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Central α-rigid rings

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Abstract. For a ring endomirphism α, we introduce the class of central α-rigid rings, which are a generalization of α-rigid rings, and investigate their properties. For a ring R, we show that R is central α-rigid if and only if RS^{-1} is central $\bar{\alpha}$-rigid. Moreover, we give an example to show that if R is central α-rigid, then $T_n(R)$ is not necessary central $\bar{\alpha}$-rigid, but $S_n(R)$ is central $\bar{\alpha}$-rigid.

1 Introduction

Throughout this article, R denotes an associative ring with identity and α be an endomorphism of a ring R. For notation $R[x]$, $R[x, x^{-1}]$, $T_n(R)$, $C(R)$ and e_{ij} denote, the polynomial ring over R, the Laurent polynomial ring over R, its upper triangular matrix ring, the center of a ring R and the matrix with (i, j)-entry 1 and elsewhere 0, respectively. A ring is reduced if it has no nonzero nilpotent elements. According to Krempa [5], an endomorphism α of a ring R is called to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring R α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced rings by Hong et al. [2]. Properties of α-rigid rings have been studied in Krempa [5, 2, 1]. So far α-rigid rings are generalized in several forms [7, 6, 4, 3].

Motivated by the above results, we investigate a generalization of α-rigid rings. A ring R is called a central α-rigid ring if for any $a, b \in R, a\alpha(a) = 0$ implies $a \in C(R)$. Clearly, all commutative rings and α-rigid rings are central α-rigid.

2 Central α-rigid rings

In this section, the central α-rigid rings are introduced as a generalization of α-rigid rings.

Definition 2.1. Let α be an endomorphism of a ring R. The ring R is called central α-rigid if for any $a \in R, a\alpha(a) = 0$ implies $a \in C(R)$.

It is clear that α-rigid rings are central α-rigid, but the converse is not always true by the following examples.

Example 2.2. Let $R = R_1 \oplus R_2$, where R_i is a commutative ring for $i = 1, 2$. Let $\alpha : R \longrightarrow R$ be an automorphism defined by $\alpha((a, b)) = (b, a)$, then $(1, 0)\alpha(1, 0) = 0$, but $(1, 0) \neq 0$. Therefore, R is not α-rigid. But R is central α-rigid, since R is commutative.

Example 2.3. Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider a ring $R = \left\{ \left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{smallmatrix}\right) \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}$ and $\alpha : R \longrightarrow R$ be an endomorphism defined by $\alpha\left(\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \bar{a} & -\bar{b} \\ 0 & \bar{a} \end{smallmatrix}\right)$. The ring R is not α-rigid. In fact $\left(\begin{smallmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{smallmatrix}\right)\alpha\left(\left(\begin{smallmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{smallmatrix}\right)\right) = 0$ but $\left(\begin{smallmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{smallmatrix}\right) \neq 0$. But it can be easily checked that R is commutative and so, it is central α-rigid ring.

The subrings of central α-rigid rings are central α-rigid. Let R_k be a ring, where $k \in \mathbb{Z}$, α_k an endomorphism of R_k and let $R = \prod_{k\in\mathbb{Z}} R_k$. Then the map $\alpha : R \longrightarrow R$ defined by $\alpha((a_k)) =$

$(\alpha_k(a_k))$ is an endomorphism of R and therefore R_k is central α_k-rigid for each $k \in \mathbb{Z}$ if and only if $R = \prod_{k\in\mathbb{Z}} R_k$ is central $\alpha-$rigid. As a result, for any idempotent $e^2 = e$ we have eR and $(1-e)R$ are central α-rigid if and only if R is central α-rigid, since $R = eR \oplus (1-e)R$.

Proposition 2.4. *Let α be an endomorphism of a ring R. Let S be a ring and $\varphi : R \longrightarrow S$ an isomorphism. Then R is central α-rigid if and only if S is central $\varphi\alpha\varphi^{-1}$ -rigid.*

Proof. Let $\alpha' = \varphi\alpha\varphi^{-1}$. Clearly, α' is an endomorphism of S. Suppose that $a' = \varphi(a)$, for $a \in R$. Since φ is an isomorphism, $a'\alpha'(a') = 0$ in S if and only if $a\alpha(a) = 0$ in R and so $a \in C(R)$ if and only if $a' \in C(S)$. Thus R is central α-rigid if and only if S is central $\varphi\alpha\varphi^{-1}$-rigid. □

Let α be an endomorphism of a ring R. The endomorphism α of R is extended to the endomorphism $\bar{\alpha} : T_n(R) \longrightarrow T_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. The following example shows that if R is central α-rigid, then $T_2(R)$ is not necessary central $\bar{\alpha}$-rigid.

