Palestine Journal of Mathematics

Vol. 6(2)(2017) , 562–568

Reccurence Relation of Generalized Mittag Leffler Function

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Communicated by Ayman Badawi

MSC 2010 Classifications: 33E12, 33B15, 11R32.

Keywords and phrases: Generalized Mittag-Leffler function; Recurrence relation: Wiman's function.

The authors are thankful to Prof. Kantesh Gupta, Malviya National Institute of Technology, Jaipur for her valuable help and constant encouragement.

Abstract. The aim of the present paper is to investigate a recurrence relation and an integral representation of generalized Mittag- Leffler function $E^{\gamma,\delta,q}_{\alpha,\beta,p}$ which can be reduced to H-function and Hyper geometric function. In the end several special cases have also been discussed.

1 Introduction

The Swedish Mathematician Gosta Mittag- Leffler [3] in 1903, introduced the function $E_\alpha(z)$, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n + 1)}, \{\alpha, z \in C; Re(\alpha) > 0\} \tag{1.1}$$

where $z \in C$ and $\Gamma(z)$ is the Gamma function: $\alpha \geq 0$.The Mittag- Leffler function in (1.1) reduces immediately to the exponential function $e^z = E_1(z)$ when $\alpha = 1$. Mittag- Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equation.

In 1905, Wiman [8] studied a function $E_{\alpha,\beta}(z)$,generalization of $E_\alpha(z)$ and defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n + \beta)}, \{\alpha, \beta, z \in C; Re(\alpha) > 0, Re(\beta) > 0\} \tag{1.2}$$

The function $E_{\alpha,\beta}(z)$ is known as Wiman function.

Prabhakar [4] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form of (see also Kilbas et al.[2])

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{(z)^n}{n!}, \{\alpha, \beta, \gamma, z \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0\} \tag{1.3}$$

where $(\gamma)_n$ is the pochammer symbol

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1 & (n = 0, \gamma \neq 0) \\ \gamma(\gamma + 1)...(\gamma + n - 1) & (n \in N, \gamma \in C) \end{cases}$$

N being the set of positive integers.

Shukla and Prajapati [6] defined and investigated the function, $E^{\gamma;q}_{\alpha,\beta}(z)$ as

$$E^{\gamma;q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{(z)^n}{n!} \tag{1.4}$$

$$\{\alpha, \beta, \gamma, z \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, q \in (0, 1) \cup N\}$$

and

$$(\gamma)_{qn} = q^{qn} \prod_{r=1}^{q} (\frac{\gamma + r - 1}{q})_n (q \in N, n \in N_0 := N \cup \{0\})$$

In the sequel of this study, Tariq and Ahmad [5] defined the function

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{(z)^n}{(\delta)_{pn}} \tag{1.5}$$

$$\{\alpha, \beta, \gamma, z \in C; min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta) > 0\}, p, q > 0, q \leq Re(\alpha) + p\}$$

It is easily seen that (1.5) is an obvious generalization of (1.1) to (1.4)

- Setting $\delta = p = 1$ it reduces to (1.4) defined by Shukla and Prajapati [6], in addition to that if $q = 1$, then we get eq. (1.3) defined by Prabhakar [4].
- On putting $\gamma = \delta = p = q = 1$ in (1.5) it reduces to Wiman's function, moreover if $\beta = 1$, Mittag-Leffler function $E_\alpha(z)$ will be the result.

2 Recurrence Relation

Theorem 1

For $(R(\alpha + a) > 0, R(\beta + s) > 0, R(c) > 0, p, q \in (0, 1) \cup N)$, we get

$$\begin{aligned} E^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p}(cz) - E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(cz) = (\beta + s)(\beta + s + 2)E^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) \\ +(\alpha + a)^2 z^2 \ddot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) + (\alpha + a)(\alpha + a + 2(\beta + s + 1))z\dot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) \end{aligned} \tag{2.1}$$

Where

$$\dot{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{d}{dz} E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$$

and

$$\ddot{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{d^2}{dz^2} E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$$

By putting $\alpha + a = k$ and $\beta + s = m$ in this theorem, we get the following corollary

Corollary 1

$$E^{\gamma,\delta,q}_{k,m+1,p}(cz) - E^{\gamma,\delta,q}_{k,m+2,p}(cz) = m(m+2)E^{\gamma,\delta,q}_{k,m+3,p}(cz) + k^2z^2\ddot{E}^{\gamma,\delta,q}_{k,m+3,p}(cz) + k(k+2m+2)z\dot{E}^{\gamma,\delta,q}_{k,m+3,p}(cz) \tag{2.2}$$

Proof of Theorem 1

By the fundamental relation of Gamma function $\Gamma(z+1) = z\Gamma z$ to (1.5), we can write

$$E^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p}(cz) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n+\beta+s\}\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} \tag{2.3}$$

and

$$E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(cz) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n+\beta+s+1\}\{(\alpha+a)n+\beta+s\}\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} \tag{2.4}$$

