Non-negative Majority Total Domination In Graphs

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Abstract. A two-valued function \( f : V \rightarrow \{-1,+1\} \) defined on the vertices of a graph \( G = (V,E) \), is a non-negative majority total dominating function if the sum of its function values over at least half the open neighbourhood is at least zero. That is, for at least half of the vertices \( v \in V \), \( f(N(v)) \geq 0 \), where \( N(v) \) consists of every vertex adjacent to \( v \). The non-negative majority total domination number of a graph \( G \), denoted \( \gamma^N_{\text{maj}}(G) \), is the minimum value of \( \sum_{v \in V(G)} f(v) \) over all non-negative majority total dominating functions \( f \) of \( G \). In this paper, we initialize the study of non-negative majority total domination in graphs.

1 Introduction

By a graph \( G = (V,E) \), we mean a finite, non-trivial, connected, and undirected graph with neither loops nor multiple edges. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1].

The study of domination is one of the fastest growing areas within graph theory. A subset \( D \) of vertices is said to be a dominating set of \( G \) if every vertex in \( V \) either belongs to \( D \) or is adjacent to a vertex in \( D \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). Survey of several advanced topics on domination are given in the book edited by Haynes et al [2].

For a real valued function \( f : V \rightarrow R \) on \( V \), weight of \( f \) is defined to be \( w(f) = \sum_{v \in V} f(v) \) and also for a subset \( S \subseteq V \), we define \( f(S) = \sum_{v \in S} f(v) \). Therefore \( w(f) = f(V) \). Majority domination was first introduced by Broere et al. in [3] and further studied in [4, 5].

A function \( f : V \rightarrow \{-1,+1\} \) is called a signed majority total dominating function if \( f(N(v)) \geq 1 \) for at least half of the vertices in graph \( G \). The signed majority total domination number of \( G \), is denoted by \( \gamma^s_{\text{maj}}(G) \) and is defined as \( \gamma^s_{\text{maj}}(G) = \min \{ w(f) \mid f \text{ is a signed majority total dominating function of } G \} \). Further, the concept of non-negative signed domination of a graph was introduced in [6]. In this paper, we initiate the study of non-negative majority total domination in graphs.

2 Common Classes of Graphs

Definition 2.1. A function \( f : V \rightarrow \{-1,+1\} \) is called a non-negative majority total dominating function (briefly NMTDF) if \( f(N(v)) \geq 0 \) for at least half of the vertices in \( G \). The non-negative majority total domination number of \( G \), denoted by \( \gamma^N_{\text{maj}}(G) \), is defined as \( \gamma^N_{\text{maj}}(G) = \min \{ w(f) \mid f \text{ is a NMTDF of } G \} \).

Let us follow throughout the paper the following terminologies. If \( f \) is a non-negative majority total dominating function of a graph \( G \), then we define the sets \( P_f,M_f \) and \( N_f \) as follows.
(i) \( P_f(G) = \{ v \in V(G) : f(v) = 1 \} \)
(ii) \( M_f(G) = \{ v \in V(G) : f(v) = -1 \} \)
(iii) \( N_f(G) = \{ v \in V(G) : f(N(v)) \geq 0 \} \)

**Theorem 2.2.** For any path \( P_n \) on \( n \geq 2 \) vertices,
\[
\gamma_{maj}^{N} (P_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor - n
\]

**Proof.** Let \( P_n = (v_1, v_2, ..., v_n) \) and let \( f \) be a non-negative majority total dominating function of \( P_n \). Then for any vertex \( v \in N_f \), at least one neighbour of \( v \) belongs to \( P_f \). Since \( |N_f| \geq \left\lfloor \frac{n}{2} \right\rfloor \), we have \( |M_f| \geq \left\lfloor \frac{n}{2} \right\rfloor \) which implies that \( |M_f| \leq n - \left\lfloor \frac{n}{2} \right\rfloor \). Hence \( |P_f| - |M_f| \geq 2 \left\lfloor \frac{n}{2} \right\rfloor - n \).