Example 2.5. Let $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\alpha : R \longrightarrow R$ be an automorphism defined by $\alpha((a,b)) = (b,a)$. Then the ring $T_2(R)$ is not central $\bar{\alpha}$-rigid. In fact

$$\begin{pmatrix} (2,2) & (2,0) \\ (0,0) & (1,0) \end{pmatrix} \alpha\left(\begin{pmatrix} (2,2) & (2,0) \\ (0,0) & (1,0) \end{pmatrix}\right) = 0$$

but $\begin{pmatrix} (2,2) & (2,0) \\ (0,0) & (1,0) \end{pmatrix} \notin C(T_2(R))$. But R is central α-rigid, since it is commutative.

Theorem 2.6. *Let α be an endomorphism of a ring R. Then R is central α-rigid if and only if*

$$S_n(R) = \left\{ \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix} | a_1, a_2, ..., a_n \in R \right\} \text{ is central } \bar{\alpha}\text{-rigid for any } n \geq 1.$$

Proof. Suppose R is central α-rigid. Let $A = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix} \in S_n(R)$ be such that $A\bar{\alpha}(A) = 0$. Therefore $a_i\alpha(a_i) = 0$ for $i = 1, 2, ..., n$. Hence $a_i \in C(R)$, since R is central α-rigid, and so $A \in C(S_n(R))$, as desired.

Conversely, let $a\alpha(a) = 0$ for any $a \in R$. Therefore $ae_{11}\alpha(ae_{11}) = 0$. Hence $ae_{11} \in C(S_n(R))$, since $S_n(R)$ is central α-rigid, and so $a \in C(R)$, as desired. □

Recall that if α is an endomorphism of a ring R, then the map $R[x] \longrightarrow R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall also denote the extended map $R[x] \longrightarrow R[x]$ by α and the image of $f \in R[x]$ by $\alpha(f)$. The ring $R[x]$ is called linear central α-rigid if for any $f(x) = a_0 + a_1x \in R[x]$, $f(x)\alpha(f(x)) = 0$ implies that $f(x) \in C(R[x])$. Now we have the following.

Theorem 2.7. *Let α be an endomorphism of a ring R. Then R is central α-rigid if and only if $R[x]$ is linear central α-rigid.*

Proof. Assume that $R[x]$ is linear central α-rigid. Then R is central α-rigid as a subring of $R[x]$. Conversely, assume that R is central α-rigid and $f(x) = a_0 + a_1x \in R[x]$ such that $f(x)\alpha(f(x)) = 0$. Then $a_0\alpha(a_0) = 0$ and $a_1\alpha(a_1) = 0$ and so $a_0, a_1 \in C(R)$, since R is central α-rigid. Therefore, $f(x) \in C(R[x])$ and hence $R[x]$ is linear central α-rigid. □

The following example, shows that there exists a non-identity endomorphism α of a ring R such that R/I is central $\bar{\alpha}$-rigid and as a ring I is central α-rigid for any nonzero proper ideal I of R, but R is not central α-rigid.

Example 2.8. Let F be a field and consider a ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and an endomorphism α of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. Notice $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin C(R)$. Thus R is not central α-rigid. Consider the ideal $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ of R. Hence R/I is central α-rigid because of $R/I \cong F$.

For an ideal I of R, if $\alpha(I) \subseteq I$ then $\bar{\alpha} : R/I \longrightarrow R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a) + I$ is an endomorphism of a factor ring R/I. The homomorphic image of a central α-rigid ring need not be central α-rigid. Consider the following example.

Example 2.9. Let D be a division ring, $R = D[x, y, z]$ and $I = \langle z^2 \rangle$ where $zx \neq xz$. Let $\alpha : R \longrightarrow R$ be an endomorphism defined by $\alpha(a_1 + a_2x + a_3y + a_4z) = a_1 + a_2y + a_3x + a_4z$, for any $a_i \in D$. Since R is domain, R is central α-rigid. On the other hand, $(z+I)\bar{\alpha}(z+I) = I$ but $z + I$ does not commute with $x + I$. Hence R/I is not central $\bar{\alpha}$-rigid.

