, a, \text{ with } a > 0,$$

and

$$X(t,x) = (a_1, ..., a_n, -a) + (x, u(t,x)) \text{ with } u(t_0, 0) = 0.$$

By formula (2.16) of section 2, we have

$$\nu = \frac{1}{v}(Du, -1) \tag{4.4}$$

and

$$\langle X, \nu\rangle = \frac{1}{v}\left(a - u + \sum_{k=1}^n (x_k + a_k) D_k u\right) , \tag{4.5}$$

where

$$v = (1 + |Du|^2)^{1/2}. \tag{4.6}$$

By our choice of coordinates we have

$$u(t_0, 0) = 0 \tag{4.7}$$

and

$$Du(t_0, 0) = (0, ..., 0). \tag{4.8}$$

By rotating the new $x_1, ..., x_n$ coordinates, we may suppose that $\max\limits_{1\le i\le n} \kappa_i(t_0,x_0)$ occurs in the x_1-direction. We have then by using formula (2.17) and (4.8)

$$\begin{aligned}\max_{1\le i\le n} \kappa_i(t_0,x_0) = \kappa_1(t_0,x_0) &= \frac{D_{11}u(t_0,0)}{v(t_0,0)(1 + (D_1u(t_0,0))^2)} \\ &= D_{11}u(t_0,0).\end{aligned}$$

On a neighborhood of $(t_0, 0)$ define the function H by

$$H = \log\left(\frac{D_{11}u}{\varphi v(1 + (D_1 u)^2)}\right)$$

where

$$\varphi = \langle X, \nu \rangle = \frac{1}{v}\left(a - u + \sum_{k=1}^{n}(x_k + a_k)D_k u\right)$$

Thus we have

$$H(t_0, 0) = h(t_0, x_0) = \max_{(t,x)\in[0,T]\times\mathbb{S}^n} h(t,x). \tag{4.9}$$

We will give an upper bound on $H(t_0, 0)$. By our choice of coordinates we have

$$D_{1\alpha}u(t_0, 0) = 0 \text{ for } \alpha > 1, \tag{4.10}$$

so by rotating the $x_2, ..., x_n$ coordinates, we may suppose that the matrix $D^2u(t_0, 0)$ is diagonal and that $D_{11}u(t_0, 0) > 0$.
We have, since H attains a local maximum at $(t_0, 0)$, that

$$DH(t_0, 0) = 0 \tag{4.11}$$

and

$$\partial_t H(t_0, 0) \geq 0 \tag{4.12}$$

since $t_0 > 0$. On the other hand, we have

$$D_\alpha H = \frac{D_{11\alpha}u}{D_{11}u} - \frac{D_\alpha v}{v} - \frac{2D_1 u D_{1\alpha}u}{1 + (D_1 u)^2} - \frac{D_\alpha \varphi}{\varphi}$$

and

$$D_\alpha \varphi = \sum_{k=1}^{n} \frac{(a_k + x_k)D_{\alpha k}u}{v} - \frac{\varphi D_\alpha v}{v}.$$

But by using (4.8) and (4.10) we have at $(t_0, 0)$

$$D_\alpha v = \sum_{k=1}^{n} \frac{D_k u D_{\alpha k} u}{v} = 0,$$

so

$$D_\alpha \varphi = a_\alpha D_{\alpha\alpha} u$$

and

$$D_\alpha H = \frac{D_{11\alpha}u}{D_{11}u} - \frac{a_\alpha D_{\alpha\alpha}u}{\varphi}$$

which together with (4.11) give at $(t_0, 0)$,

$$\frac{D_{11\alpha}u}{D_{11}u} - \frac{a_\alpha D_{\alpha\alpha}u}{\varphi} = 0. \tag{4.13}$$

Differentiating once again, we get at $(t_0, 0)$

$$D_{\alpha\alpha} v = (D_{\alpha\alpha}u)^2$$

and

$$D_\alpha\left(\frac{D_\alpha \varphi}{\varphi}\right) = \frac{1}{\varphi}\left(D_{\alpha\alpha}u + \sum_{k=1}^{n} a_k D_{\alpha\alpha k}u\right) - \frac{(a_\alpha D_{\alpha\alpha}u)^2}{\varphi^2} - (D_{\alpha\alpha}u)^2.$$

