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ON THE GLOBAL ASYMPTOTIC STABILITY OF SOLUTIONS TO NEUTRAL EQUATIONS OF FIRST ORDER

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Abstract In this paper,we consider a nonlinear first order neutral differential equation. By using fixed point theory, we give some new conditions to ensure that the zero solution of the considered equation is globally asymptotically stable in C^1. Our result includes and improves some results in the literature. We also give an example to demonstrate the correctness of the obtained result by using MATLAB-Simulink.

1 Introduction

It is well known that neutral differential equations have many applications in science and engineering. Indeed, these kind of equations are used as models of steam or water pipes, heat exchangers (see [15]), lossless transmission lines, partial element equivalent circuits (see [16]) and control of constrained manipulators with delay measurements in mechanical engineering (see [17]). In addition, the asymptotic stability of delay differential equations of neutral type have been investigated intensively during the past decades, (see, e.g.,[2] , [5], [7], [14] and references therein). However, to the best of our knowledge,there are only a few results on the global asymptotic stability of solutions of neutral differential equations(see [14].Thus, it is worth to investigate the global asymptotic stability of solutions for that kinds of equations.As distinguished from this line, in 2004, Raffoul [11] obtained stability results about the zero solution of the nonlinear neutral differential equation with functional delay

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q(x(t), x(t - g(t))). \tag{1.1}$$

In [8], Jin and Luo considered the linear scalar neutral delay differential equation with variable delay

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau(t) - b(t)x(t - \tau(t)). \tag{1.2}$$

The authors gave some new sufficient conditions to ensure that the zero solution is asymptotic stable by means of fixed point theory. In [9], Djoudi and Khemis dealt with the stability of the solutions of nonlinear neutral differential equations with unbounded delay in following form

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau(t)) - b(t)g(x(t - \tau(t))) \tag{1.3}$$

by using fixed point techniques.

In [14] Liu and Yang considered the following non-linear neutral differential equation,

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau_1(t)) + q(t, x(t), x(t - \tau_2(t))). \tag{1.4}$$

By using fixed point theory,they gave some new conditions to ensure that the zero solution of equation (1.4) is global asymptotical stable in C^1.

In this paper, we are concerned with the globally asymptotically stability of the zero solution for the following neutral equation with variable delay

$$x'(t) = -a(t)x(t) + b(t)g(x(t)) + c(t)f(x'(t - \tau(t))) + q(t, x(t), x(t - \tau(t))). \tag{1.5}$$

We suppose that the following assumptions hold:

C_1) $a, b, c \in C([0,\infty), R), g, f \in C(R, R), q \in C([0,\infty), R \times R, R)$ such that for constants

$L_g, L_f > 0$ the functions f, g satisfy Lipschitz condition and $f(0) = g(0) = 0$,$\tau_i \in C([0,\infty),(0,\infty))$ with $t - \tau_i(t) \to \infty$ as $t \to \infty$, $(i = 1, 2)$,
$C_2)$ $q(t, 0, 0) = 0$, and there exist functions $L_1, L_2 \in C([0,\infty),(0,\infty))$ such that

$$|q(t, x_1, y_1) - q(t, x_2, y_2)| \leq L_1(t)|x_1 - y_1| + L_2(t)|x_2 - y_2|,$$

for all $x_i, y_i \in R, i = 1, 2$,
$C_3)$ the function a is bounded and $\lim_{t\to\infty}\inf \int_0^t a(s)ds > -\infty$,
$C_4)$ there exists a constant $\eta \in (0, 1)$ such that

$$\int_0^t e^{-\int_u^t a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \leq \eta$$

and

$$|a(t)|\int_0^t e^{-\int_u^t a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du$$
$$+|b(t)|L_g + L_f|c(t)| + L_1(t) + L_2(t) \leq \eta, \qquad t \geq 0.$$

