Palestine Journal of Mathematics

Vol. 6(2)(2017) , 585–590

ON CERTAIN MIXED GENERATING FUNCTIONS OF JACOBI POLYNOMIALS

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Communicated by Ayman Badawi

MSC 2010 Classifications: 33C20, 33C45, 33C65, 33C90.

Keywords and phrases: Legendre polynomials, Gegenbauer polynomials, Tchebcheff polynomials, Generalized Laguerre or Sonine polynomials, Jacobi and Lagrange's polynomials, Lauricella function, Appel's and generalized Gauss function, Pochhammer symbol.

Abstract. This paper deals with general expansions which gives as special cases involving Jacobi and Laguerre polynomials, Lauricella, Appell and generalized Gauss function. The results unify and extended Exton's generating function [2] and Feldheim's expansion [3]. Also of interest are mixed generating functions which are partly unilateral and partly bilateral.

1 Introduction

Now consider a generating function [7, p. 106 (11)].

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\frac{t^n}{(-\alpha-\beta)_n} = exp\left[-\frac{1}{2}(x+1)t\right]\ {}_1F_1\left[\begin{array}{c} -\beta\,; \\ \\ -\alpha-\beta\,; \end{array} t\right] \tag{1.1}$$

where $P_n^{(\alpha,\beta)}$ is a Jacobi polynomial [1; p. 170 (16)]

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{array}{c} -n,\ 1+\alpha+\beta+n\,; \\ \\ 1+\alpha\,; \end{array} -\frac{1}{2}(1-x)\right] \tag{1.2}$$

or equivalently,

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}}{k!\ (n-k)!\ (1+\alpha)_k(1+\alpha+\beta)_n}\left(\frac{x-1}{x}\right)^k. \tag{1.3}$$

where ${}_pF_q$ denote generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters defined by [7; p. 42 (1)].

$$\begin{aligned} {}_pF_q\left[\begin{array}{cccc} (\alpha_1), & (\alpha_2), & \cdots & (\alpha_p); \\ \\ (\beta_1), & (\beta_2), & \cdots & (\beta_q); \end{array} x\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{x^n}{n!} \\ &= {}_pF_q(\alpha_1,\cdots\alpha_p;\beta_1,\cdots\beta_q;x) \end{aligned} \tag{1.4}$$

where $(a)_n$ is the Pochhammer symbol, defined by

$$(a)_n = \left\{\begin{array}{cc} 1 & if n=0 \\ a(a+1)(a+2)\cdots(a+n-1) & if n=1,2,3,\cdots \end{array}\right\} \tag{1.5}$$

the denominator parameters are neither zero nor a negative integers the numerator parameters may be zero and negative integers. We list below a number of important polynomials which can be expressed in terms of Jacobi polynomial for different values of α and β.

$$P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(z) = \frac{(\mu+\frac{1}{2})_n}{(2\mu)_n}\, C_n^{\mu}(z), \tag{1.6}$$

where $C_n^{\mu}(z)$ is the Gegenbauer polynomial (see [6],[7]),

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(z) = \frac{(\frac{1}{2})_n}{n!}\, T_n(z), \tag{1.7}$$

$$P_n^{(\frac{1}{2},\frac{1}{2})}(z) = \frac{(\frac{3}{2})_n}{(n+1)!}\, U_n(z), \tag{1.8}$$

where $T_n(z)$ and $U_n(z)$ are the Tchebicheff polynomial of first and second kind, (see [6],[7]) and

$$P_n^{(0,0)}(z) = P_n(z), \tag{1.9}$$

where $P_n(z)$ is the Legendre polynomial (see [6],[7]).

Jacobi polynomial is an important class of orthogonal polynomial which is a generalization of ultraspherical polynomials. This class contains many special functions commonly encountered in the applications, e.g. Legendre, Gegenbauer, Tchebcheff, Laguerre and Bessel polynomials.