Let α be an automorphism of a ring R. Suppose that there exists the classical right quotient ring $Q(R)$ of R. Then for any $ab^{-1} \in Q(R)$ where $a, b \in R$ with b regular, the induced map $\bar{\alpha} : Q(R) \longrightarrow Q(R)$ defined by $\bar{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ is also an automorphism. Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements and let RS^{-1} be the localization of R at S.

Theorem 2.10. *Let α be an automorphism of a ring R. Then R is central α-rigid, if and only if RS^{-1} is central $\bar{\alpha}$-rigid.*

Proof. Suppose that R is central α-rigid. Let $(as^{-1})\bar{\alpha}(as^{-1}) = 0$, for any $(as^{-1}) \in RS^{-1}$. Let $as^{-1} = c^{-1}a'$ with c regular element in R. Then we have $(c^{-1}a')\bar{\alpha}(c^{-1}a') = 0$. Therefore, $a'\alpha(a') = 0$. Since R is central α-rigid, $a' \in C(R)$. and so as^{-1} is central in R. Thus RS^{-1} is central $\bar{\alpha}$-rigid. Conversely, assume that RS^{-1} is central $\bar{\alpha}$-rigid ring. Then R is central α-rigid as a subring of RS^{-1}. □

Corollary 2.11. *Let R be a ring and α an automorphism of R. Then the following are equivalent:*
$(1)R$ is central α-rigid.
$(2)R[x]$ is linear central α-rigid.
$(3)R[x, x^{-1}]$ is linear central α-rigid.

Proof. Let $S = \{1, x, x^2, x^3, x^4, \cdots\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Then the proof follows from Theorem 2.7 and Theorem 2.10. □

Theorem 2.12. *The class of central α-rigid rings is closed under direct limits with injective maps.*

Proof. Let $D = \{R_i, \alpha_{ij}\}$ be direct system of central $\alpha_{ij}-$rigid rings R_i, for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \longrightarrow R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \underline{\lim} R_i$ be a direct limit of D and let $\alpha : R \longrightarrow R$ be an automorphism defined by $\alpha(\underline{\lim} R_i) = \underline{\lim}\alpha_{ij}(R_i)$. Also $L_i : R_i \longrightarrow R$ and $L_j\alpha_{ij} = L_i$ where every L_i is injective. We will show that R is an central α-rigid ring. Take $a, b \in R$. Then $a = L_i(a_i), b = L_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define

$$a + b = L_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \text{ and } ab = L_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$$

where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k. Then R forms a ring with $0 = L_i(0)$ and $1 = L_i(1)$. Now let $a \in R$ be nonzero element such that $a\alpha(a) = 0$. There is $k \in I$ such that $a \in R_k$. Hence we get $a\alpha_{ij}(a) = 0$ in R_k. Since R_k is central $\alpha_{ij}-$rigid, so $a \in C(R_k)$. Therefore $ac_k = c_ka$, for any $c_k \in R_i$. Put $c = L_k(c_k)$. Then $ac = ca$, for any $c \in R$. Thus R is central α-rigid ring. □

References

[1] Y. Hirano, On the uniqueness of rings of coefficients in skew polynomial rings, Publ. Math. Debrecen, 54(1999), 489-495.

[2] C.Y. Hong, N.K. Kim and T.K. Kwak, Ore extensions of Baer and p.p.-rings, J. Pure Appl. Algebra, 151(2000), 215-226.

[3] A. R. Nasr-Isfahani, A. Moussavi, On weakly rigid rings, Glasgow Math. J. 51 (2009) 425-440.

[4] D. JokanoviÄĞ, Properties of Armendariz rings and weak Armendariz rings, Publ. de lâĂŹinstitut math., 85(99) (2009), 131-137.

[5] J. Krempa, Some examples of reduced rings, Algebra Colloq., 3(1996), 289- 300.

[6] L. Ouyang, Extensions of generalized α-rigid rings, Int. Electronic J. of Algebra, 3 (2008), 103-116.

[7] H. Kose, B. Ungor, S. Halicioglu, A generalization of reduced rings, Hacettepe J. Math. Statis., 41(5) (2012), 689-696.

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