Palestine Journal of Mathematics

Vol. 6(2)(2017) , 380–395

Strongly divided rings with zero-divisors

Ahmed Ayache and David E. Dobbs

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13A15; Secondary 13G05, 13B21, 13B99.

Keywords and phrases: Commutative ring, zero-divisor, total quotient ring, overring, prime ideal, idealization, integral domain, torsion-free module, divided ring, pseudo-valuation domain, pseudo-valuation ring, going-down ring

Abstract. This work is motivated by the recent study of the class of strongly divided (integral) domains. We compare two alternatives for the definition of a strongly divided (commutative unital) ring. Neither of these alternatives implies the other. For each of these alternatives, we seek analogues (or failures of analogues) of results on strongly divided domains, especially for idealizations. For instance, each pseudo-valuation ring satisfies the conditions in both of the alternative definitions. One byproduct is the determination of all the integrally closed overrings of an idealization.

1 Introduction

All rings considered below are commutative with identity; all modules, subrings, inclusions of rings, and ring/algebra-homomorphisms are unital. If R is a ring, then Spec(R) denotes the set of all prime ideals of R; "dimension(al)" in regard to R refers to the Krull dimension of R, which is denoted by dim(R); and Nil(R) denotes the set of nilpotent elements of R. If R is a ring and E is an R-module, then $Z(E) := Z_R(E) := \{r \in R \mid$ there exists $e \in E \setminus \{0\}$ such that $re = 0\}$, the set of zero-divisors of E (with respect to R); and tq$(R) := R_{R\setminus Z(R)}$, the total quotient ring of R. If R is a (commutative integral) domain, we write qf(R), instead of tq(R), to denote the quotient field of R. If R is any ring, then an *overring* of R is any R-subalgebra of tq(R), that is, any ring B such that $R \subseteq B \subseteq$ tq(R). By a *proper simple overring* of a ring R, we mean any ring of the form $R[u]$ where $u \in$ tq$(R) \setminus R$. As in [8], a ring R is said to be a *treed ring* if Spec(R), when viewed as a poset under inclusion, is a tree; that is, if no maximal ideal of R can contain incomparable prime ideals of R. The main purpose of this paper is to introduce, study and compare two classes of quasi-local rings (possibly with nontrivial zero-divisors) that are analogues of a class of quasi-local domains that was introduced in [4]. The next paragraph recalls some salient facts about that class of domains.

As in [4], a quasi-local domain (R, m) is said to be a *strongly divided domain* if, whenever B is an overring of R and Q is a prime ideal of B such that $Q \cap R \neq m$, then Q is a prime ideal of R (that is, $Q \subseteq R$). It was shown in [4] that the class of strongly divided domains fits properly between the class of divided domains (in the sense of [9]) and the class of pseudo-valuation domains (or PVDs, in the sense of [15]). The first result in [4] consists of the following three assertions [4, Proposition 2.1 (a), (b), (c)]: each strongly divided domain is a divided domain; each quasi-local domain of dimension at most 1 is a strongly divided domain; and each PVD is a strongly divided domain. After introducing two concepts that generalize the "strongly divided" notion from the context of domains to that of arbitrary rings, we will study those generalizations and the extent to which they admit analogues of the three results that were just stated. We will also seek analogues of [4, Corollary 3] and [4, Corollary 4], which state the following: a quasi-local integrally closed domain R is a strongly divided domain (resp., PVD) if and only if each proper simple overring of R is a treed domain (resp., a going-down domain, in the sense of [8], [13]).

It is now straightforward to get the definition of the first generalization being introduced here: simply replace "domain" with "ring" in the earlier definition of a strongly divided domain. More precisely put: a quasi-local ring (R, m) is said to be a *strongly divided ring in the first sense* if, whenever B is an overring of R and Q is a prime ideal of B such that $Q \cap R \neq m$, then Q is a prime ideal of R (that is, $Q \subseteq R$). It is manifest that a domain is a strongly divided ring in the first sense if and only if it is a strongly divided domain. Getting the definition of the second generalization being introduced here will not be straightforward, but we will get to it after it has been motivated by a discussion of some additional background material in the next two paragraphs.

The following implications are known and none of them can be reversed: pseudo-valuation domain ⇒ strongly divided domain ⇒ divided domain ⇒ going-down domain ⇒ treed domain (see [4], [9], [8], [14]). Apart from "strongly divided", the other domain-theoretic concepts which have just been mentioned have well established generalizations to rings, and it is well known that pseudo-valuation ring (or PVR, in the sense of [2]) ⇒ divided ring (in the sense of [9], [5]) ⇒ going-down ring (in the sense of [9]) ⇒ treed ring (with none of these implications being reversible). Moreover, if $\mathcal{P}$ stands for any of "pseudo-valuation", "divided", "going-down" or "treed", then a domain is a $\mathcal{P}$-ring if and only if it is a $\mathcal{P}$-domain. We noted above that the same behavior holds in passing from strongly divided domains to strongly divided rings in the first sense. We believe that it should also hold for the concept that is about to be introduced, namely, a "strongly divided ring in the second sense." A clue as to how to formulate this definition is provided next by recalling the genesis of the definition of going-down rings given in [11].