On the other hand, define the function \( g : V \rightarrow \{-1, +1\} \) by
\[
g(v_i) = \begin{cases} +1 & \text{if } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil \text{ and } i \equiv 2 \text{(mod 3)} \\ -1 & \text{otherwise} \end{cases}
\]

Then we can verify that \( g(N(v)) \geq 0 \) for at least half of the vertices in \( G \) with weight \( 2 \left\lfloor \frac{n}{2} \right\rfloor - n \). Hence \( \gamma_{maj}^{N} (P_n) \leq w(f) = 2 \left\lfloor \frac{n}{2} \right\rfloor - n \). Consequently, the result follows. \( \square \)

**Corollary 2.3.** For any negative integer \( k \), there exists a graph \( G \) for which \( \gamma_{maj}^{N} (G) = k \).

**Theorem 2.4.** For \( n \geq 3 \), an integer \( \gamma_{maj}^{N} (C_n) = \gamma_{maj}^{N} (P_n) \).

**Proof.** Let \( C_n = (v_1, v_2, ..., v_n) \) be the cycle on \( n \) vertices. Then \( C_n - v_1 v_n \) is a path on \( n \) vertices and also the function \( g \) defined on \( P_n = C_n - v_1 v_n \) as in Theorem 2.2, would be a non-negative majority total domination for the cycles \( C_n \) so that \( \gamma_{maj}^{N} (C_n) \leq \gamma_{maj}^{N} (P_n) \). We now show that \( \gamma_{maj}^{N} (C_n) \geq \gamma_{maj}^{N} (P_n) \). Let \( f \) be a minimum non-negative majority total domination of \( C_n \). For \( n \geq 3 \), by Theorem 2.2, \( \gamma_{maj}^{N} (P_n) < 0 \). Therefore, \( |P_f| - |M_f| = f(V) = \gamma_{maj}^{N} (C_n) \leq \gamma_{maj}^{N} (P_n) \) which in turn implies that \( |M_f| > |P_f| \). This means that \( M_f \) must contain two adjacent vertices \( v_i, v_j \). Consider now the path \( P \) on \( n \) vertices obtained from \( C_n \) by removing the edge \( v_i v_j \). The number of non-negative open neighborhood sums under \( f \) on \( P \) is the same as that of \( f \) on \( C_n \). It follows that \( f \) is a non-negative majority total dominating function of \( P \) and hence \( \gamma_{maj}^{N} (P_n) = \gamma_{maj}^{N} (P) \leq f(V) = \gamma_{maj}^{N} (C_n) \). \( \square \)

**Theorem 2.5.** For any complete graph \( K_n (n \geq 2) \), we have
\[
\gamma_{maj}^{N} (K_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** Let \( f \) be a non-negative majority total dominating function of \( K_n \). Then \( |P_f| + |M_f| = n \) and \( |P_f| - |M_f| = f(V) \). Now, consider a vertex \( v \) of \( K_n \) with \( f(N(v)) \geq 0 \). Certainly, \( f(V) = f(N(v)) + f(v) \geq 0 \) which means that \( |P_f| - |M_f| \geq -1 \). It follows that \( |P_f| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( |M_f| \leq \left\lfloor \frac{n-1}{2} \right\rfloor \). Thus \( \gamma_{maj}^{N} (K_n) \geq \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n+1}{2} \right\rfloor \). That is, \( \gamma_{maj}^{N} (K_n) \geq \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \)

Now, consider the function \( g : V \rightarrow \{-1, +1\} \) that assigns the value -1 for \( \left\lfloor \frac{n}{2} \right\rfloor \) vertices of \( K_n \) and the value +1 for the remaining vertices. Obviously, \( g \) is a non-negative majority total dominating function of \( K_n \), so that \( \gamma_{maj}^{N} (K_n) \leq n - 2 \left\lfloor \frac{n}{2} \right\rfloor \).