Let $d_{t_0} = inf_{t\in[t_0,\infty)}\{t - \tau_1(t), t - \tau_2(t)\}$ and denote $C^1_{t_0} = C^1([d_{t_0}, t_0], R)$ with the norm defined by

$$|x|_{t_0} = max_{t\in[d_{t_0},t_0]}|x(t), |x'(t)|$$

for $x \in C^1_{t_0}$ and

$$\Phi_{t_0} = \{\varphi \in C^1_{t_0} : \varphi'_-(t_0) = -a(t_0)\varphi(t_0) + b(t_0)g(\varphi(t_0))$$
$$+c(t_0)f(\varphi'(t_0 - \tau_1(t_0)) + q(t_0, \varphi(t_0)), \varphi(t_0 - \tau_2(t_0)))\}.$$

2 Preliminaries

We now give some basic information.

Definition 2.1. ([14]) For each $(t_0, \varphi) \in [0,\infty) \times \Phi_{t_0}, x$ said to be a solution of Eq. (1.5) through (t_0, φ) if $x \in C^1([d_{t_0},\infty))$ satisfies Eq. (1.5) on (t_0, φ) and $x(t) = \varphi(t)$ for $t \in [d_{t_0}, t_0]$. We denote such a solution by $x(t) = x(t, t_0, \varphi)$.

Definition 2.2. ([14]) The zero solution of Eq. (1.5) is said to be stable in C^1 if, for any $t_0 \in [0,\infty), \epsilon > 0$, there is a $\delta = \delta(\epsilon, t_0)$ such that $\varphi \in \Phi_{t_0}$ and $|\varphi|_{t_0} < \delta$ implies

$$max_{s\in[d_{t_0},t]}\{|x(s)|, |x'(s)|\} < \epsilon$$

for $t \geq t_0$.

Definition 2.3. ([14]) The zero solution of Eq.(1.5) is said to be globally asymptotically stable in C^1 if it is stable in C^1,and for any $t_0 \in [0,\infty)$, $\varphi \in \Phi_{t_0}$ implies

$$lim_{t\to\infty}x(t, t_0, \varphi) = lim_{t\to\infty}x'(t, t_0, \varphi) = 0.$$

3 Stability

We give our stability results by the following theorem.

Theorem 3.1. *We suppose that assumptions (C_1)–(C_4) hold. Then the zero solution of Eq. (1.5) is globally asymptotically stable in C^1 if and only if*

$$\int_0^\infty a(s)ds = \infty. \tag{3.1}$$

Proof. ($\Leftarrow$:) We assume that $\int_0^\infty a(s)ds = \infty$ and for any $t_0 \in [0,\infty)$,

$$X = \{x \in C^1([d_{t_0},\infty)) : lim_{t\to\infty}x(t) = lim_{t\to\infty}x'(t) = 0\},$$

and $\|x\|_{t_0} = max_{t\in[d_{t_0},\infty)}\{|x(t)|, |x'(t)|\}$ for $x \in X$. It is clear that X is a Banach space.

For any $\varphi \in \Phi_{t_0}$ we suppose that

$$D = \{x \in X : x(t) = \varphi(t) \quad \texttt{for} \quad t \in [d_{t_0}, t_0]\}.$$

It follows that D is a non-empty and closed subset of X.

Define the mapping $T : D \to C([d_{t_0},\infty))$ by $(Tx)(t) = \varphi(t)$ for $t \in [d_{t_0}, t_0]$ and

$$\begin{aligned}(Tx)(t) &= \varphi(t_0)e^{-\int_{t_0}^t a(s)ds} \\ &+ \int_{t_0}^t e^{-\int_u^t a(u)du}[b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) \\ &+ q(u,x(u),x(u-\tau_2(u)))]du\end{aligned} \tag{3.2}$$

for $t \in [t_0,\infty)$.