Equation (2.4) can be written as follows:

$$E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(cz) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} [\frac{1}{\{(\alpha+a)n+\beta+s\}} - \frac{1}{\{(\alpha+a)+\beta+s+1\}}]$$

$$E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(cz) = E^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p} - \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n+\beta+s+1\}\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} \tag{2.5}$$

For convenience we denote summation in (2.5) by S,

$$S = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n+\beta+s+1\}\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} \tag{2.6}$$

$$= E^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p}(cz) - E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(cz)$$

Applying a simple identity

$$\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1}$$

$$u = ((\alpha+a)n+\beta+s+1)$$

to (2.6)

$$S = \sum_{n=0}^{\infty} \frac{\{(\alpha+a)n+\beta+s\}(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}}$$

$$+ \sum_{n=0}^{\infty} \frac{\{(\alpha+a)n+\beta+s\}\{(\alpha+a)n+\beta+s+1\}(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}}$$

$$\begin{aligned} S = (\alpha+a)\sum_{n=0}^{\infty}\frac{n(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \\ +(\beta+s)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \\ +(\alpha+a)^2\sum_{n=0}^{\infty}\frac{n^2(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \\ +u\sum_{n=0}^{\infty}\frac{n(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \\ +v\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \end{aligned} \tag{2.7}$$

where $u = (\alpha+a)(2\beta+2s+1)$ and $v = (\beta+s)(\beta+s+1)$

Now express each summation on right hand side of (2.7) as follows:

$$\frac{d^2}{dz^2}(z^2E^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}) = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \tag{2.8}$$

From (2.8) we get

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{n^2(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \\ &= z^2\ddot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) + 4z\dot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) \\ &\quad -3\sum_{n=0}^{\infty}\frac{n(\gamma)_{qn}(cz)^n}{\Gamma\{(\alpha+a)n+\beta+s+3\}(\delta)_{pn}} \end{aligned} \tag{2.9}$$

Considering

$$\frac{d}{dz}(zE^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p}) = \sum_{n=0}^{\infty}\frac{(n+1)(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} \tag{2.10}$$

Similarly we get

$$\sum_{n=0}^{\infty}\frac{n(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} = zE^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) \tag{2.11}$$

Using (2.9) and (2.11) we have

$$\sum_{n=0}^{\infty}\frac{n^2(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n+\beta+s+3)(\delta)_{pn}} = z^2\ddot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) + z\dot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) \tag{2.12}$$

Using (2.11) and (2.12) in (2.7), we get

$$S=(\alpha+a)^2[z^2\ddot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz)+z\dot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz)]$$
$$+(\alpha+a+u)z\dot{E}^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz)-(\beta+s+v)E^{\gamma,\delta,q}_{\alpha+a,\beta+s+3,p}(cz) \tag{2.13}$$

From (2.6) and (2.13) we get the proof of theorem 1

3 Integral Representation

Theorem 2

We get

$$\int_0^1 t^{\beta+s}E^{\gamma,\delta,q}_{\alpha+a,\beta+s,p}(t^{\alpha+a})dt=\frac{1}{c^n}[E^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p}(c)-E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(c)] \tag{3.1}$$

$(R(\alpha+a)>0, R(\beta+s)>0, R(\gamma)>0, q\in(0,1)\cup N)$

Setting $\alpha+a=k\in N$ and $\beta+s=m\in N$ in (3.1) yields

Corollary 2

$$\int_0^1 t^m E^{\gamma,\delta,q}_{k,m,p}(t^k)dt=\frac{1}{c^n}[E^{\gamma,\delta,q}_{k,m+1,p}(c)-E^{\gamma,\delta,q}_{k,m+2,p}(c)] \tag{3.2}$$

Where

$k,m\in N$

Proof

Putting z=1 in (2.6) gives

$$\sum_0^\infty \frac{(\gamma)_{qn}(c)^n}{\Gamma((\alpha+a)n+\beta+s)\{(\alpha+a)n+\beta+s+1)\}(\delta)_{pn}}$$
$$=[E^{\gamma,\delta,q}_{\alpha+a,\beta+s+1,p}(c)-E^{\gamma,\delta,q}_{\alpha+a,\beta+s+2,p}(c)] \tag{3.3}$$

It is easy to find that

$$\int_0^1 t^{\beta+s}E^{\gamma,\delta,q}_{\alpha+a,\beta+s,p}(t^{\alpha+a})dt=\sum_0^\infty \frac{(\gamma)_{qn}(z)^{(\alpha+a)n+\beta+s+1}}{\{(\alpha+a)n+\beta+s+1)\}\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} \tag{3.4}$$