That is, \( \gamma_{maj}^{N} (K_n) \leq \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \)

**Theorem 2.6.** For any complete bipartite graph \( K_{r,s} (s \geq r \geq 1) \)
\[
\gamma_{maj}^{N} (K_{r,s}) = \begin{cases} -s & \text{if } r \text{ is even} \\ 1 - s & \text{if } r \text{ is odd} \end{cases}
\]
Proof. Let \((U, W)\) be the bipartition of \(K_{r,s}\) with \(|U| = r\) and \(|W| = s\). Let \(f\) be a minimum non-negative majority total dominating function of \(K_{r,s}\). Then \(W\) contains a vertex \(x\) with \(f(N(x)) \geq 0\) when \(r < s\). Certainly, when \(r = s\), either \(U\) or \(W\) contains such a vertex \(x\). Without loss of generality assume that \(W\) contains such a vertex \(x\). This implies that \(f(U) \geq 0\). If \(U^+\) and \(U^-\) denote the set of vertices that are assigned with +1 and -1 respectively, then \(f(U) = |U^+| - |U^-|\) so that \(|U^+| - |U^-| \geq 0\). Obviously, \(|U^+| + |U^-| = r\). Using these, we get \(|U^+| \geq \left\lceil \frac{r}{2} \right\rceil\) and \(|U^-| \leq \left\lfloor \frac{r}{2} \right\rfloor\) and consequently \(f(U) \geq \left\lceil \frac{r}{2} \right\rceil - \left\lfloor \frac{r}{2} \right\rfloor\).

We now claim that every vertex of \(W\) receives the value -1 under \(f\). If not, there exists a vertex \(w \in W\) with \(f(w) = +1\). Now the function \(g : V(K_{r,s}) \to \{-1, +1\}\) obtained from \(f\) by replacing \(f(w)\) by -1, is a non-negative majority total dominating function with \(w(g) = w(f) - 2\), which is a contradiction to the minimality of \(f\). Hence every vertex of \(W\) receives -1 under \(f\) so that \(f(W) = -s\). Thus \(f(V) = f(U) + f(W) \geq \left\lceil \frac{r}{2} \right\rceil - \left\lfloor \frac{r}{2} \right\rfloor - s\). That is,

\[
\gamma_{maj}^{Nt}(K_{r,s}) \geq \begin{cases} 
-s & \text{if } r \text{ is even} \\
1 - s & \text{if } r \text{ is odd}
\end{cases}
\]

Now, the function that assigns the value +1 to \(\left\lceil \frac{r}{2} \right\rceil\) vertices of \(U\) and the value -1 for the remaining vertices of \(K_{r,s}\) is a non-negative majority total dominating function of \(K_{r,s}\) with weight \(\left\lceil \frac{r}{2} \right\rceil - \left\lfloor \frac{r}{2} \right\rfloor - s\). This proves the result. \(\square\)

3 Bounds

In this section, we discuss some bounds for the non-negative majority total domination.

Theorem 3.1. A NMTDF for a graph \(G\) is minimal only if for every vertex \(v \in V\) with \(f(v) = 1\), there exists a vertex \(w \in N(v)\) with \(f(N(u)) \in \{0, 1\}\).

Proof. Let \(f\) be a minimal NMTDF and assume that there is a vertex \(v\) with \(f(v) = 1\) and \(f(N(u)) \notin \{0, 1\}\) for every vertex \(u \in N(v)\). Now, define a new function \(g : V \to \{-1, +1\}\) by \(g(v) = -1\) and \(g(w) = f(w)\) for all \(w \neq v\). Then for all \(u \in N(v)\), either \(f(N(u)) \leq -1\), in which case \(g(N(u)) = f(N(u)) - 2 \leq -3\) or \(f(N(u)) \geq 2\), in which case \(g(N(u)) = f(N(u)) - 2 \geq 0\). For \(w \notin N(v)\), we have \(g(N(w)) = f(N(w))\). Thus \(|N_g| = |N_f|\) and so \(g\) is an NMTDF on \(G\). Since \(w(g) < w(f)\), the minimality of \(f\) is contradicted. \(\square\)

Theorem 3.2. Let \(G\) be a graph with the degree sequence \((d_1, d_2, ..., d_n)\) such that \(d_1 \leq d_2 \leq \ldots \leq d_n\). Then \(\gamma_{maj}^{Nt}(G) \geq -n + \frac{r}{m} \sum_{j=1}^{m} \left\lceil \frac{d_j}{2} \right\rceil\).