Initially, we show that $T : D \to D$ is a self-mapping. In view of (3.2),we can get

$$\begin{aligned}(Tx)'(t) &= \varphi(t_0)a(t)e^{-\int_{t_0}^t a(s)ds} \\ &+ b(t)g(x(t)) + c(t)f(x'(t-\tau_1(t))) + q(t,x(t),x(t-\tau_2(t))) \\ &- a(t)\int_{t_0}^t e^{-\int_u^t a(u)du}[b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) \\ &+ q(u,x(u),x(u-\tau_2(u)))]du \\ &= -a(t)(Tx)(t) + b(u)g(x(t)) + c(t)f(x'(t-\tau_1(t))) \\ &+ q(t,x(t),x(t-\tau_2(t))), \qquad t > t_0.\end{aligned} \tag{3.3}$$

Then

$$\begin{aligned}(Tx)'_+(t_0) = -a(t_0)\varphi(t_0) + b(t_0)g(\varphi(t_0)) + c(t_0)f(\varphi'(t_0 - \tau_1(t_0))) \\ +q(t_0,\varphi(t_0),\varphi(t_0-\tau_2(t_0))) = \varphi'_-(t_0).\end{aligned}$$

Therefore, $Tx \in C^1([d_{t_0},\infty))$ for $x \in D$ and

$$lim_{t\to\infty}x(t) = lim_{t\to\infty}x'(t) = 0.$$

The former estimates, assumption (C_1) and definition of the limit implies that

$$lim_{t\to\infty}t - \tau_i(t) = \infty, i = 1,2.$$

Hence, for any $\epsilon > 0$, there exists $T > 0$ such that

$$max\{|x(t)|, |x(t-\tau_2(t))|, |x'(t-\tau_1(t))|\} < \epsilon, \quad t \geq T. \tag{3.4}$$

In view of (3.2),(3.4) and assumptions (C_1) and (C_4), it follows for $t > T$ and $x \in D$ that

$$\begin{aligned}|(Tx)(t)| &\leq |\varphi(t_0)|e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^T e^{-\int_u^t a(u)du} \\ &\times [b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) + q(u,x(u),x(u-\tau_2(u)))]du \\ &+ \int_T^t e^{-\int_u^t a(u)du}[|b(u)||g(x(u)) - g(0)| + |c(u)f(x'(u-\tau_1(u))) - c(u)f(0)| \\ &+ |q(u,x(u),x(u-\tau_2(u))) - q(u,0,0)|]du \\ &\leq e^{-\int_{t_0}^t a(s)ds}\Bigg[|\varphi(t_0)| + \int_{t_0}^T e^{-\int_{t_0}^u a(u)du} \\ &\times [b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) + q(u,x(u),x(u-\tau_2(u)))]\Bigg]du\end{aligned}$$

$$
\begin{aligned}
&+ \quad \int_T^t e^{-\int_u^t a(s)ds}\Big[|b(u)|L_g|x(u)| + L_f|c(u)||x'(u-\tau_1(u))| \\
&+ \quad L_1(u)|x(u)| + L_2(u)|x(u-\tau_2(u))|\Big]du \\
&\le \quad e^{-\int_{t_0}^t a(s)ds}\Big[|\varphi(t_0)| + \int_{t_0}^T e^{-\int_{t_0}^u a(u)du} \\
&\times \quad [b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) + q(u,x(u),x(u-\tau_2(u)))]\Big]du \\
&+ \quad \epsilon\int_T^t e^{-\int_u^t a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\
&\le \quad e^{-\int_{t_0}^t a(s)ds}\Big[|\varphi(t_0)| + \int_{t_0}^T e^{-\int_{t_0}^u a(u)du} \\
&\times \quad [b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) + q(u,x(u),x(u-\tau_2(u)))]\Big]du + \eta\epsilon.
\end{aligned}
$$

On the other hand,it may be followed from (3.1),there exists $T_1 > T$ such that for $t > T_1$,

$$
e^{-\int_{t_0}^t a(s)ds}\Big[|\varphi(t_0)| + \int_{t_0}^T e^{-\int_{t_0}^u a(u)du}
$$

$$
\times[b(u)g(x(u))+c(u)f(x'(u-\tau_1(u)))+q(u,x(u),x(u-\tau_2(u)))]\Big]du < \epsilon.
$$