So

$$D_{\alpha\alpha}H = \frac{D_{11\alpha\alpha}u}{D_{11}u} - \left(\frac{D_{11\alpha}u}{D_{11}u}\right)^2 - 2\,(D_{1\alpha}u)^2 + \frac{(a_\alpha D_{\alpha\alpha}u)^2}{\varphi^2}$$
$$-\frac{1}{\varphi}\left(D_{\alpha\alpha}u + \sum_{k=1}^{n} a_k D_{\alpha\alpha k}u\right)$$

at $(t_0, 0)$. And using (4.13) we obtain then

$$D_{\alpha\alpha}H = \frac{D_{11\alpha\alpha}u}{D_{11}u} - 2\left(D_{1\alpha}u\right)^2 - \frac{1}{a}\left(D_{\alpha\alpha}u + \sum_{k=1}^{n} a_k D_{\alpha\alpha k}u\right) \tag{4.14}$$

at $(t_0, 0)$ for $\alpha = 1, ..., n$, where we have used the fact that $a = \varphi(t_0, 0)$.

Now if we differentiate equation (2.19) in the x_1 direction, we get

$$D_1\partial_t u = -\frac{1}{\sqrt{1+|Du|^2}}\left(\frac{1}{F} - f\right)\sum_{k=1}^{n} D_k u D_{k1}u$$

$$+\frac{\sqrt{1+|Du|^2}}{F^2}\sum_{i,j=1}^{n} F_{ij}D_1 a_{ij} \ + \ \sqrt{1+|Du|^2}\left(D_1 f + D_{n+1}fD_1 u\right).$$

Differentiating once again in the x_1 direction and using (4.7), (4.8) and (4.10) we get at $(t_0, 0)$

$$D_{11}\partial_t u = -\left(\frac{1}{F} - f\right)\left(D_{11}u\right)^2 + \frac{1}{F^2}\sum_{i,j=1}^{n} F_{ij}D_{11}a_{ij} - \frac{2}{F^3}\left(\sum_{i,j=1}^{n} F_{ij}D_1 a_{ij}\right)^2$$

$$+\frac{1}{F^2}\sum_{j,j,r,s=1}^{n} F_{ij,rs}D_1 a_{ij}D_1 a_{rs} + D_{11}f + D_{n+1}fD_{11}u. \tag{4.15}$$

But since log F is concave, we have

$$-\frac{2}{F^3}\left(\sum_{i,j=1}^{n} F_{ij}D_1 a_{ij}\right)^2 + \frac{1}{F^2}\sum_{j,j,r,s=1}^{n} F_{ij,rs}D_1 a_{ij}D_1 a_{rs} \le 0,$$

so it follows from (4.15) that at $(t_0, 0)$

$$D_{11}\partial_t u \le -\left(\frac{1}{F} - f\right)\left(D_{11}u\right)^2 + \frac{1}{F^2}\sum_{i,j=1}^{n} F_{ij}D_{11}a_{ij} + D_{11}f + D_{n+1}fD_{11}u \tag{4.16}$$

Now from the definition of the matrix $[a_{ij}]$ in (2.17), we have at $(t_0, 0)$ by using (4.7) and (4.8),

$$D_{11}a_{ij} = D_{11ij}u - (D_{11}u)^2 D_{ij}u - 2D_{1i}uD_{1j}uD_{11}u,$$

and since D^2u is diagonal at $(t_0, 0)$, then we have at this point

$$D_{11}a_{11} = D_{1111}u - 3(D_{11}u)^3 \tag{4.17}$$

and

$$D_{11}a_{\alpha\alpha} = D_{11\alpha\alpha}u - D_{\alpha\alpha}u(D_{11}u)^2 \tag{4.18}$$

for $\alpha = 2, ..., n$. Combining (4.16), (4.17) and (4.18) we obtain, since $[F_{ij}]$ is diagonal at $(t_0, 0)$,

$$D_{11}\partial_t u \le -\left(\frac{1}{F} - f\right)(D_{11}u)^2 + \frac{1}{F^2}\left(\sum_{\alpha=1}^{n} F_{\alpha\alpha}D_{11\alpha\alpha}u - (D_{11}u)^2\sum_{\alpha=2}^{n} F_{\alpha\alpha}D_{\alpha\alpha}u\right)$$