Proof. Let \(g\) be a non-negative majority total dominating function of \(G\). Then \(g(N(v)) \geq 0\) for at least half of the vertices say \(v_{i1}, v_{i2}, ..., v_j \left\lceil \frac{d_j}{2} \right\rceil\) with corresponding degrees \(d_{j1}, d_{j2}, ..., d_j \left\lceil \frac{d_j}{2} \right\rceil\) respectively in \(G\). Let \(f(x) = \frac{(g(x) + 1)}{4}\) for all vertices in \(G\). Then \(f\) is a 0-1 valued function. First,

\[
\sum_{i=1}^{\left\lceil \frac{d_i}{2} \right\rceil} f(N(v_{ij})) = \sum_{i=1}^{\left\lceil \frac{d_i}{2} \right\rceil} g(N(v_{ij})) + d_{ij} \\
\geq \sum_{i=1}^{\left\lceil \frac{d_i}{2} \right\rceil} d_{ij} \\
\geq \sum_{j=1}^{\left\lceil \frac{d_j}{2} \right\rceil} d_j 
\]

On the other hand,

\[
\sum_{i=1}^{\left\lceil \frac{d_i}{2} \right\rceil} f(N(v_{ij})) \leq \sum_{j=1}^{n} f(N(v_j)) \\
= \sum_{j=1}^{n} \deg v_j f(v_j) \\
\leq d_n f(V) 
\]

Therefore, \(f(V) \geq \frac{1}{4n} \sum_{j=1}^{\left\lceil \frac{d_j}{2} \right\rceil} d_j\). Also since \(\gamma_{maj}^{Nt}(G) = g(V) = 2f(V) - n\), we
have $\gamma_{maj}^{N_t}(G) \geq -n + \frac{2}{\Delta} \sum_{j=1}^{\left\lceil \frac{n}{2} \right\rceil} \left\lfloor \frac{d_j}{2} \right\rfloor$. □

Theorem 3.3. If $G$ is a graph of order $n$, then

$$\gamma_{maj}^{N_t}(G) \geq \begin{cases} \frac{n\delta - 2\Delta n}{\Delta + \delta} & \text{if } n \text{ is even} \\ \frac{n\delta + \Delta (1 - 2n)}{\Delta + \delta} & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $f$ be a $\gamma_{maj}^{N_t}(G)$-function on $G$. Let $P_f = P_\Delta \cup P_\delta \cup P_\Theta$ where $P_\Delta$ and $P_\delta$ are sets of all vertices of $P_f$ with degree equal to $\Delta$ and $\delta$, respectively, and $P_\Theta$ contains all other vertices in $P_f$. Let $M_f = M_\Delta \cup M_\delta \cup M_\Theta$ where $M_\Delta$, $M_\delta$ and $M_\Theta$ are defined similarly. Further, for $i \in \{\Delta, \delta, \Theta\}$, let $V_i$ be defined by $V_i = P_i \cup M_i$. Thus, $n = |V_\Delta| + |V_\delta| + |V_\Theta|.$

Since at least half of the vertices $v \in V$, $f(N(v)) \geq 0$, we have

$$\sum_{v \in V} f(N(v)) \geq 0 \left\lceil \frac{n}{2} \right\rceil - \Delta(n - \left\lceil \frac{n}{2} \right\rceil) = \Delta\left(\left\lceil \frac{n}{2} \right\rceil - n\right).$$

The sum $\sum_{v \in V} f(N(v))$ counts the value $f(v)$ exactly $\deg v$ times for each vertex $v \in V$. That is

$$\sum_{v \in V} f(N(v)) = \sum_{v \in V} f(v) \deg v.$$

Thus

$$\sum_{v \in V} f(v) \deg v \geq \Delta\left(\left\lceil \frac{n}{2} \right\rceil - n\right).$$

By splitting the sum into the six summations and replacing $f(v)$ with the corresponding value of $+1$ or $-1$ yields

$$\sum_{v \in P_\Delta} \deg v + \sum_{v \in P_\delta} \deg v + \sum_{v \in P_\Theta} \deg v - \sum_{v \in M_\Delta} \deg v - \sum_{v \in M_\delta} \deg v - \sum_{v \in M_\Theta} \deg v \geq \Delta\left(\left\lceil \frac{n}{2} \right\rceil - n\right).$$

We know that $\deg v = \Delta$ for all $v \in \{P_\Delta, M_\Delta\}$ and $\deg v = \delta$ for all $v \in \{P_\delta, M_\delta\}$. Also, for any vertex $v \in \{P_\Theta, M_\Theta\}$, $\delta + 1 \leq \deg v \leq \Delta - 1$.