Thus, $lim_{t\to\infty}(Tx)(t) = 0$ for $x \in D$.
In addition, we have from (3.3) and (C_2) that

$$
\begin{aligned}
|(Tx)'(t)| \quad &\le \quad |a(t)(Tx)(t)| + |g(x(t)) - g(0)| \\
&+ \quad |c(t)f(x'(t-\tau_1(t))) - c(t)f(0)| \\
&+ \quad |q(t,x(t),x(t-\tau_2(t))) - q(t,0,0)| \\
&\le \quad |a(t)(Px)(t)| + L_g|x(t)| + L_f|c(t)x'(t-\tau_1(t))| \\
&+ \quad |L_1(t)|x(t)| + L_2(t)|x(t-\tau_2(t))|.
\end{aligned}
$$

This together (C_3) and (C_4) leads that $lim_{t\to\infty}(Tx)'(t) = 0$ for $x \in D$. Therefore, $Tx \in D$.
Now, we show that $T : D \to D$ is a contraction mapping. For $x, y \in D$, it follows from (3.1) and (C_2) and (C_4) that

$$
\begin{aligned}
|(Tx)(t) - (Ty)(t)| \quad &= \quad \int_{t_0}^T e^{-\int_u^t a(u)du}[|g(x(u)) - g(y(t))| \\
&+ \quad |c(u)||f(x'(u-\tau_1(u))) - f(y'(u-\tau_1(u)))| \\
&+ \quad |q(u,x(u),x(u-\tau_2(u))) - q(u,y(u),y(u-\tau_2(u)))|]du \\
&\le \quad \int_{t_0}^T e^{-\int_u^t a(u)du}[L_g|x(t) - y(t)| \\
&+ \quad L_f|c(u)||x'(u-\tau_1(u)) - y'(u-\tau_1(u))| \\
&+ \quad |L_1(u)|x(u) - y(u)| + L_2(u)|x(u-\tau_2(u)) - y(u-\tau_2(u))|]du \\
&\le \quad \|x-y\|_{t_0}\int_{t_0}^T e^{-\int_u^t a(u)du}[L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\
&\le \quad \eta\|x-y\|_{t_0}, t \in [t_0,\infty).
\end{aligned}
$$

In addition,we can get

$$\begin{aligned}
|(Tx)(t)-(Ty)(t)| &= |a(t)||(Tx)(t)-(Ty)(t)|+|g(x(t))-g(y(t))| \qquad (3.5)\\
&+ |c(t)||f(x'(t-\tau_1(t)))-f(y'(t-\tau_1(t)))|\\
&+ |q(t,x(t),x(t-\tau_2(t)))-|q(t,y(t),y(t-\tau_2(t)))|\\
&\le \|x-y\|_{t_0}\big[|a(t)|\int_{t_0}^{T} e^{-\int_u^t a(u)du}\\
&\times [L_g+L_f|c(u)|+L_1(u)+L_2(u)+L_g+L_f|c(t)|+L_1(t)+L_2(t)]du\\
&\le \eta\|x-y\|_{t_0}.
\end{aligned}$$

When we consider the above discussion,we can reach that $T : D \to D$ is a contraction mapping. By the contraction mapping principle,T has a unique fixed point x in D, which is a unique solution of Eq. (1.5) through (t_0,φ) and satisfies

$$lim_{t\to\infty}x(t) = lim_{t\to\infty}x'(t) = 0.$$

Finally, we prove that the zero solution of Eq. (1.5) is stable in C^1.