Pathan and Kamarujjama [5; p. 2 (2.3)] and Khan [4; p. 67 (2.3)] introduced a generating relation involving product of three Laguerre polynomials in the form

$$exp\left[-\left(u+v-\frac{wv}{u}\right)x\right](1+u)^{\alpha}(1+v)^{\beta}\left(1-\frac{wv}{u}\right)^{\gamma}$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} L_{(m+r)}^{\alpha-(m+r)}(x) L_{(n-r)}^{\beta-(n-r)}(x) L_r^{(\gamma-r)}(x)(-w)^r \tag{1.10}$$

and

$$exp\left(u+v-\frac{wv}{u}\right) {}_0F_1\left[\begin{matrix} \text{—}; & \\ 1+\alpha; & \end{matrix} -xu\right] {}_0F_1\left[\begin{matrix} \text{—}; & \\ 1+\beta; & \end{matrix} -xv\right] {}_0F_1\left[\begin{matrix} \text{—}; & \\ 1+\gamma; & \end{matrix} -\frac{xwv}{u}\right]$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} \frac{L_{(m+r)}^{(\alpha)}(x) L_{(n-r)}^{(\beta)}(x) L_r^{(\gamma)}(x)(-w)^r}{(1+\alpha)_{m+r}(1+\beta)_{n-r}(1+\gamma)_r} \tag{1.11}$$

where $m^* = max(0,-m)$, so that all factorials of negative integers have meaning and $L_n^{(\alpha)}(x)$ is known as generalized Laguerre or Sonine polynomial, defined as [6; p. 200 (1)]

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n; 1+\alpha; x)$$

Equation (1.10) and (1.11) is infact generalization of number of results due to Feldheim [3].

Our work is motivated by a number of results of Exton [2], Feldhein [3], Pathan and Kamarujjama [5] and Khan [4] on Lagurre polynomials.

The object of the present paper is to derive a general expansion for the product of Jacobi polynomial using series rearrangement technique.

2 Generating Relation for the Product of Jacobi Polynomials

In this paper we derive a generating relation involving the product of three Jacobi polynomials which generalize many known result of Feldheim [3] and Exton [2]. To obtain our main results, consider the product

$$S(u,v,w) = exp\left[-\left(u+v-\frac{wv}{u}\right)\frac{1}{2}(1+x)\right]\ {}_1F_1[-\beta_1;-\alpha_1-\beta_1;u]$$
$$\times {}_1F_1[-\beta_2;-\alpha_2-\beta_2;v]\ {}_1F_1\left[-\beta_3;-\alpha_3-\beta_3;\frac{wv}{u}\right] \tag{2.1}$$

Now expanding the right hand member of (2.1) as a multiple series with the help of (1.1), we get

$$S(u,v,w) =$$
$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{P_s^{(\alpha_1-s,\beta_1-s)}(x)\ P_k^{(\alpha_2-k,\beta_2-k)}(x)\ P_r^{(\alpha_3-r,\beta_3-r)}(x)}{(-\alpha_1-\beta_1)_s\ (-\alpha_2-\beta_2)_k\ (-\alpha_3-\beta_3)_r}u^{s-r}v^{k+r}(-w)^r \tag{2.2}$$

Replacing s-r and k+r by m and n respectively then after rearrangement justified by the absolute convergence of the above series, it follows that

$$S(u,v,w) = \sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty}u^m v^n$$
$$\times\sum_{r=0}^{\infty}\frac{P_{m+r}^{(\alpha_1-(m+r),\beta_1-(m+r))}(x)\ P_{n-r}^{(\alpha_2-(n-r),\beta_2-(n-r))}(x)\ P_r^{(\alpha_3-r,\beta_3-r)}(x)(-w)^r}{(-\alpha_1-\beta_1)_{m+r}\ (-\alpha_2-\beta_2)_{n-r}\ (-\alpha_3-\beta_3)_r} \tag{2.3}$$

where $m^* = max(0,-m)$, so that all factorial of negative integers have meaning.

3 Special Cases

Equation (2.3) gives many generating functions for well known polynomials. We are presenting only some interesting special cases here.

(i) Setting $x = 1$ in (2.3) and using the result [6, p. 254 (1)], we get

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!} \tag{3.1}$$

$$exp\left[-\left(u+v-\frac{wv}{u}\right)\right]\ {}_1F_1[-\beta_1;-\alpha_1-\beta_1;u]{}_1F_1[-\beta_2;-\alpha_2-\beta_2;v]\ {}_1F_1\left[-\beta_3;-\alpha_3-\beta_3;\frac{wv}{u}\right]$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m\ v^n}{m!\ n!}\frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(1+\alpha_1-m)\Gamma(1+\alpha_2-n)}$$
$$\times_4 F_4\left[\begin{array}{c}-n,\ m-\alpha_1,\ 1+\alpha_2+\beta_2-n,\ -\alpha_3;\\ \\ 1+m,\ m-\alpha_1-\beta_1,\ 1+\alpha_2-n,\ \alpha_3-\beta_3;\end{array} w\right] \tag{3.2}$$