Therefore, we have

$$\Delta |P_\Delta| + \delta |P_\delta| + (\Delta - 1) |P_\Theta| - \Delta |M_\Delta| - \delta |M_\delta| - (\delta + 1) |M_\Theta| \geq \Delta\left(\left\lceil \frac{n}{2} \right\rceil - n\right).$$

For $i \in \{\Delta, \delta, \Theta\}$, we replace $|P_i|$ with $|V_i| - |M_i|$ in the above inequality, we have

$$\Delta |V_\Delta| + \delta |V_\delta| + (\Delta - 1) |V_\Theta| \geq \Delta\left(\left\lceil \frac{n}{2} \right\rceil - n\right) + 2\Delta |M_\Delta| + 2\delta |M_\delta| + (\Delta + \delta) |M_\Theta|.$$

It follows that

$$(2n - \left\lceil \frac{n}{2} \right\rceil) \Delta \geq 2\Delta |M_\Delta| + 2\delta |M_\delta| + (\Delta + \delta) |M_\Theta| + (\Delta - \delta) |V_\Theta| + |V_\Theta|$$

$$= 2\Delta |M_\Delta| + 2\delta |M_\delta| + (\Delta + \delta) |M_\Theta| + (\Delta - \delta) |P_\delta| + |M_\delta| + (|P_\delta| + |M_\delta|)$$

$$= 2\Delta |M_\Delta| + (\Delta + \delta) |M_\delta| + (\Delta + \delta + 1) |M_\Theta| + (\Delta - \delta) |P_\delta| + |P_\delta|$$

$$\geq (\Delta + \delta) |M_\Delta| + (\Delta + \delta) |M_\delta| + (\Delta + \delta) |M_\Theta| = (\Delta + \delta) |M_f|.$$ 

Therefore $|M_f| \leq \frac{2n - \left\lceil \frac{n}{2} \right\rceil}{\Delta + \delta}$.

Hence,$\gamma_{maj}^{N_t}(G) = |P_f| - |M_f| = n - 2 |M_f| \geq n - 2 \frac{2n - \left\lceil \frac{n}{2} \right\rceil}{\Delta + \delta}$. □

Theorem 3.4. Let $G$ be a graph of order $n$ and let $k$ be any integer. Then $\gamma_{maj}^{N_t}(G) = k$ if and only if there exists a partition $(P_f, M_f)$ of $V$ for which

(i) $|N(x) \cap P_f| - |N(x) \cap M_f| \geq 0$ for at least half of the vertices of $G$.

(ii) $|P_f| - |M_f| = k$.

(iii) For any $P' \subseteq P_f$ and any $M' \subseteq M_f$ satisfying $|P'| > |M'|$, we have

$$\left(\left\lceil \frac{n}{2} \right\rceil \sum_{x \in V} \left[ 2\left( |N(x) \cap P'| - |N(x) \cap M'| \right) > |N(x) \cap P_f| - |N(x) \cap P'| \right] \right) > n - \left\lceil \frac{n}{2} \right\rceil.$$
Proof. Suppose \( \gamma_{\text{maj}}^{N}(G) = k \). Let \( f \) be a NMTDF of \( G \) such that \( f(V) = \gamma_{\text{maj}}^{N}(G) = k \). Then \( (P_f, M_f) \) constitutes a partition of \( V \). For each \( x \in N_f \), \( |N(x) \cap P_f| - |N(x) \cap M_f| \geq 0 \). Since \( |N_f| \geq \left\lceil \frac{n}{2} \right\rceil \), condition (i) holds. Since \( f(V) = |P_f| - |M_f| \), condition (ii) holds. To verify condition (iii), consider any \( P' \subseteq P_f \) and \( M' \subseteq M_f \) such that \( |P'| > |M'| \). Let \( X = (P_f - P') \cup M' \) and \( Y = (M_f - M') \cup P' \). Now, define a function \( g : V \to \{-1, +1\} \) by \( g(x) = 1 \) for every \( x \in X \) and \( g(x) = -1 \) for every \( x \in Y \). Then