We now suppose the following conditions are true:

$$K = sup_{t\in[t_0,\infty)}e^{-\int_{t_0}^t a(s)ds}, A = sup_{t\in[t_0,\infty)}|a(t)|.$$

From (6) and (C_3) ,we get $K, A \in (0,\infty)$. For any $\epsilon > 0$, let $\delta > 0$ such that

$$\delta < \epsilon min\{1, \frac{1-\eta}{K}, \frac{1-\eta}{KA}\}.$$

If $x(t) = x(t, t_0, \varphi)$ is a solution of Eq.(5) with $|\varphi|_{t_0} < \delta$, then $x(t) = (Tx)(t)$ on $[t_0,\infty)$. We claim that $\|x\|_{t_0} < \epsilon$. Otherwise, there exists $t_1 > t_0$ such that

$$max\{|x(t_1)|, |x'(t_1)|\} = \epsilon$$

and

$$max\{|x(t)|, |x'(t)|\} < \epsilon$$

for $t \in [d_{t_0}, t_1)$. If $|x(t_1)| = \epsilon$, then it follows from (3.2) and (C_2) and (C_4) that

$$\begin{aligned}
|x(t_1)| &\le |\varphi(t_0)|e^{-\int_{t_0}^{t_1} a(s)ds} + \int_{t_0}^{t_1} e^{-\int_u^{t_1} a(s)ds}\\
&\times [b(u)g(x(u))+c(u)f(x'(u-\tau_1(u)))+q(u,x(u),x(u-\tau_2(u)))]du\\
&\le K\delta+\epsilon\int_{t_0}^{t_1} e^{-\int_u^{t_1} a(s)ds}[L_g+L_f|c(u)|+L_1(u)+L_2(u)]du\\
&\le K\delta+\epsilon\eta\\
&< \epsilon.
\end{aligned}$$

This is a contradiction.

If $|x'(t_1)| = \epsilon$, then it follows from (3.3) and (C_2) and (C_4) that

$$\begin{aligned} |x'(t_1)| &\leq |\varphi(t_0)||a(t_1)|e^{-\int_{t_0}^{t_1}}a(s)ds + |b(t_1)||g(x(t_1))| \\ &+ |c(t_1)||f(x'(t_1 - \tau_1(t_1)))| + |q(t_1, x(t_1), x(t_1 - \tau_2(t_1)))| \\ &+ |a(t)| \int_{t_0}^{t_1} e^{-\int_u^t a(s)ds} \\ &\times [b(u)g(x(u)) + c(u)f(x'(u - \tau_1(u))) + q(u, x(u), x(u - \tau_2(u)))]du \\ &\leq KA\delta + \epsilon|a(t)| \int_{t_0}^{t_1} e^{-\int_u^t a(u)du} \\ &\times [|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u) + |b(t_1)|L_g + L_f|c(t_1)| \\ &+ L_1(t_1) + L_2(t_1)]du \\ &< \epsilon. \end{aligned}$$

This is a contraction,too. Hence, the zero solution of Eq. (1.5) is stable in C^1. This, together with

$$lim_{t\to\infty}x(t) = lim_{t\to\infty}x'(t) = 0$$

implies that the zero solution of Eq. (1.5) is globally asymptotically stable in C^1.

($\Rightarrow$) : We now assume that the zero solution of Eq. (1.5) is globally asymptotically stable in C^1. We prove that (3.1) holds. In the contrary to this fact, let us assume that (3.1) does not hold. Then there exists a positive constant l such that

$l = lim_{t\to\infty}inf \int_0^t a(s)ds$, $K_0 = sup_{t\in[t_0,\infty)}e^{-\int_{t_0}^t a(s)ds}$ and $A_0 = sup_{t\in[t_0,\infty)}|a(t)|$.

Hence it follows from (C_3) that $l \in (-\infty, \infty), K_0 \in (0, \infty)$ and $A_0 \in [0, \infty)$. So, there exists an increasing sequence $\{t_n\} \subset [0, \infty)$ such that $lim_{n\to\infty}t_n = \infty$ and

$$l = lim_{n\to\infty} \int_0^{t_n} a(s)ds. \tag{3.6}$$

Denote

$$I_n = \int_0^{t_n} e^{\int_0^u a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du, n = 1, 2, ...$$

From (C_4), it follows that

$$\begin{aligned} l_n &= e^{-\int_0^{tn} a(s)ds} \int_0^{t_n} e^{-\int_u^{tn} a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\ &\leq \eta e^{-\int_0^{tn} a(s)ds}. \end{aligned}$$

This together with (3.5) implies that the sequence $\{l_n\}$ is bounded. Further, there exists a convergent subsequence. For shortness of notation, we may assume that $\{l_n\}$ is convergent.

