

ON CERTAIN MIXED GENERATING FUNCTIONS OF JACOBI POLYNOMIALS

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Abstract. This paper deals with general expansions which gives as special cases involving Jacobi and Laguerre polynomials, Lauricella, Appell and generalized Gauss function. The results unify and extended Exton’s generating function [2] and Feldheim’s expansion [3]. Also of interest are mixed generating functions which are partly unilateral and partly bilateral.

1 Introduction

Now consider a generating function [7, p. 106 (11)].

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) \frac{t^n}{(-\alpha - \beta)_n} = \exp\left[-\frac{1}{2}(x+1)t\right] {}_1F_1\left[\begin{matrix} -\beta; \\ -\alpha - \beta; \end{matrix} t\right] \tag{1.1}$$

where $P_n^{(\alpha, \beta)}$ is a Jacobi polynomial [1; p. 170 (16)]

$$P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} {}_2F_1\left[\begin{matrix} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha; \end{matrix} -\frac{1}{2}(1 - x)\right] \tag{1.2}$$

or equivalently,

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \frac{(1 + \alpha)_n (1 + \alpha + \beta)_{n+k}}{k! (n - k)! (1 + \alpha)_k (1 + \alpha + \beta)_n} \left(\frac{x - 1}{x}\right)^k. \tag{1.3}$$

where ${}_pF_q$ denote generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters defined by [7; p. 42 (1)].

$${}_pF_q\left[\begin{matrix} (\alpha_1), (\alpha_2), \dots, (\alpha_p); \\ (\beta_1), (\beta_2), \dots, (\beta_q); \end{matrix} x\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n x^n}{(\beta_1)_n \dots (\beta_q)_n n!} = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; x) \tag{1.4}$$

where $(a)_n$ is the Pochhammer symbol, defined by

$$(a)_n = \begin{cases} 1 & \text{if } n = 0 \\ a(a + 1)(a + 2) \dots (a + n - 1) & \text{if } n = 1, 2, 3, \dots \end{cases} \tag{1.5}$$

the denominator parameters are neither zero nor a negative integers the numerator parameters may be zero and negative integers. We list below a number of important polynomials which can be expressed in terms of Jacobi polynomial for different values of α and β .

$$P_n^{(\mu-\frac{1}{2}, \mu-\frac{1}{2})}(z) = \frac{(\mu + \frac{1}{2})_n}{(2\mu)_n} C_n^\mu(z), \tag{1.6}$$

where $C_n^\mu(z)$ is the Gegenbauer polynomial (see [6],[7]),

$$P_n^{(-\frac{1}{2}, -\frac{1}{2})}(z) = \frac{(\frac{1}{2})_n}{n!} T_n(z), \tag{1.7}$$

$$P_n^{(\frac{1}{2}, \frac{1}{2})}(z) = \frac{(\frac{3}{2})_n}{(n+1)!} U_n(z), \tag{1.8}$$

where $T_n(z)$ and $U_n(z)$ are the Tchebicheff polynomial of first and second kind, (see [6],[7]) and

$$P_n^{(0,0)}(z) = P_n(z), \tag{1.9}$$

where $P_n(z)$ is the Legendre polynomial (see [6],[7]).

Jacobi polynomial is an important class of orthogonal polynomial which is a generalization of ultraspherical polynomials. This class contains many special functions commonly encountered in the applications, e.g. Legendre, Gegenbauer, Tchebcheff, Laguerre and Bessel polynomials.

Pathan and Kamarujjama [5; p. 2 (2.3)] and Khan [4; p. 67 (2.3)] introduced a generating relation involving product of three Laguerre polynomials in the form

$$\begin{aligned} & \exp \left[- \left(u + v - \frac{wv}{u} \right) x \right] (1 + u)^\alpha (1 + v)^\beta \left(1 - \frac{wv}{u} \right)^\gamma \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} L_{(m+r)}^{\alpha-(m+r)}(x) L_{(n-r)}^{\beta-(n-r)}(x) L_r^{\gamma-r}(x) (-w)^r \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} & \exp \left(u + v - \frac{wv}{u} \right) {}_0F_1 \left[\begin{matrix} -; \\ 1 + \alpha; \end{matrix} \middle| -xu \right] {}_0F_1 \left[\begin{matrix} -; \\ 1 + \beta; \end{matrix} \middle| -xv \right] {}_0F_1 \left[\begin{matrix} -; \\ 1 + \gamma; \end{matrix} \middle| -\frac{xwv}{u} \right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} \frac{L_{(m+r)}^{(\alpha)}(x) L_{(n-r)}^{(\beta)}(x) L_r^{(\gamma)}(x) (-w)^r}{(1 + \alpha)_{m+r} (1 + \beta)_{n-r} (1 + \gamma)_r} \end{aligned} \tag{1.11}$$