\[
g(V) = |X| - |Y| = (|P_f| - |P'| + |M'|) - (|M_f| - |M'| + |P'|) = |P_f| - |M_f| - 2(|P'| - |M'|) = |P_f| - |M_f| = f(V) = \gamma_{\text{maj}}^{N}(G).
\]

Thus \( g \) is not a NMTDF of \( G \) and hence \( |N_g| < \left\lceil \frac{n}{2} \right\rceil \). Consequently, \( |x \in V|g(N(x) < 0)| = |V - N_g| = n - |N_g| > n - \left\lceil \frac{n}{2} \right\rceil \). Also,

\[
g(N(x)) = |N(x) \cap X| - |N(x) \cap Y| = |N(x) \cap P_f| - |N(x) \cap M_f| - 2(|N(x) \cap P'| - |N(x) \cap M'|).
\]

Hence we obtain condition (iii).

For the sufficiency, suppose there is a partition \((P_f, M_f)\) of \( V \) such that conditions (i), (ii) and (iii) hold. Define a function \( f : V \to \{-1, +1\} \) by \( f(x) = 1 \) for every \( x \in P_f \) and \( f(x) = -1 \) for every \( x \in M_f \). Then by condition (i), \( f(N(x)) = |N(x) \cap P_f| - |N(x) \cap M_f| \geq 0 \) for at least half vertices of \( G \). Thus \( f \) is NMTDF of \( G \) so that by condition(ii) \( \gamma_{\text{maj}}^{N}(G) \leq |P_f| - |M_f| = k \).

We now show that \( \gamma_{\text{maj}}^{N}(G) \geq |P_f| - |M_f| \). Suppose to the contrary, \( \gamma_{\text{maj}}^{N}(G) < |P_f| - |M_f| \). Let \( g \) be a NMTDF of \( G \) such that \( \gamma_{\text{maj}}^{N}(G) = g(V) \). Let \( X = \{x \in V|g(x) = 1\} \) and \( Y = \{x \in V|g(x) = -1\} \). Let \( P' = P_f - X \) and \( M' = M_f - Y \). Then \( P' \subseteq P_f, M' \subseteq M_f, X = (P_f - P') \cup M' \) and \( Y = (M_f - M') \cup P' \). Moreover,

\[
|P_f| - |M_f| + 2(|M'| - |P'|) = |P_f| - |P'| + |M'| - |M_f| + |M'| - |P'|
\]

\[
= |X| - |Y| = \gamma_{\text{maj}}^{N}(G)
\]

\[
< |P_f| - |M_f|, \text{ so that } |P'| > |M'|.
\]

Therefore by condition (iii),

\[
|V - N_g| = |\{x \in V|g(N(x) < 0)\}|
\]

\[
= |\{x \in V|2(|N(x) \cap P'| - |N(x) \cap M'|) > |N(x) \cap P_f| - |N(x) \cap M_f|\}| > n - \left\lceil \frac{n}{2} \right\rceil .
\]

Thus, \( |N_g| < \left\lceil \frac{n}{2} \right\rceil \), contradicting the fact that \( g \) is NMTDF of \( G \). Hence, \( \gamma_{\text{maj}}^{N}(G) \geq |P_f| - |M_f| \). \( \square \)

4 Trees

In this section, we determine upper bound of non-negative majority total domination of a tree. By assigning +1 to the center of a star and -1 to all the leaves we obtain a NMTDF of the star. Thus

Proposition 4.1. For \( n \geq 3 \), \( \gamma_{\text{maj}}^{N}(K_{1,n-1}) = 2 - n \).

Hence the Non-negative majority total domination number of a tree can be arbitrarily large negative.