Therefore, there exists a positive integer m such that for any integer $n > m$,

$$\int_{t_m}^{t_n} e^{\int_0^u a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du < \frac{1-\eta}{8B(e^{-l}+1)}, \tag{3.7}$$

and

$$e^{\int_{tm}^{tn} a(s)ds} > \frac{1}{2}, \quad e^{-\int_0^{tn} a(s)ds} < e^{-l} + 1, \quad e^{\int_0^{tm} a(s)ds} < e^l + 1, \tag{3.8}$$

where $B = max\{K_0(e^l + 1), A_0K_0(e^l + 1), 1\}$.

For any $\delta > 0$ consider the solution $x(t) = x(t, t_m, \varphi)$ of Eq. (5) with $|\varphi|_{t_m} < \delta$ and

$|\varphi(t_m)| > \frac{\delta}{2}$. It follows from (3.2),(3.3),(3.6) ,C_3 and C_4 for $t \in [t_m, \infty)$ that

$$\begin{aligned}
|x(t)| &= \varphi(t_m)e^{-\int_{t_m}^{t} a(s)ds} \\
&+ \int_{t_m}^{t} e^{-\int_u^t a(u)du}[b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) \\
&+ q(u, x(u), x(u-\tau_2(u)))]du \\
&\leq \varphi(t_m)e^{-\int_u^t a(s)ds}e^{-\int_0^{t_m} a(s)ds} \\
&+ \|x\|_{t_m}\int_{t_m}^{t} e^{-\int_u^t a(u)du}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\
&\leq K_0(e^l+1)\delta + \|x\|_{t_m}\int_{t_m}^{t} e^{-\int_u^t a(u)du}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\
&\leq B\delta + \eta\|x\|_{t_m}
\end{aligned}$$

and

$$\begin{aligned}
|x'(t)| &= \varphi(t_m)a(t)e^{-\int_{t_m}^{t} a(s)ds} \\
&+ |b(t)g(x(t)) + c(t)f(x'(t-\tau_1(t))) + q(t, x(t), x(t-\tau_2(t)))| \\
&+ |a(t)|\int_{t_m}^{t} e^{-\int_u^t a(s)ds}[|b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) \\
&+ q(u, x(u), x(u-\tau_2(u)))|]du \\
&\leq A_0K_0(e^l+1)\delta \\
&+ \|x\|_{t_m}\{|a(t)|\int_{t_m}^{t} e^{-\int_u^t a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\
&+ |b(t)|L_g + L_f|c(t)| + L_1(t) + L_2(t)\} \\
&\leq B\delta + \eta\|x\|_{t_m}.
\end{aligned}$$

Hence,

$$\|x\|_{t_m} \leq B\delta + \eta\|x\|_{t_m},$$

so that

$$\|x\|_{t_m} \leq \frac{B}{1-\eta}\delta. \tag{3.9}$$

For $n > m$, we can obtain (3.2), (3.6)-(3.8) and C_2 that for $n > m$,

$$\begin{aligned}
|x(t_n)| &\geq |\varphi(t_m)|e^{-\int_{t_m}^{t_n} a(s)ds} \\
&- e^{-\int_0^{t_n} a(s)ds}\int_{t_m}^{t_n} e^{-\int_0^u a(s)ds}[|b(u)g(x(u)) + c(u)f(x'(u-\tau_1(u))) \\
&+ q(u, x(u), x(u-\tau_2(u)))|]du \\
&\geq |\varphi(t_m)|e^{-\int_{t_m}^{t_n} a(s)ds} \\
&- \|x\|_{t_m}e^{-\int_0^{t_n} a(s)ds}\int_{t_m}^{t_n} e^{-\int_0^u a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\
&> \frac{1}{4}\delta - \frac{B}{1-\eta}\delta(e^{-l}+1)\frac{1-\eta}{8B(e^{-l}+1)} = \frac{1}{8}\delta.
\end{aligned}$$

This contradicts to the fact that

$$lim_{t\to\infty}t_n = \infty.$$

Thus the zero solution of Eq. (5) is globally asymptotically stable in C^1. The proof is complete.