$$-3\frac{F_{11}}{F^2}(D_{11}u)^3 + D_{11}f + D_{n+1}fD_{11}u. \tag{4.19}$$

But from (4.14) we have

$$D_{11\alpha\alpha}u = D_{11}uD_{\alpha\alpha}H + 2D_{11}u\left(D_{1\alpha}u\right)^2 + \frac{D_{11}u}{a}\left(D_{\alpha\alpha}u + \sum_{k=1}^{n} a_k D_{\alpha\alpha k}u\right),$$

which gives by replacing in (4.19)

$$D_{11}\partial_t u \le -\left(\frac{1}{F}-f\right)(D_{11}u)^2+\frac{D_{11}u}{F^2}\sum_{\alpha=1}^n F_{\alpha\alpha}D_{\alpha\alpha}H-\frac{(D_{11}u)^2}{F^2}\sum_{\alpha=1}^n F_{\alpha\alpha}D_{\alpha\alpha}u$$
$$+\frac{D_{11}u}{aF^2}\sum_{\alpha=1}^n F_{\alpha\alpha}D_{\alpha\alpha}u+\frac{D_{11}u}{aF^2}\sum_{\alpha,k=1}^n F_{\alpha\alpha}a_kD_{\alpha\alpha k}u+D_{11}f+D_{n+1}fD_{11}u, \tag{4.20}$$

and since F is homogenous of degree k we have at $(t_0,0)$

$$\sum_{\alpha=1}^n F_{\alpha\alpha}D_{\alpha\alpha}u=kF.$$

So it follows from (4.20) that at $(t_0,0)$

$$D_{11}\partial_t u \le -\left(\frac{k+1}{F}-f\right)(D_{11}u)^2+k\frac{D_{11}u}{aF}+\frac{D_{11}u}{F^2}\sum_{\alpha=1}^n F_{\alpha\alpha}D_{\alpha\alpha}H$$
$$+\frac{D_{11}u}{aF^2}\sum_{\alpha,k=1}^n F_{\alpha\alpha}a_kD_{\alpha\alpha k}u+D_{11}f+D_{n+1}fD_{11}u. \tag{4.21}$$

Since H achieves a local maximum at $(t_0,0)$, then the matrix $[D_{ij}H]$ is negative semi-definite at $(t_0,0)$, and since $[F_{ij}]$ is positive semi-definite and diagonal at $(t_0,0)$, then we have at $(t_0,0)$

$$\sum_{\alpha=1}^n F_{\alpha\alpha}D_{\alpha\alpha}H\le 0.$$

Then using the fact that $D_{11}u(t_0,0)>0$, we get from (4.21) at $(t_0,0)$,

$$D_{11}\partial_t u \le -\left(\frac{k+1}{F}-f\right)(D_{11}u)^2+k\frac{D_{11}u}{aF}+\frac{D_{11}u}{aF^2}\sum_{\alpha,k=1}^n F_{\alpha\alpha}a_kD_{\alpha\alpha k}u+D_{11}f+D_{n+1}fD_{11}u. \tag{4.22}$$

Let us prove that the first term in the right side of (4.22) is negative,that is

$$\frac{k+1}{F}-f\ge 0. \tag{4.23}$$

If $0<k\le 1$, then by Proposition 3.2, we have a

$$\frac{1}{F}-f\ge 0 \tag{4.24}$$

since $\frac{1}{F}-f=\frac{\rho}{\sqrt{\rho^2+|\nabla\rho|^2}}\partial_t\rho\ge 0$. It is clear that (4.24) implies (4.23) since $F>0$. Now if $k>1$, then by Proposition 3.2 we have

$$0\le -\left(\frac{1}{F}-f\right)=-\frac{\rho}{\sqrt{\rho^2+|\nabla\rho|^2}}\partial_t\rho\le|\partial_t\rho|\le\frac{R_2}{R_1}\max_{x\in\mathbb{S}^n}\left|\mathcal{F}[\rho_0](x)\right|,$$

that is,

$$\frac{1}{F}\ge f-\frac{R_2}{R_1}\max_{x\in\mathbb{S}^n}\left|\mathcal{F}[\rho_0](x)\right|. \tag{4.25}$$