(ii) Setting $x = -1$ in (2.3) and using the result [6, p. 257], we get

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(1+\beta)_n}{n!} \tag{3.3}$$

$${}_1F_1[-\beta_1; -\alpha_1-\beta_1; u]\,{}_1F_1[-\beta_2; -\alpha_2-\beta_2; v]\,{}_1F_1\left[-\beta_3; -\alpha_3-\beta_3; \frac{wv}{u}\right]$$

$$= (-1)^{m+n}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m\, v^n}{m!\, n!}\frac{\Gamma(1+\beta_1)\Gamma(1+\beta_2)}{\Gamma(1+\beta_1-m)\Gamma(1+\beta_2-n)(-\alpha_1-\beta_1)_m(-\alpha_2-\beta_2)_n}$$
$$\times {}_4F_4\left[\begin{array}{c} -n,\ m-\beta_1,\ 1+\alpha_2+\beta_2-n,\ -\beta_3; \\ \\ 1+m,\ m-\alpha_1-\beta_1,\ 1+\beta_2-n,\ \alpha_3-\beta_3; \end{array} w\right] \tag{3.4}$$

(iii) In view of the relation [7; p. 441 (16)]

$$P_n^{(\alpha-n,\beta-n)}(x) = g_n^{-\alpha,-\beta}\left(-\frac{x+1}{2}, -\frac{x-1}{2}\right) \tag{3.5}$$

equation (2.3) yields an interesting result

$$exp\left[-\left(u+v-\frac{wv}{u}\right)\frac{1}{2}(1+x)\right]\ {}_1F_1[-\beta_1; -\alpha_1-\beta_1; u]\,{}_1F_1[-\beta_2; -\alpha_2-\beta_2; v]$$
$${}_1F_1\left[-\beta_3; -\alpha_3-\beta_3; \frac{wv}{u}\right] = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty}\frac{g_{m+r}^{(-\alpha_1,-\beta_1)}\left(-\frac{x+1}{2}, -\frac{x-1}{2}\right)}{(-\alpha_1-\beta_1)_{m+r}}$$
$$\frac{g_{n-r}^{(-\alpha_2,-\beta_2)}\left(-\frac{x+1}{2}, -\frac{x-1}{2}\right)}{(-\alpha_2-\beta_2)_{n-r}}\frac{g_r^{(-\alpha_3,-\beta_3)}\left(-\frac{x+1}{2}, -\frac{x-1}{2}\right)}{(-\alpha_3-\beta_3)_r} \tag{3.6}$$

where $g_n^{(\alpha,\beta)}$ is Lagrange's polynomial [1; p. 267 see also 7; p. 85 (25)].

(iv) On replacing u, v and w by ut, vt and wt respectively, multiply both side by $t^{\gamma-1}$ in equation (2.3) and then taking Laplace transform, we get

$$z^{-\gamma}\ F_A^{(3)}\left[\gamma, -\beta_1, -\beta_2, -\beta_3; -\alpha_1-\beta_1, -\alpha_2-\beta_2, -\alpha_3-\beta_3; \frac{u}{z}, \frac{v}{z}, \frac{wv}{z}\right]$$
$$= \sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n (\gamma)_{m+n}\sum_{r=0}^{\infty}\frac{(\gamma+m+n)_r(-w)^r}{(-\alpha_1-\beta_1)_{m+r}\,(-\alpha_2-\beta_2)_{n-r}\,(1+\alpha_3)_r}$$
$$\times P_{m+r}^{(\alpha_1-(m+r),\beta_1-(m+r))}(x)\ P_{n-r}^{(\alpha_2-(n-r),\beta_2-(n-r))}(x)\ P_r^{(\alpha_3-r,\beta_3-r)}(x) \tag{3.7}$$

where

$$z = \left[\left(u+v-\frac{wv}{u}\right)\left(\frac{1+x}{2}\right)-1\right]$$

and $F_A^{(n)}$ is Lauricella function defined by [7; p. 60 (1)]

$$F_A^{(n)}[a, b_1, b_2, \cdots, b_n; c_1, c_2, \cdots, c_n; x_1, \cdots, x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \tag{3.8}$$

$$| x_1 | + \cdots + | x_n | < 1 .$$

(v) If we set $w = 0$, equation (2.3) becomes

$$exp\left[-(u+v)\frac{1}{2}(1+x)\right]\ {}_1F_1[-\beta_1; -\alpha_1-\beta_1; u]{}_1F_1[-\beta_2; -\alpha_2-\beta_2; v]$$
$$= \sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m v^n}{(-\alpha_1-\beta_1)_m\,(-\alpha_2-\beta_2)_n}P_m^{(\alpha_1-m,\beta_1-m)}(x)\ P_n^{(\alpha_2-n,\beta_2-n)}(x) \tag{3.9}$$

