

Reccurrence Relation of Generalized Mittag Leffler Function

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Abstract. The aim of the present paper is to investigate a recurrence relation and an integral representation of generalized Mittag- Leffler function $E_{\alpha,\beta,p}^{\gamma,\delta,q}$ which can be reduced to H-function and Hyper geometric function. In the end several special cases have also been discussed.

1 Introduction

The Swedish Mathematician Gosta Mittag- Leffler [3] in 1903, introduced the function $E_{\alpha}(z)$, defined as

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n + 1)}, \{\alpha, z \in C; Re(\alpha) > 0\} \tag{1.1}$$

where $z \in C$ and $\Gamma(z)$ is the Gamma function: $\alpha \geq 0$.The Mittag- Leffler function in (1.1) reduces immediately to the exponential function $e^z = E_1(z)$ when $\alpha = 1$. Mittag- Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equation.

In 1905, Wiman [8] studied a function $E_{\alpha,\beta}(z)$,generalization of $E_{\alpha}(z)$ and defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n + \beta)}, \{\alpha, \beta, z \in C; Re(\alpha) > 0, Re(\beta) > 0\} \tag{1.2}$$

The function $E_{\alpha,\beta}(z)$ is known as Wiman function.

Prabhakar [4] introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ in the form of (see also Kilbas et al.[2])

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{(z)^n}{n!}, \{\alpha, \beta, \gamma, z \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0\} \tag{1.3}$$

where $(\gamma)_n$ is the pochhammer symbol

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1 & (n = 0, \gamma \neq 0) \\ \gamma(\gamma + 1)\dots(\gamma + n - 1) & (n \in N, \gamma \in C) \end{cases}$$

N being the set of positive integers.

Shukla and Prajapati [6] defined and investigated the function, $E_{\alpha,\beta}^{\gamma,q}(z)$ as

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{(z)^n}{n!} \tag{1.4}$$

$$\{\alpha, \beta, \gamma, z \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, q \in (0, 1) \cup N\}$$

and

$$(\gamma)_{qn} = q^{qn} \prod_{r=1}^q \left(\frac{\gamma + r - 1}{q} \right)_n \quad (q \in N, n \in N_0 := N \cup \{0\})$$

In the sequel of this study, Tariq and Ahmad [5] defined the function

$$E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{(z)^n}{(\delta)_{pn}} \tag{1.5}$$

$$\{\alpha, \beta, \gamma, z \in C; \min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0, p, q > 0, q \leq Re(\alpha) + p\}$$

It is easily seen that (1.5) is an obvious generalization of (1.1) to (1.4)

- Setting $\delta = p = 1$ it reduces to (1.4) defined by Shukla and Prajapati [6], in addition to that if $q = 1$, then we get eq. (1.3) defined by Prabhakar [4].
- On putting $\gamma = \delta = p = q = 1$ in (1.5) it reduces to Wiman’s function, moreover if $\beta = 1$, Mittag-Leffler function $E_{\alpha}(z)$ will be the result.

2 Recurrence Relation

Theorem 1

For $(R(\alpha + a) > 0, R(\beta + s) > 0, R(c) > 0, p, q \in (0, 1) \cup N)$, we get

$$E_{\alpha+a, \beta+s+1, p}^{\gamma, \delta, q}(cz) - E_{\alpha+a, \beta+s+2, p}^{\gamma, \delta, q}(cz) = (\beta + s)(\beta + s + 2)E_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) + (\alpha + a)^2 z^2 \ddot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) + (\alpha + a)(\alpha + a + 2(\beta + s + 1))z \dot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) \tag{2.1}$$

Where

$$\dot{E}_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \frac{d}{dz} E_{\alpha, \beta, p}^{\gamma, \delta, q}(z)$$

and

$$\ddot{E}_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \frac{d^2}{dz^2} E_{\alpha, \beta, p}^{\gamma, \delta, q}(z)$$

By putting $\alpha + a = k$ and $\beta + s = m$ in this theorem, we get the following corollary