Now it is easy to see that (4.23) is a consequence of (4.25) and the second part of condition 4.1 in Proposition 4.1. Thus it follows from (4.22) and (4.23) that at $(t_0,0)$,

$$\partial_t D_{11}u\le k\frac{D_{11}u}{aF}+\frac{D_{11}u}{aF^2}\sum_{\alpha,k=1}^n F_{\alpha\alpha}a_kD_{\alpha\alpha k}u+D_{11}f+D_{n+1}fD_{11}u. \tag{4.26}$$

On the other hand, since at $(t_0, 0)$ we have

$$D_k a_{ij} = D_{ijk} u,$$

then by differentiating equation (2.19) we get at $(t_0, 0)$

$$D_k \partial_t u = \frac{1}{F^2} \sum_{i,j=1}^{n} F_{ij} D_{ijk} u + D_k f = \frac{1}{F^2} \sum_{\alpha=1}^{n} F_{\alpha\alpha} D_{\alpha\alpha k} u + D_k f \tag{4.27}$$

since $[F_{ij}]$ is diagonal at $(t_0, 0)$.

Now differentiating H with respect to t, we see that at $(t_0, 0)$

$$\partial_t H = \frac{\partial_t D_{11} u}{D_{11} u} - \frac{\partial_t \varphi}{\varphi} = \frac{\partial_t D_{11} u}{D_{11} u} + \frac{1}{a} \partial_t u - \frac{1}{a} \sum_{k=1}^{n} a_k D_k \partial_t u$$

and using equation (2.18) and (4.27) we obtain then at $(t_0, 0)$

$$\partial_t H = \frac{\partial_t D_{11} u}{D_{11} u} - \frac{1}{a} \left(\frac{1}{F} - f \right) - \frac{1}{aF^2} \sum_{\alpha,k=1}^{n} F_{\alpha\alpha} a_k D_{\alpha\alpha k} u - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f. \tag{4.28}$$

Thus we obtain from (4.26) and (4.28) at $(t_0, 0)$

$$\partial_t H \leq \frac{k-1}{aF} + \frac{D_{11} f}{D_{11} u} + D_{n+1} f - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f + \frac{1}{a} f \tag{4.29}$$

Since by (4.12) we have $\partial_t H(t_0, 0) \geq 0$, then it follows from (4.29) that

$$0 \leq \frac{k-1}{aF} + \frac{D_{11} f}{D_{11} u} + D_{n+1} f - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f + \frac{1}{a} f. \tag{4.30}$$

And since

$$D_{n+1} f(a_1, .., a_n, -a) - \frac{1}{a} \sum_{k=1}^{n} a_k D_k f(a_1, .., a_n, -a) = -\frac{1}{a} \rho \partial_\rho f(a_1, .., a_n, -a),$$

then (4.30) becomes

$$0 \leq \frac{k-1}{aF} + \frac{D_{11} f}{D_{11} u} - \frac{1}{a} \rho \partial_\rho f + \frac{1}{a} f \tag{4.31}$$

But by Proposition 3.2 we have $\partial_t \rho \geq 0$ if $k \leq 1$, and $\partial_t \rho \leq 0$ if $k > 1$. This implies that $\dfrac{1}{F(a_{ij})} - f(\rho x) \geq 0$ if $k \leq 1$, and $\dfrac{1}{F(a_{ij})} - f(\rho x) \leq 0$ if $k > 1$. That is,

$$\frac{k-1}{F(a_{ij})} \leq (k-1) f(\rho x). \tag{4.32}$$

It follows from (4.31) and (4.32) that at $(t_0, 0)$

$$\frac{1}{a} \left(\rho \partial_\rho f - k f \right) \leq \frac{D_{11} f}{D_{11} u} \tag{4.33}$$

Since f satisfies (1.7), then $\rho \partial_\rho f - k f > 0$, which implies

$$\delta_0 = \min_{(\rho,x) \in [R_1, R_2] \times \mathbb{S}^n} \left(\rho \partial_\rho f(\rho x) - k f(\rho x) \right) > 0,$$

and since $R_1 \leq \rho(t, x) \leq R_2$ by Proposition 3.1, then $\rho \partial_\rho f - k f \geq \delta_0$. Thus we get from (4.33) at $(t_0, 0)$

$$\frac{\delta_0}{a} \leq \frac{D_{11}f}{D_{11}u} \leq \frac{C}{D_{11}u}\,, \tag{4.34}$$

where

$$C = \|f\|_{C^2(A_{R_1,R_2})},\ \text{ with } A_{R_1,R_2} = \left\{X \in \mathbb{R}^{n+1}\ :\ R_1 \leq |X| \leq R_2\right\}.$$