For $z=1$ in (3.4)

$$\int_0^1 t^{\beta+s}E^{\gamma,\delta,q}_{\alpha+a,\beta+s,p}(t^{\alpha+a})dt=\sum_0^\infty \frac{(\gamma)_{qn}}{\{(\alpha+a)n+\beta+s+1)\}\Gamma((\alpha+a)n+\beta+s)(\delta)_{pn}} \tag{3.5}$$

On comparing (3.3) with the identity obtained in (3.5) is seen to yields (3.1) in theorem 3

4 Special Cases

(i) Setting $\alpha = 1, q = 1, p = 1, \delta = 1, a = 0$ in (2.1) we get the following interesting relation

$$\begin{aligned}(\beta+s+2)(\beta+s+1)\,{}_1F_1[\gamma,\beta+s+1,cz] - {}_1F_1[\gamma,\beta+s+2,cz] \\ = (\beta+s)(\beta+s+2)\,{}_1F_1[\gamma,\beta+s+3,cz] - z^2\,{}_1\ddot{F}_1[\gamma,\beta+s+3,cz] \\ +\{1+2(\beta+s+1)\}z\,{}_1\dot{F}_1[\gamma,\beta+s+3,cz]\end{aligned} \tag{4.1}$$

(ii) Setting $\delta = p = c = 1$ in (2.1), we get a known recurrence relation of $E^{\gamma,q}_{\alpha,\beta}(z)$ by Shukla and Prajapati [[7],p.134,eq(2.1)].

$$\begin{aligned}E^{\gamma,q}_{\alpha+a,\beta+s+1}(z) - E^{\gamma,q}_{\alpha+a,\beta+s+2}(z) = (\beta+s)(\beta+s+2)E^{\gamma,q}_{\alpha+a,\beta+s+3}(z) \\ +(\alpha+a)^2 z^2 \ddot{E}^{\gamma,q}_{\alpha+a,\beta+s+3}(z) + (\alpha+a)(\alpha+a+2(\beta+s+1))z\dot{E}^{\gamma,q}_{\alpha+a,\beta+s+3}(z)\end{aligned} \tag{4.2}$$

where

$$\dot{E}^{\gamma,q}_{\alpha,\beta}(z) = \frac{d}{dz}E^{\gamma,q}_{\alpha,\beta}(z)$$

and

$$\ddot{E}^{\gamma,q}_{\alpha,\beta}(z) = \frac{d^2}{dz^2}E^{\gamma,q}_{\alpha,\beta}(z)$$

(iii) Putting $a = 0, \delta = \gamma = q = 1; \beta + s = m \in N, p = 1$ in (2.1), reduces to a known recurrence relation by Gupta and Debnath [1] of $E_{\alpha,\beta}(z)$

$$\begin{aligned}E_{\alpha,m+1}(z) = E_{\alpha,m+2}(z) + m(m+2)E_{\alpha,m+3}(z) + \alpha^2 z^2 \ddot{E}_{\alpha,m+3}(z) \\ +\alpha(\alpha+2m+2)z\dot{E}_{\alpha,m+3}(z)\end{aligned} \tag{4.3}$$

Where

$$\dot{E}_{\alpha,\beta}(z) = \frac{d}{dz}E_{\alpha,\beta}(z)$$

$$\ddot{E}_{\alpha,\beta}(z) = \frac{d^2}{dz^2}E_{\alpha,\beta}(z)$$

(iv) Substituting $\delta = 1, c = 1, p = 1$ in (3.1), we get integral representation of $E^{\gamma,q}_{\alpha,\beta}(z)$ by Shukla and Prajapati [7]

$$\int_0^1 t^{\beta+s}E^{\gamma,q}_{\alpha+a,\beta+s}(t^{\alpha+a})dt = [E^{\gamma,q}_{\alpha+a,\beta+s+1}(1) - E^{\gamma,q}_{\alpha+a,\beta+s+2}(1)] \tag{4.4}$$

(v) Substituting $\gamma = 2, q = 1, \alpha = 1, a = 0, \beta + s = 1, c = 1, z = 1, p = 1, \delta = 1$ in (3.1),we get

$$\int_0^1 tE^{2,1,1}_{1,1,1}(t)dt = [E^{2,1,1}_{1,2,1}(1) - E^{2,1,1}_{1,3,1}(1)] \tag{4.5}$$

Putting $\gamma = 1, \delta = 1, q = 1, c = 1, k = 1, m = 1, p = 1$ in (3.1) we get

$$\int_0^1 tE_{1,1,1}^{1,1,1}(t)dt = E_{1,2,1}^{1,1,1}(1) - E_{1,3,1}^{1,1,1}(1) \tag{4.6}$$

or

$$\int_0^1 te^t dt = E_{1,2}(1) - E_{1,3}(1) \tag{4.7}$$

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Received: December 12, 2015.
Accepted: October 10, 2016.