Corollary 1

$$E_{k,m+1,p}^{\gamma,\delta,q}(cz) - E_{k,m+2,p}^{\gamma,\delta,q}(cz) = m(m+2)E_{k,m+3,p}^{\gamma,\delta,q}(cz) + k^2 z^2 \dot{E}_{k,m+3,p}^{\gamma,\delta,q}(cz) + k(k+2m+2)z \dot{E}_{k,m+3,p}^{\gamma,\delta,q}(cz) \tag{2.2}$$

Proof of Theorem 1

By the fundamental relation of Gamma function $\Gamma(z + 1) = z\Gamma z$ to (1.5), we can write

$$E_{\alpha+a,\beta+s+1,p}^{\gamma,\delta,q}(cz) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n + \beta + s\}\Gamma((\alpha+a)n + \beta + s)(\delta)_{pn}} \tag{2.3}$$

and

$$E_{\alpha+a,\beta+s+2,p}^{\gamma,\delta,q}(cz) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n + \beta + s + 1\}\{(\alpha+a)n + \beta + s\}\Gamma((\alpha+a)n + \beta + s)(\delta)_{pn}} \tag{2.4}$$

Equation (2.4) can be written as follows:

$$E_{\alpha+a,\beta+s+2,p}^{\gamma,\delta,q}(cz) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n + \beta + s)(\delta)_{pn}} \left[\frac{1}{\{(\alpha+a)n + \beta + s\}} - \frac{1}{\{(\alpha+a)n + \beta + s + 1\}} \right]$$

$$E_{\alpha+a,\beta+s+2,p}^{\gamma,\delta,q}(cz) = E_{\alpha+a,\beta+s+1,p}^{\gamma,\delta,q}(cz) - \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n + \beta + s + 1\}\Gamma((\alpha+a)n + \beta + s)(\delta)_{pn}} \tag{2.5}$$

For convenience we denote summation in (2.5) by S,

$$S = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\{(\alpha+a)n + \beta + s + 1\}\Gamma((\alpha+a)n + \beta + s)(\delta)_{pn}} \tag{2.6}$$

$$= E_{\alpha+a,\beta+s+1,p}^{\gamma,\delta,q}(cz) - E_{\alpha+a,\beta+s+2,p}^{\gamma,\delta,q}(cz)$$

Applying a simple identity

$$\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1}$$

$$u = ((\alpha+a)n + \beta + s + 1)$$

to (2.6)

$$S = \sum_{n=0}^{\infty} \frac{\{(\alpha+a)n + \beta + s\}(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n + \beta + s + 3)(\delta)_{pn}}$$

$$+ \sum_{n=0}^{\infty} \frac{\{(\alpha+a)n + \beta + s\}\{(\alpha+a)n + \beta + s + 1\}(\gamma)_{qn}(cz)^n}{\Gamma((\alpha+a)n + \beta + s + 3)(\delta)_{pn}}$$

$$\begin{aligned}
 S &= (\alpha + a) \sum_{n=0}^{\infty} \frac{n(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \\
 &+ (\beta + s) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \\
 &+ (\alpha + a)^2 \sum_{n=0}^{\infty} \frac{n^2(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \\
 &+ u \sum_{n=0}^{\infty} \frac{n(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \\
 &+ v \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}}
 \end{aligned} \tag{2.7}$$

where $u = (\alpha + a)(2\beta + 2s + 1)$ and $v = (\beta + s)(\beta + s + 1)$

Now express each summation on right hand side of (2.7) as follows:

$$\frac{d^2}{dz^2} (z^2 E_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}) = \sum_{n=0}^{\infty} \frac{(n + 2)(n + 1)(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \tag{2.8}$$

From (2.8) we get

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \frac{n^2(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \\
 &= z^2 \ddot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) + 4z \dot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) \\
 &- 3 \sum_{n=0}^{\infty} \frac{n(\gamma)_{qn}(cz)^n}{\Gamma\{(\alpha + a)n + \beta + s + 3\}(\delta)_{pn}}
 \end{aligned} \tag{2.9}$$