We recall that by definition of H, we have $D_{11}u = ae^H$ at $(t_0, 0)$. It follows from (4.34) that

$$e^{H(t_0,0)} \leq \delta_0^{-1} C$$

or equivalently

$$H(t_0, 0) \leq \log \frac{C}{\delta_0}\,. \tag{4.35}$$

Thus the estimate (4.3) is proved by taking

$$C_0 = \max\left(\log \frac{C}{\delta_0}\,,\ \max_{x\in S^n} h(0,x)\right).$$

(4.3) implies then, for any $(t, x) \in [0, T] \times \mathbb{S}^n$,

$$h(t, x) \leq C_0. \tag{4.36}$$

We have by (4.2)

$$\max_{1\leq i\leq n} \kappa_i = \langle X, \nu\rangle e^h$$

and since by Proposition 3.1 we have

$$\langle X, \nu\rangle = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}} \leq \rho \leq R_2,$$

then we get from (4.36) the upper bound

$$\max_{1\leq i\leq n} \kappa_i \leq R_2 e^{C_0}. \tag{4.37}$$

Now, to get a lower bound on the principal curvatures, it suffices to observe that $\kappa_1+\cdots+\kappa_n > 0$ since $\kappa = (\kappa_1, ..., \kappa_n) \in \Gamma$, and then use the upper bound (4.37). Indeed, we have for all $i = 1, ..., n$,

$$0 < \kappa_1 + \cdots + \kappa_n \leq \kappa_i + (n-1)R_2e^{C_0}$$

so

$$\kappa_i \geq -(n-1)R_2e^{C_0}.$$

The proof of Proposition 4.1 is complete.

□

The previous proposition allows us to get higher order estimates on our solutions.

Proposition 4.2. Let $\rho : [0, T] \times \mathbb{S}^n \to (0, +\infty)$ be an admissible solution of (2.10) as in Proposition 4.1. Then for any $m \in \mathbb{N}$, there exist two positive constants C_m and λ_m depending only on m, f, F, r_1, r_2 and ρ_0 such that

$$\|\rho\|_{C^m([0,T]\times\mathbb{S}^n)} \leq C_m \tag{4.38}$$

and for all $t \in [0, T]$,

$$\|\partial_t\rho(t, .)\|_{C^m(\mathbb{S}^n)} \leq C_m e^{-\lambda_m t}. \tag{4.39}$$

Moreover, there exists a compact set $K \subset M(\Gamma)$ depending only on f, F, r_1, r_2 and ρ_0, sucht that for any $(t, x) \in [0, T] \times \mathbb{S}^n$,

$$[a_{ij}(t, x)] \in K\,, \tag{4.40}$$

where the cone $M(\Gamma)$ is defined in section 2, and the matrix $[a_{ij}]$ is given by (2.4) in section 2.

Proof. The principal curvatures κ_i of the hypersurface M_t parametrized by $X(t,x)=\rho(t,x)x$, are the eigenvalues of the matrix $[a_{ij}]$ (see section 2) defined by

$$[a_{ij}]=[g^{ij}]^{\frac{1}{2}}[h_{ij}][g^{ij}]^{\frac{1}{2}} \tag{4.41}$$

where $[g^{ij}]^{\frac{1}{2}}$ is the positive square root of $[g^{ij}]$ which is given by

$$[g^{ij}]^{\frac{1}{2}}=\rho^{-1}\left[\delta_{ij}-\frac{\nabla_i\rho\nabla_j\rho}{\sqrt{\rho^2+|\nabla\rho|^2}(\rho+\sqrt{\rho^2+|\nabla\rho|^2})}\right] \tag{4.42}$$

and $[h_{ij}]$ is the matrix representing the second fondamental form of M_t, given by

$$h_{ij}=\left(\rho^2+|\nabla\rho|^2\right)^{-\frac{1}{2}}(\rho^2\delta_{ij}+2\nabla_i\rho\nabla_j\rho-\rho\nabla_{ij}\rho). \tag{4.43}$$