where $m^* = \max(0, -m)$, so that all factorials of negative integers have meaning and $L_n^{(\alpha)}(x)$ is known as generalized Laguerre or Sonine polynomial, defined as [6; p. 200 (1)]

$$L_n^{(\alpha)}(x) = \frac{(1 + \alpha)_n}{n!} {}_1F_1(-n; 1 + \alpha; x)$$

Equation (1.10) and (1.11) is infact generalization of number of results due to Feldheim [3].

Our work is motivated by a number of results of Exton [2], Feldheim [3], Pathan and Kamarujjama [5] and Khan [4] on Lagurre polynomials.

The object of the present paper is to derive a general expansion for the product of Jacobi polynomial using series rearrangement technique.

2 Generating Relation for the Product of Jacobi Polynomials

In this paper we derive a generating relation involving the product of three Jacobi polynomials which generalize many known result of Feldheim [3] and Exton [2]. To obtain our main results, consider the product

$$S(u, v, w) = \exp \left[- \left(u + v - \frac{wv}{u} \right) \frac{1}{2} (1 + x) \right] {}_1F_1[-\beta_1; -\alpha_1 - \beta_1; u] \times {}_1F_1[-\beta_2; -\alpha_2 - \beta_2; v] {}_1F_1 \left[-\beta_3; -\alpha_3 - \beta_3; \frac{wv}{u} \right] \tag{2.1}$$

Now expanding the right hand member of (2.1) as a multiple series with the help of (1.1), we get

$$S(u, v, w) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{P_s^{(\alpha_1-s, \beta_1-s)}(x) P_k^{(\alpha_2-k, \beta_2-k)}(x) P_r^{(\alpha_3-r, \beta_3-r)}(x)}{(-\alpha_1 - \beta_1)_s (-\alpha_2 - \beta_2)_k (-\alpha_3 - \beta_3)_r} u^{s-r} v^{k+r} (-w)^r \tag{2.2}$$

Replacing s-r and k+r by m and n respectively then after rearrangement justified by the absolute convergence of the above series, it follows that

$$S(u, v, w) = \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} u^m v^n \times \sum_{r=0}^{\infty} \frac{P_{m+r}^{(\alpha_1-(m+r), \beta_1-(m+r))}(x) P_{n-r}^{(\alpha_2-(n-r), \beta_2-(n-r))}(x) P_r^{(\alpha_3-r, \beta_3-r)}(x) (-w)^r}{(-\alpha_1 - \beta_1)_{m+r} (-\alpha_2 - \beta_2)_{n-r} (-\alpha_3 - \beta_3)_r} \tag{2.3}$$

where $m^* = \max(0, -m)$, so that all factorial of negative integers have meaning.

3 Special Cases

Equation (2.3) gives many generating functions for well known polynomials. We are presenting only some interesting special cases here.

(i) Setting $x = 1$ in (2.3) and using the result [6, p. 254 (1)], we get

$$P_n^{(\alpha, \beta)}(1) = \frac{(1 + \alpha)_n}{n!} \tag{3.1}$$

$$\begin{aligned} & \exp \left[- \left(u + v - \frac{wv}{u} \right) \right] {}_1F_1[-\beta_1; -\alpha_1 - \beta_1; u] {}_1F_1[-\beta_2; -\alpha_2 - \beta_2; v] {}_1F_1 \left[-\beta_3; -\alpha_3 - \beta_3; \frac{wv}{u} \right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{u^m v^n}{m! n!} \frac{\Gamma(1 + \alpha_1) \Gamma(1 + \alpha_2)}{\Gamma(1 + \alpha_1 - m) \Gamma(1 + \alpha_2 - n)} \\ & \quad \times {}_4F_4 \left[\begin{matrix} -n, m - \alpha_1, 1 + \alpha_2 + \beta_2 - n, -\alpha_3; \\ 1 + m, m - \alpha_1 - \beta_1, 1 + \alpha_2 - n, \alpha_3 - \beta_3; \end{matrix} \right. \left. w \right] \tag{3.2} \end{aligned}$$