Theorem 4.2. For any tree \( T \) of order \( n \geq 2 \), \( \gamma_{\text{maj}}^{N}(T) \leq 2 \left\lceil \frac{n}{4} \right\rceil - n \).

Proof. We proceed by induction on the order \( n \geq 2 \) of a tree \( T \). If \( n \in \{2, 3\} \), then \( T = P_n \) and the result follows from Theorem 2.2. This proves the base cases when \( n = 2 \) or \( n = 3 \). For \( n \geq 4 \), assume that every nontrivial tree \( T' \) of order \( n' < n \), \( \gamma_{\text{maj}}^{N}(T') \leq 2 \left\lceil \frac{n'}{4} \right\rceil - n' \). Let \( T \) be
a tree of order \( n \). If \( T \) is a star, then by Proposition 4.1, \( \gamma_{\text{maj}}^{NT}(T) = 2 - n \leq 2 \left\lceil \frac{n}{2} \right\rceil - n \). Hence the desired result follows if \( T \) is a star. Thus we assume that \( \text{diam}(T) \geq 3 \).

Let \( T \) be rooted at a leaf \( r \) of a longest path. Let \( v \) be a vertex at distance \( \text{diam}(T) - 1 \) from \( r \) on a longest path starting at \( r \) and let \( w \) be the parent of \( v \). Let \( |N(v) - \{ w \}| = m \). Then \( m \geq 1 \). Let \( T' = T - (N(v) - \{ w \}) \). Then \( T' \) has order \( n' = n - m \). Since \( \text{diam}(T) \geq 3 \), we have \( n' \geq 2 \). Let \( f' \) be a \( \gamma_{\text{maj}}^{NT}(T') \) - function. Let \( f : V \rightarrow \{-1, +1\} \) be the function defined by \( f(u) = -1 \) for every child of \( v \) and every vertex whose open neighborhood sum is at least zero in \( T' \) also has open neighborhood sum at least zero in \( T \), while each child of \( v \) has \( f(N(u)) \geq 0 \).

Hence \( \left\lceil \frac{n'}{2} \right\rceil + m \geq \left\lceil \frac{n}{2} \right\rceil \) vertices of \( T \) has open neighborhood sum at least zero and so \( f \) is a NMTDF of \( T \). Thus \( \gamma_{\text{maj}}^{NT}(T) \leq f(V(T)) = f'(V(T')) - m \). By the inductive hypothesis, \( \gamma_{\text{maj}}^{NT}(T') \leq 2 \left\lceil \frac{n'}{2} \right\rceil - n' = 2 \left\lceil \frac{n-m}{2} \right\rceil - n + m \) and so \( \gamma_{\text{maj}}^{NT}(T) \leq 2 \left\lceil \frac{n-m}{2} \right\rceil - n + m - m \). Since \( m \geq 1 \), \( \gamma_{\text{maj}}^{NT}(T) \leq 2 \left\lceil \frac{n-1}{2} \right\rceil - n \leq 2 \left\lceil \frac{n}{2} \right\rceil - n \). Hence the desired result follows. \( \square \)

As an immediate consequence of Theorem 3.4, we have the following result.

**Corollary 4.3.** Let \( T \) be a tree of order \( n \). Then \( \gamma_{\text{maj}}^{NT}(T) = 2 \left\lceil \frac{n}{2} \right\rceil - n \) if and only if there exists a partition \(( P_f, M_f) \) of \( V \) for which

(i) \( |N(x) \cap P_f| - |N(x) \cap M_f| \geq 0 \) for at least half of the vertices of \( T \).

(ii) \( |P_f| - |M_f| = 2 \left\lceil \frac{n}{2} \right\rceil - n \).

(iii) For any \( P' \subseteq P_f \) and any \( M' \subseteq M_f \) satisfying \( |P'| > |M'| \), we have

\[ \left\{ x \in V \mid 2 \left( |N(x) \cap P'| - |N(x) \cap M'| \right) > |N(x) \cap P_f| - |N(x) \cap M_f| \right\} > n - 2 \left\lceil \frac{n}{2} \right\rceil. \]

**References**


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