It is clear from Proposition 4.1, Proposition 3.1 and Proposition 3.3 by using (4.41), (4.42) and (4.43) that

$$\sup_{t\in[0,T]}\|\rho(t,.)\|_{C^2(\mathbb{S}^n)}\leq C, \tag{4.44}$$

where C depends only on f,r_1,r_2 and ρ_0. In order to get higher order estimates, let us first prove (4.40). By Proposition 3.2 we have

$$|\partial_t\rho|\leq Ce^{-\lambda t}\leq C, \tag{4.45}$$

where the constant C depends only on f,r_1,r_2 and ρ_0. Since ρ satisfies (2.10), then it follows from (4.45)

$$\frac{1}{F(a_{ij})}-f(\rho x)\leq\left|\frac{1}{F(a_{ij})}-f(\rho x)\right|\frac{\sqrt{\rho^2+|\nabla\rho|^2}}{\rho}=|\partial_t\rho|\leq C$$

that is,

$$\frac{1}{F(a_{ij})}\leq f(\rho x)+C\leq C_0$$

or equivalently

$$F(a_{ij})\geq\frac{1}{C_0}\ , \tag{4.46}$$

where

$$C_0=C+\max_{R_1\leq|X|\leq R_2}|f(X)|\ .$$

Since $F\equiv 0$ on $\partial M(\Gamma)$, it follows from (4.46) that there exists a constant $\delta_0>0$ depending only on f,F,R_1,R_2 and ρ_0 such that

$$\mathrm{dist}\Big([a_{ij}],\partial M(\Gamma)\Big)\geq\delta_0, \tag{4.47}$$

where $\partial M(\Gamma)$ is the boundary of the cone $M(\Gamma)$ and $\mathrm{dist}\big([a_{ij}],\partial M(\Gamma)\big)$ is the distance of $[a_{ij}]$ to $\partial M(\Gamma)$. It is clear from (4.47) that there exists a compact set $K\subset M(\Gamma)$ depending only on f,F,r_1,r_2 and ρ_0 such that $[a_{ij}]\in K$. Thus (4.40) is proved.

Let us now prove the estimates (4.38) and (4.39). Since F satisfies (1.3)(or equivalently (2.6)), it follows from (4.40) and the estimate (4.44) that equation (2.10) is uniformly parabolic. Since by hypothesis the function $\log F$ is concave, then we can apply a result of B. Andrews [2] (Theorem 6, p.3), which is a generalisation of the result of N. Krylov [7] on fully nonlinear parabolic equations, to obtain the estimate

$$\|\partial_t\rho\|_{C^\alpha([0,T]\times\mathbb{S}^n)}+\|\nabla_{ij}\rho\|_{C^\alpha([0,T]\times\mathbb{S}^n)}\leq C, \tag{4.48}$$

where $C^\alpha([0,T]\times\mathbb{S}^n)$ is the parabolic Hölder's space, and where the constants $C>0$, $\alpha\in(0,1)$ depend only on f,F,r_1,r_2 and ρ_0. The higher order estimates (4.38) follows from (4.48) and the standard theory of linear parabolic equations (see [8]). In order to prove (4.39) we use the following well known interpolation inequality, which is valid on any compact Riemannian manifold M,

$$\|\nabla u\|^2_{L^\infty(M)} \leq 4\|u\|_{L^\infty(M)}\|\nabla^2 u\|_{L^\infty(M)}, \quad u \in C^\infty(M), \tag{4.49}$$

where ∇u and $\nabla^2 u$ denote respectively the gradient and the hessian of u. It suffices to apply (4.49) first to $u = \partial_t \rho$ and iterate it on the spatial higher order derivatives of $\partial_t \rho$ and using (4.38) and (3.11) to get (4.39). This achieves the proof of Proposition 4.2.

□

Now we are in position to prove our main result.