Considering

$$\frac{d}{dz} (z E_{\alpha+a, \beta+s+1, p}^{\gamma, \delta, q}) = \sum_{n=0}^{\infty} \frac{(n + 1)(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} \tag{2.10}$$

Similarly we get

$$\sum_{n=0}^{\infty} \frac{n(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} = z E_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) \tag{2.11}$$

Using (2.9) and (2.11) we have

$$\sum_{n=0}^{\infty} \frac{n^2(\gamma)_{qn}(cz)^n}{\Gamma((\alpha + a)n + \beta + s + 3)(\delta)_{pn}} = z^2 \ddot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) + z \dot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) \tag{2.12}$$

Using (2.11) and (2.12) in (2.7), we get

$$\begin{aligned}
 S &= (\alpha + a)^2 [z^2 \ddot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) + z \dot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz)] \\
 &+ (\alpha + a + u) z \dot{E}_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz) - (\beta + s + v) E_{\alpha+a, \beta+s+3, p}^{\gamma, \delta, q}(cz)
 \end{aligned}
 \tag{2.13}$$

From (2.6) and (2.13) we get the proof of theorem 1

3 Integral Representation

Theorem 2

We get

$$\int_0^1 t^{\beta+s} E_{\alpha+a, \beta+s, p}^{\gamma, \delta, q}(t^{\alpha+a}) dt = \frac{1}{c^n} [E_{\alpha+a, \beta+s+1, p}^{\gamma, \delta, q}(c) - E_{\alpha+a, \beta+s+2, p}^{\gamma, \delta, q}(c)]
 \tag{3.1}$$

$(R(\alpha + a) > 0, R(\beta + s) > 0, R(\gamma) > 0, q \in (0, 1) \cup N)$

Setting $\alpha + a = k \in N$ and $\beta + s = m \in N$ in (3.1) yields

Corollary 2

$$\int_0^1 t^m E_{k, m, p}^{\gamma, \delta, q}(t^k) dt = \frac{1}{c^n} [E_{k, m+1, p}^{\gamma, \delta, q}(c) - E_{k, m+2, p}^{\gamma, \delta, q}(c)]
 \tag{3.2}$$

Where

$k, m \in N$

Proof

Putting $z=1$ in (2.6) gives

$$\begin{aligned}
 \sum_0^\infty \frac{(\gamma)_{qn}(c)^n}{\Gamma((\alpha + a)n + \beta + s)\{(\alpha + a)n + \beta + s + 1\}(\delta)_{pn}} \\
 = [E_{\alpha+a, \beta+s+1, p}^{\gamma, \delta, q}(c) - E_{\alpha+a, \beta+s+2, p}^{\gamma, \delta, q}(c)]
 \end{aligned}
 \tag{3.3}$$

It is easy to find that

$$\int_0^1 t^{\beta+s} E_{\alpha+a, \beta+s, p}^{\gamma, \delta, q}(t^{\alpha+a}) dt = \sum_0^\infty \frac{(\gamma)_{qn}(z)^{(\alpha+a)n+\beta+s+1}}{\{(\alpha + a)n + \beta + s + 1\}\Gamma((\alpha + a)n + \beta + s)(\delta)_{pn}}
 \tag{3.4}$$

For $z = 1$ in (3.4)

$$\int_0^1 t^{\beta+s} E_{\alpha+a, \beta+s, p}^{\gamma, \delta, q}(t^{\alpha+a}) dt = \sum_0^\infty \frac{(\gamma)_{qn}}{\{(\alpha + a)n + \beta + s + 1\}\Gamma((\alpha + a)n + \beta + s)(\delta)_{pn}}
 \tag{3.5}$$

On comparing (3.3) with the identity obtained in (3.5) is seen to yields (3.1) in theorem 3