(vi) For $x = 0$, equation (3.9) reduces to

$$exp\left[-\frac{1}{2}(u+v)\right]\ {}_1F_1[-\beta_1; -\alpha_1-\beta_1; u]{}_1F_1[-\beta_2; -\alpha_2-\beta_2; v]$$
$$= \sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m v^n}{m!\,n!}\,{}_2F_1\left[\begin{matrix}-n, & -\alpha_1\ ; \\ & \\ -\alpha_1-\beta_1; & \end{matrix}\ \frac{1}{2}\right]{}_2F_1\left[\begin{matrix}-n, & -\alpha_2\ ; \\ & \\ -\alpha_2-\beta_2; & \end{matrix}\ \frac{1}{2}\right] \tag{3.10}$$

(vii) On replacing u and v by ut and vt, multiplying both side by $t^{\gamma-1}$, and then taking Laplace transform, in equation (3.2), (3.4) and (3.10), we get

$$F_A^{(3)}\left[\gamma, -\beta_1, -\beta_2, -\beta_3; -\alpha_1-\beta_1, -\alpha_2-\beta_2, -\alpha_3-\beta_3; \frac{u}{z}, \frac{v}{z}, \frac{wv}{z}\right]$$
$$= \sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}(\gamma)_{m+n}\frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{(-\alpha_1-\beta_1)_m\,(-\alpha_2-\beta_2)_n\,\Gamma(1+\alpha_1-m)\Gamma(1+\alpha_2-n)}$$
$$\times{}_5F_4\left[\begin{matrix}-n,\ m-\alpha_1,\ 1+\alpha_2+\beta_2-n,\ -\alpha_3, \gamma+m+n; \\ \\ 1+m,\ m-\alpha_1-\beta_1,\ 1+\alpha_2-n,\ \alpha_3-\beta_3;\end{matrix}\ w\right] \tag{3.11}$$

and

$$F_A^{(3)}\left[\gamma, -\beta_1, -\beta_2, -\beta_3; -\alpha_1-\beta_1, -\alpha_2-\beta_2, -\alpha_3-\beta_3; \frac{u}{z}, \frac{v}{z}, \frac{wv}{z}\right]$$
$$= \sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty}(-1)^{m+n}\frac{u^m}{m!}\frac{v^n}{n!}(\gamma)_{m+n}\frac{\Gamma(1+\beta_1)\Gamma(1+\beta_2)}{(-\alpha_1-\beta_1)_m\,(-\alpha_2-\beta_2)_n\,\Gamma(1+\beta_1-m)\Gamma(1+\beta_2-n)}$$
$$\times{}_5F_4\left[\begin{matrix}-n,\ m-\beta_1,\ 1+\alpha_2+\beta_2-n,\ -\beta_3, \gamma+m+n; \\ \\ 1+m,\ m-\alpha_1-\beta_1,\ 1+\beta_2-n,\ \alpha_3-\beta_3;\end{matrix}\ w\right] \tag{3.12}$$

and

$$\left(\frac{u}{2}+\frac{v}{2}-1\right)^{-\gamma} F_2\left[\gamma,-\beta_1,-\beta_2;-\alpha_1-\beta_1,-\alpha_2-\beta_2;u,v\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m v^n}{m!\,n!}\,{}_2F_1\left[\begin{array}{ll}-m,\quad -\alpha_1\,; & \\ & \frac{1}{2}\\ \quad -\alpha_1-\beta_1; & \end{array}\right]{}_2F_1\left[\begin{array}{ll}-n,\quad -\alpha_2\,; & \\ & \frac{1}{2}\\ \quad -\alpha_2-\beta_2; & \end{array}\right] \tag{3.13}$$

where F_2 is Appell function of two variables is defined by [7; p. 53 (5)]

$$F_2\left[a,b,b';c,c';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\,\frac{x^m y^n}{m!n!} \tag{3.14}$$

$$|x|+|y|<1$$

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Received: November 28, 2015.
Accepted: May 17, 2016.