(ii) Setting $x = -1$ in (2.3) and using the result [6, p. 257], we get

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(1 + \beta)_n}{n!} \tag{3.3}$$

$$\begin{aligned} & {}_1F_1[-\beta_1; -\alpha_1 - \beta_1; u] {}_1F_1[-\beta_2; -\alpha_2 - \beta_2; v] {}_1F_1\left[-\beta_3; -\alpha_3 - \beta_3; \frac{wv}{u}\right] \\ &= (-1)^{m+n} \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{u^m v^n}{m! n!} \frac{\Gamma(1 + \beta_1)\Gamma(1 + \beta_2)}{\Gamma(1 + \beta_1 - m)\Gamma(1 + \beta_2 - n)(-\alpha_1 - \beta_1)_m(-\alpha_2 - \beta_2)_n} \\ & \quad \times {}_4F_4 \left[\begin{matrix} -n, m - \beta_1, 1 + \alpha_2 + \beta_2 - n, -\beta_3; \\ 1 + m, m - \alpha_1 - \beta_1, 1 + \beta_2 - n, \alpha_3 - \beta_3; \end{matrix} \right] w \end{aligned} \tag{3.4}$$

(iii) In view of the relation [7; p. 441 (16)]

$$P_n^{(\alpha-n,\beta-n)}(x) = g_n^{-\alpha,-\beta} \left(-\frac{x+1}{2}, -\frac{x-1}{2} \right) \tag{3.5}$$

equation (2.3) yields an interesting result

$$\begin{aligned} & \exp \left[-\left(u + v - \frac{wv}{u}\right) \frac{1}{2}(1 + x) \right] {}_1F_1[-\beta_1; -\alpha_1 - \beta_1; u] {}_1F_1[-\beta_2; -\alpha_2 - \beta_2; v] \\ & {}_1F_1\left[-\beta_3; -\alpha_3 - \beta_3; \frac{wv}{u}\right] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} \frac{g_{m+r}^{(-\alpha_1,-\beta_1)} \left(-\frac{x+1}{2}, -\frac{x-1}{2} \right)}{(-\alpha_1 - \beta_1)_{m+r}} \\ & \quad \frac{g_{n-r}^{(-\alpha_2,-\beta_2)} \left(-\frac{x+1}{2}, -\frac{x-1}{2} \right)}{(-\alpha_2 - \beta_2)_{n-r}} \frac{g_r^{(-\alpha_3,-\beta_3)} \left(-\frac{x+1}{2}, -\frac{x-1}{2} \right)}{(-\alpha_3 - \beta_3)_r} \end{aligned} \tag{3.6}$$

where $g_n^{(\alpha,\beta)}$ is Lagrange’s polynomial [1; p. 267 see also 7; p. 85 (25)].

(iv) On replacing u, v and w by ut, vt and wt respectively, multiply both side by $t^{\gamma-1}$ in equation (2.3) and then taking Laplace transform, we get

$$\begin{aligned} & z^{-\gamma} F_A^{(3)} \left[\gamma, -\beta_1, -\beta_2, -\beta_3; -\alpha_1 - \beta_1, -\alpha_2 - \beta_2, -\alpha_3 - \beta_3; \frac{u}{z}, \frac{v}{z}, \frac{wv}{z} \right] \\ &= \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} u^m v^n (\gamma)_{m+n} \sum_{r=0}^{\infty} \frac{(\gamma + m + n)_r (-w)^r}{(-\alpha_1 - \beta_1)_{m+r} (-\alpha_2 - \beta_2)_{n-r} (1 + \alpha_3)_r} \\ & \quad \times P_{m+r}^{(\alpha_1-(m+r),\beta_1-(m+r))}(x) P_{n-r}^{(\alpha_2-(n-r),\beta_2-(n-r))}(x) P_r^{(\alpha_3-r,\beta_3-r)}(x) \end{aligned} \tag{3.7}$$

where

$$z = \left[\left(u + v - \frac{wv}{u}\right) \left(\frac{1+x}{2}\right) - 1 \right]$$

and $F_A^{(n)}$ is Lauricella function defined by [7; p. 60 (1)]

$$\begin{aligned} & F_A^{(n)}[a, b_1, b_2, \dots, b_n; c_1, c_2, \dots, c_n; x_1, \dots, x_n] \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n} x_1^{m_1} \dots x_n^{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n} m_1! \dots m_n!}, \end{aligned} \tag{3.8}$$

$$|x_1| + \dots + |x_n| < 1.$$

(v) If we set $w = 0$, equation (2.3) becomes

$$\begin{aligned} & \exp \left[-(u + v) \frac{1}{2}(1 + x) \right] {}_1F_1[-\beta_1; -\alpha_1 - \beta_1; u] {}_1F_1[-\beta_2; -\alpha_2 - \beta_2; v] \\ &= \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} \frac{u^m v^n}{(-\alpha_1 - \beta_1)_m (-\alpha_2 - \beta_2)_n} P_m^{(\alpha_1 - m, \beta_1 - m)}(x) P_n^{(\alpha_2 - n, \beta_2 - n)}(x) \end{aligned} \tag{3.9}$$