Proof of Theorem 1.1 and Theorem 1.2 . Let $X_0(x) = \rho_0(x)x$ satisfies conditions (1.10) in Theorem 1.1 or conditions (1.13) in Theorem 1.2. Let $X : [0,T] \times \mathbb{S}^n \to \mathbb{R}^{n+1}$ a local solution of (1.2). As we saw in section 2, X is given by

$$X(t,x) = \rho(t,\varphi(t,x))\varphi(t,x), \ (t,x) \in [0,T] \times \mathbb{S}^n \tag{4.50}$$

where ρ satisfies (2.10) and $\varphi(t,.) : \mathbb{S}^n \to \mathbb{S}^n$ is a diffeomorphism satisfying the ODE

$$\begin{cases} \partial_t \varphi(t,x) = Z(t,\varphi(t,x)) \\ \varphi(0,x) = x, \end{cases} \tag{4.51}$$

with

$$\begin{aligned} Z(t,y) &= -\left(\frac{1}{F(a_{ij}(t,y))} - f(\rho(t,y)y)\right) \frac{\nabla\rho(t,y)}{\rho\sqrt{|\nabla\rho(t,y)|^2 + \rho^2(t,y)}} \\ &= -\frac{\partial_t \rho(t,y)\nabla\rho(t,y)}{|\nabla\rho(t,y)|^2 + \rho^2(t,y)} \ , \ (t,y) \in [0,T] \times \mathbb{S}^n. \end{aligned} \tag{4.52}$$

Since X_0 satisfies condition (1.10) in Theorem 1.1 or condition (1.13) in Theorem 1.2, then it is easy to check that the hypothesis of Proposition 4.1(and then Proposition 4.2) concerning ρ_0 are satisfied. We can then apply Proposition 4.2 to the function ρ given above. If we differentiate equation (4.51) and use the estimates (4.38)-(4.39) in Proposition 4.2, then it is not difficult to see that for any $m \in \mathbb{N}$, we have

$$\|\varphi\|_{C^m([0,T]\times\mathbb{S}^n,\ \mathbb{S}^n)} \leq C_m \tag{4.53}$$

and for any $t \in [0,T]$,

$$\|\partial_t \varphi(t,.)\|_{C^m(\mathbb{S}^n,\ \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t}, \tag{4.54}$$

where C_m and λ_m are positive constants depending only on m, f, F, r_1, r_2 and X_0. It follows from (4.50) by using the estimates (4.38)-(4.39) in Proposition 4.2 and (4.53)-(4.54) that, for any $m \in \mathbb{N}$,

$$\|X\|_{C^m([0,T]\times\mathbb{S}^n,\ \mathbb{R}^{n+1})} \leq C_m \tag{4.55}$$

and for all $t \in [0,T]$,

$$\|\partial_t X(t,.)\|_{C^m(\mathbb{S}^n,\ \mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t}, \tag{4.56}$$

with new constants C_m and λ_m depending only on m, f, F, r_1, r_2 and X_0. Also by Proposition 4.2 there exists a compact set $K \subset M(\Gamma)$ depending only on m, f, F, r_1, r_2 and X_0 such that for any $(t,x) \in [0,T] \times \mathbb{S}^n$, we have

$$[a_{ij}(t,x)] \in K \subset M(\Gamma) \ , \tag{4.57}$$

where the matrix $[a_{ij}]$ is given by (2.4). Since the constant C_m in (4.55) and the compact set K in (4.57) are independant of T, then X can be extended to $[0,+\infty)$ as a solution of (1.2). The estimates (4.55), (4.56) and (4.57) become then

$$\|X\|_{C^m([0,+\infty)\times\mathbb{S}^n,\ \mathbb{R}^{n+1})} \leq C_m \tag{4.58}$$

$$\|\partial_t X(t,.)\|_{C^m(\mathbb{S}^n,\,\mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t} \text{ for all } t \in [0,+\infty) \tag{4.59}$$

and

$$[a_{ij}(t,x)] \in K \subset M(\Gamma) \text{ for all } t \in [0,+\infty). \tag{4.60}$$

Now it is clear from (4.58) and (4.59) that there exists a map $X_\infty \in C^\infty(\mathbb{S}^n,\mathbb{R}^{n+1})$ such that $X(t,.) \to X_\infty$ as $t \to +\infty$ in $C^m(\mathbb{S}^n,\mathbb{R}^{n+1})$ for all $m \in \mathbb{N}$, and satisfying

$$\|X(t,.) - X_\infty\|_{C^m(\mathbb{S}^n,\,\mathbb{R}^{n+1})} \leq C_m e^{-\lambda_m t} \text{ for all } t \in [0,+\infty).$$