4 Special Cases

(i) Setting $\alpha = 1, q = 1, p = 1, \delta = 1, a = 0$ in (2.1) we get the following interesting relation

$$\begin{aligned}
 & (\beta + s + 2)(\beta + s + 1) \text{I} F \text{I} [\gamma, \beta + s + 1, cz] - \text{I} F \text{I} [\gamma, \beta + s + 2, cz] \\
 & = (\beta + s)(\beta + s + 2) \text{I} F \text{I} [\gamma, \beta + s + 3, cz] - z^2 \text{I} \dot{F} \text{I} [\gamma, \beta + s + 3, cz] \\
 & \quad + \{1 + 2(\beta + s + 1)\}z \text{I} \dot{F} \text{I} [\gamma, \beta + s + 3, cz] \tag{4.1}
 \end{aligned}$$

(ii) Setting $\delta = p = c = 1$ in (2.1), we get a known recurrence relation of $E_{\alpha,\beta}^{\gamma,q}(z)$ by Shukla and Prajapati [[7],p.134,eq(2.1)].

$$\begin{aligned}
 & E_{\alpha+a,\beta+s+1}^{\gamma,q}(z) - E_{\alpha+a,\beta+s+2}^{\gamma,q}(z) = (\beta + s)(\beta + s + 2)E_{\alpha+a,\beta+s+3}^{\gamma,q}(z) \\
 & + (\alpha + a)^2 z^2 \ddot{E}_{\alpha+a,\beta+s+3}^{\gamma,q}(z) + (\alpha + a)(\alpha + a + 2(\beta + s + 1))z \dot{E}_{\alpha+a,\beta+s+3}^{\gamma,q}(z) \tag{4.2}
 \end{aligned}$$

where

$$\dot{E}_{\alpha,\beta}^{\gamma,q}(z) = \frac{d}{dz} E_{\alpha,\beta}^{\gamma,q}(z)$$

and

$$\ddot{E}_{\alpha,\beta}^{\gamma,q}(z) = \frac{d^2}{dz^2} E_{\alpha,\beta}^{\gamma,q}(z)$$

(iii) Putting $a = 0, \delta = \gamma = q = 1; \beta + s = m \in N, p = 1$ in (2.1), reduces to a known recurrence relation by Gupta and Debnath [1] of $E_{\alpha,\beta}(z)$

$$\begin{aligned}
 E_{\alpha,m+1}(z) & = E_{\alpha,m+2}(z) + m(m + 2)E_{\alpha,m+3}(z) + \alpha^2 z^2 \ddot{E}_{\alpha,m+3}(z) \\
 & \quad + \alpha(\alpha + 2m + 2)z \dot{E}_{\alpha,m+3}(z) \tag{4.3}
 \end{aligned}$$

Where

$$\dot{E}_{\alpha,\beta}(z) = \frac{d}{dz} E_{\alpha,\beta}(z)$$

$$\ddot{E}_{\alpha,\beta}(z) = \frac{d^2}{dz^2} E_{\alpha,\beta}(z)$$

(iv) Substituting $\delta = 1, c = 1, p = 1$ in (3.1), we get integral representation of $E_{\alpha,\beta}^{\gamma,q}(z)$ by Shukla and Prajapati [7]

$$\int_0^1 t^{\beta+s} E_{\alpha+a,\beta+s}^{\gamma,q}(t^{\alpha+a}) dt = [E_{\alpha+a,\beta+s+1}^{\gamma,q}(1) - E_{\alpha+a,\beta+s+2}^{\gamma,q}(1)] \tag{4.4}$$

(v) Substituting $\gamma = 2, q = 1, \alpha = 1, a = 0, \beta + s = 1, c = 1, z = 1, p = 1, \delta = 1$ in (3.1), we get

$$\int_0^1 t E_{1,1,1}^{2,1,1}(t) dt = [E_{1,2,1}^{2,1,1}(1) - E_{1,3,1}^{2,1,1}(1)] \tag{4.5}$$

Putting $\gamma = 1, \delta = 1, q = 1, c = 1, k = 1, m = 1, p = 1$ in (3.1) we get

$$\int_0^1 tE_{1,1,1}^{1,1,1}(t)dt = E_{1,2,1}^{1,1,1}(1) - E_{1,3,1}^{1,1,1}(1) \quad (4.6)$$

or

$$\int_0^1 te^t dt = E_{1,2}(1) - E_{1,3}(1) \quad (4.7)$$

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