(vi) For $x = 0$, equation (3.9) reduces to

$$\begin{aligned} & \exp \left[-\frac{1}{2}(u + v) \right] {}_1F_1[-\beta_1; -\alpha_1 - \beta_1; u] {}_1F_1[-\beta_2; -\alpha_2 - \beta_2; v] \\ &= \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} \frac{u^m v^n}{m! n!} {}_2F_1 \left[\begin{matrix} -n, -\alpha_1; \\ -\alpha_1 - \beta_1; \end{matrix} \quad \frac{1}{2} \right] {}_2F_1 \left[\begin{matrix} -n, -\alpha_2; \\ -\alpha_2 - \beta_2; \end{matrix} \quad \frac{1}{2} \right] \end{aligned} \tag{3.10}$$

(vii) On replacing u and v by ut and vt , multiplying both side by $t^{\gamma-1}$, and then taking Laplace transform, in equation (3.2), (3.4) and (3.10), we get

$$\begin{aligned} & F_A^{(3)} \left[\gamma, -\beta_1, -\beta_2, -\beta_3; -\alpha_1 - \beta_1, -\alpha_2 - \beta_2, -\alpha_3 - \beta_3; \frac{u}{z}, \frac{v}{z}, \frac{wv}{z} \right] \\ &= \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} \frac{u^m v^n}{m! n!} (\gamma)_{m+n} \frac{\Gamma(1 + \alpha_1)\Gamma(1 + \alpha_2)}{(-\alpha_1 - \beta_1)_m (-\alpha_2 - \beta_2)_n \Gamma(1 + \alpha_1 - m)\Gamma(1 + \alpha_2 - n)} \\ & \quad \times {}_5F_4 \left[\begin{matrix} -n, m - \alpha_1, 1 + \alpha_2 + \beta_2 - n, -\alpha_3, \gamma + m + n; \\ 1 + m, m - \alpha_1 - \beta_1, 1 + \alpha_2 - n, \alpha_3 - \beta_3; \end{matrix} \quad w \right] \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & F_A^{(3)} \left[\gamma, -\beta_1, -\beta_2, -\beta_3; -\alpha_1 - \beta_1, -\alpha_2 - \beta_2, -\alpha_3 - \beta_3; \frac{u}{z}, \frac{v}{z}, \frac{wv}{z} \right] \\ &= \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} (-1)^{m+n} \frac{u^m v^n}{m! n!} (\gamma)_{m+n} \frac{\Gamma(1 + \beta_1)\Gamma(1 + \beta_2)}{(-\alpha_1 - \beta_1)_m (-\alpha_2 - \beta_2)_n \Gamma(1 + \beta_1 - m)\Gamma(1 + \beta_2 - n)} \\ & \quad \times {}_5F_4 \left[\begin{matrix} -n, m - \beta_1, 1 + \alpha_2 + \beta_2 - n, -\beta_3, \gamma + m + n; \\ 1 + m, m - \alpha_1 - \beta_1, 1 + \beta_2 - n, \alpha_3 - \beta_3; \end{matrix} \quad w \right] \end{aligned} \tag{3.12}$$

and

$$\left(\frac{u}{2} + \frac{v}{2} - 1\right)^{-\gamma} F_2[\gamma, -\beta_1, -\beta_2; -\alpha_1 - \beta_1, -\alpha_2 - \beta_2; u, v]$$

$$= \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} \frac{u^m v^n}{m! n!} {}_2F_1 \left[\begin{matrix} -m, & -\alpha_1; \\ & -\alpha_1 - \beta_1; \end{matrix} \quad \frac{1}{2} \right] {}_2F_1 \left[\begin{matrix} -n, & -\alpha_2; \\ & -\alpha_2 - \beta_2; \end{matrix} \quad \frac{1}{2} \right] \quad (3.13)$$

where F_2 is Appell function of two variables is defined by [7; p. 53 (5)]

$$F_2[a, b, b'; c, c'; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m y^n}{m! n!} \quad (3.14)$$

$$|x| + |y| < 1$$

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