Since $X(t,.)$ is a smooth starshaped embedding, then it is easy to see that X_∞ is also a smooth starshaped embedding, and from (4.60) we deduce that the principal curvatures of X_∞ lie in Γ. By passing to the limit in equation (1.2) and using (4.59), we see that X_∞ satisfies

$$\frac{1}{F(\kappa(X_\infty))} - f(X_\infty) = 0.$$

This achieves the proofs of Theorem 1.1 and Theorem 1.2.

□

Proof of Corollary 1.1 . As in Remark 1.1, if we take $X_0(x) = rx$, where $0 < r \leq r_1$ with r_1 as in (1.8), then by using (1.7) and (1.8) one easily checks that condition (1.10) in Theorem 1.1 is satisfied by X_0. Thus the evolution problem (1.2) admits a global solution $X(t,.)$ which converges as $t \to +\infty$, to a solution X_∞ of

$$\frac{1}{F(\kappa(X_\infty))} = f(X_\infty) \tag{4.61}$$

which is smooth starshaped embedding satisfying $\kappa(X_\infty) \in \Gamma$. It remains then to prove that X_∞ is the unique starshaped solution of (4.61) such that $\kappa(X_\infty) \in \Gamma$. Let X_1 and X_2 two starshaped solutions of (4.61) such that $\kappa(X_l) \in \Gamma,\ l = 1,2$. We have then

$$\frac{1}{F(\kappa(X_l))} = f(X_l)\ ,\ l = 1,2. \tag{4.62}$$

Let ρ_l $(l = 1,2)$ be the radial function of X_l, and set $u_l(x) = \log \rho_l(x)$. Then we have by using formula (2.4) of section 2,

$$\frac{1}{F(a_{ij}(u_l))} = f(e^{u_l}x)\ ,\ l = 1,2, \tag{4.63}$$

where the matrix $[a_{ij}(u_l)]$ is given by

$$[a_{ij}(u_l)] = \frac{e^{-u_l}}{\sqrt{1+|\nabla u_l|^2}}[\gamma_{ij}][b_{ij}][\gamma_{ij}] \tag{4.64}$$

with

$$\begin{cases} b_{ij} = \delta_{ij} + \nabla_i u_l \nabla_j u_l - \nabla_{ij} u_l \\ \\ \gamma_{ij} = \delta_{ij} - \dfrac{\nabla_i u_l \nabla_j u_l}{\sqrt{1+|\nabla u_l|^2}\left(1+\sqrt{1+|\nabla u_l|^2}\right)}\ ,\ l = 1,2. \end{cases} \tag{4.65}$$

We shall prove that for any $x \in \mathbb{S}^n$, we have

$$u_1(x) \geq u_2(x). \tag{4.66}$$

It is clear that (4.66) would imply that $u_1 = u_2$, and then $\rho_1 = \rho_2$. To prove (4.66) define a function $u : \mathbb{S}^n \to \mathbb{R}$ by $u(x) = u_1(x) - u_2(x)$, and let $x_0 \in \mathbb{S}^n$ a point where u achieves its minimum. Then we have at x_0 that $\nabla u = 0$ and the matrix $\nabla^2 u$ is positive semi-definite, that is, $\nabla u_1 = \nabla u_2$ and $\nabla^2 u_1 \geq \nabla^2 u_2$ (in the sense of operators) at x_0. This implies by using (4.64) and (4.65) that at x_0,

$$e^{u_1}[a_{ij}(u_1)] \leq e^{u_2}[a_{ij}(u_2)] \tag{4.67}$$

in the sense of operators. Since the function F is monotone (by (1.3) or equivalently (2.6)) and homogenous of degree k, it follows from (4.63) and (4.67) that

$$e^{-ku_1(x_0)} f(e^{u_1(x_0)} x_0) \geq e^{-ku_2(x_0)} f(e^{u_2(x_0)} x_0)$$

which implies by using (1.7) that $u_1(x_0) \geq u_2(x_0)$ or equivalently $u(x_0) \geq 0$. This proves (4.66) and the proof of Corollary 1.1 is complete.

□

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