4 Example and Remark

Example 4.1. Consider the following first order neutral equation

$$\begin{aligned} x'(t) &= -\frac{1}{1+t}x(t) + \frac{1}{20(1+t)}(1-e^{-x^2(t)}) + \frac{1}{20(1+t)}(1-e^{x'(t-(5+4sint))}) \\ &+ \ln(1+\frac{|x(t)|+|x(t-(5+4cost))|}{20(1+t)}), \end{aligned} \tag{4.1}$$

where $a(t) = \frac{1}{1+t}, b(t) = \frac{1}{20(1+t)}, c(t) = \frac{1}{20(1+t)}, g(x) = 1-e^{-x^2(t)}, f(x') = 1-e^{x'(t-(5+4sint))}$ $,\tau_1 = 5+4sint$ and $\tau_2 = 5+4cost$.

Obviously $a, b, c \in C([0,\infty), R), g, f \in C(R, R), q \in C([0,\infty)R\times R, R)$ and $\tau_i \in C([0,\infty),[0,\infty))$ with $t-\tau_i \to \infty$ as $t \to \infty, i = 1, 2$.

A simple calculation shows that

$$|a(t)| \le 1, \qquad \int_0^\infty a(s)ds = \infty, \quad t \in [0,\infty).$$

Assume that $L_1(t) = L_2(t) = \frac{1}{20(1+t)}$. Then (C_2) holds. Also let $L_f = 2, L_g = 3$ and $\eta = 7/10$. Hence, we can get

$$\begin{aligned} &\int_0^t e^{-\int_u^t a(s)ds}[L_g|b(u)| + L_f|c(u)| + L_1(u) + L_2(u)]du \\ &\qquad = \int_0^t \frac{1+u}{1+t}\frac{7}{20(1+u)} = \frac{7}{20}\frac{t}{1+t} \le \eta \end{aligned}$$

and

$$\begin{aligned} |a(t)|\int_0^t & e^{-\int_u^t a(s)ds}[|b(u)|L_g + L_f|c(u)| + L_1(u) + L_2(u)]du \\ &+|b(t)|L_g + L_f|c(t)| + L_1(t) + L_2(t) \\ &< \frac{7}{20} + \frac{7}{20}\frac{t}{1+t} \le \eta. \end{aligned}$$

The above discussion shows that assumptions $(C_1)-(C_4)$ hold. This leads that the zero solution of (5) is global asymptotically stable in C^1.

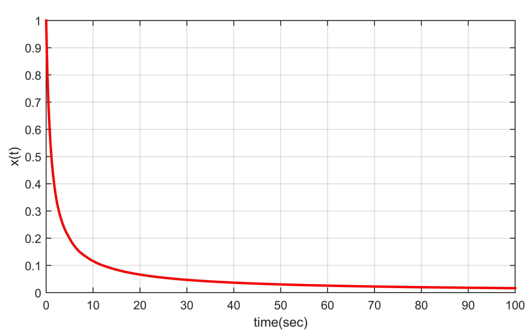

Figure 1. Trajectory solution $x(t)$ of Eq. (4.1)

Remark 4.2. Theorem 1 includes and generalizes the result of Liu and Yang [14]. In fact, when chose $g(x) = 0$ and $f(x'(t-\tau(t))) = x'(t-\tau(t))$ our conditions reduce to that of Liu and Yang [14,Theorem 2.1].

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