One could have simply taken *verbatim* the definition of a going-down domain (in terms of the going-down property GD of ring extensions [17, page 28]) and then changed the word "domain" to "ring" throughout in that definition (which is akin to how we obtained the definition of a "strongly divided ring in the first sense"). Instead, it was decided that a ring R would be defined to be a going-down ring if and only if R/P is a going-down domain for all $P \in \text{Spec}(R)$. It was shown in [11] that these two candidates for the definition of a going-down ring are equivalent for any ring R such that $Z(R) = \text{Nil}(R)$, but in general, examples in [11] showed that neither of these candidates implies the other.

In accordance with the style of the definition that was chosen in [11], we now say that a quasi-local ring R is a *strongly divided ring in the second sense* if R/P is a strongly divided domain for all $P \in \text{Spec}(R)$. Thanks to [4, Proposition 2 (a)], a domain is strongly divided ring in the second sense if and only if it is a strongly divided domain. In particular, the two senses of "strongly divided ring" agree in the context of domains. We turn next to the question of finding a natural ring-theoretic context beyond that of domains where one can compare these two senses of the "strongly divided ring" concept.

Our main focus will be on the context of idealizations. Recall that if R is a ring and E is an R-module, then the *idealization* $A := R(+)E$ is the ring whose addition is that of $R \oplus E$ and whose multiplication is given by $(r_1, e_1)(r_2, e_2) = (r_1r_2, r_1e_2 + r_2e_1)$ for all $r_1, r_2 \in R$ and $e_1, e_2 \in E$. It is customary to view R as a subring of A via the injective (unital) ring-homomorphism $R \to A$, $r \mapsto (r, 0)$. This focus has been chosen for several reasons: A is never a domain (provided that $E \neq 0$); the two newly introduced definitions may seem hard to work with because the construction of the total quotient ring of either a factor ring or a ring of fractions can behave pathologically, but such is not the case for idealizations; and idealizations will provide us ample opportunity to compare the two definitions and to see the extent to which either of them admits analogues of the five results from [4] which were recalled in the second paragraph.

Most of the next section builds on some results for idealizations which are due to J. A. Huckaba [16] and which are restated for convenience in Proposition 2.1. As above, let R be a ring and E an R-module, with $A := R(+)E$. Corollary 2.2 (c) infers an easy but useful characterization of when A is integrally closed. Then Theorem 2.3 determines A', which turns out to be an idealization, from which it follows that each overring of A' is also an idealization. Also, Proposition 2.4 collects some basic and useful facts about the interaction of the "torsion-free" and "divisible" concepts from module theory with a condition that arose naturally in Corollary 2.2.

Our main results are given in Section 3. Proposition 3.2 shows that, in testing whether a given ring (resp, domain) is a strongly divided ring in the first sense (resp., strongly divided domain), it suffices to restrict attention to the integrally closed overrings B of R. This result is used in the proof of Theorem 3.4, whose part (a) characterizes the idealizations $A := R(+)E$ which are strongly divided rings in the first sense. In case E is a torsion-free R-module, that characterization simplifies to a particularly elegant form in Theorem 3.4 (b).Turning next to the target analogues, we note that the possibility of an analogue of [4, Proposition 1 (a)] fails utterly, as Example 3.23 (resp., Remarks 3.29 and 3.32 (b)) constructs an idealization which is a strongly divided ring in the first sense and a strongly divided ring in the second sense but is not a divided ring (resp., not a PVR).

Proposition 3.13 records that the two senses of "strongly divided ring" agree when the ambient ring is a domain (where they each serve to characterize strongly divided domains). But the "second sense" concept is clearly the more tractable one, as Proposition 3.14 notes that an idealization $A := R(+)E$ is a strongly divided ring in the second sense if and only if R is a strongly divided ring in the second sense. In fact, the two concepts are inequivalent. Our path to showing this involves the search for analogues of [4, Proposition 1 (b)]. Indeed, Proposition 3.16 (b) shows that if R is any ring such that $\dim(R) \leq 1$, then R is a strongly divided ring in the second sense. In addition, Proposition 3.3 observes that each zero-dimensional ring (more generally, each total quotient ring) is a strongly divided ring in the first sense. However, in view of [4, Proposition 1 (b)], a special case of Example 3.19 leads to the construction of a one-dimensional idealization A which is not a strongly divided ring in the first sense (for instance, one could take $A = R(+)R$, where R is any quasi-local one-dimensional domain). Given an earlier comment about a result in [11], it is somewhat surprising (to us) that this example satisfies $Z(A) = \text{Nil}(A)$. Another result along these lines (this time, not using an idealization) is given in Example 3.21, which constructs a one-dimensional local (Noetherian) ring R which is (a strongly divided ring in the second sense but) not a strongly divided ring in the first sense. Interestingly, while the ring A in Example 3.19 has two prime ideals, the ring R in Example 3.21 has three prime ideals. One upshot of these one-dimensional examples is that "strongly divided ring in the second sense" does not imply "strongly divided ring in the first sense." To complete a summary on these points, we note that Example 3.20 constructs an idealization which is a strongly divided ring in the first sense but is not a strongly divided ring in the second sense.

The best possible analogues of [4, Proposition 1 (c)] are given in Theorem 3.24: each pseudo-valuation ring is a strongly divided ring in both senses. Finally, with respect to analogues of [4, Corollary 3] and [4, Corollary 4]: we achieve the former, for strongly divided rings in either sense, in Corollaries 3.11 and 3.17, for integrally closed idealizations $A = R(+)E$ arising from a quasi-local domain R and a vector space E over the quotient field of R; and we achieve the latter in Proposition 3.28 for integrally closed idealizations $A = R(+)E$ arising from a quasi-local domain R and a torsion-free R-module E.

As usual, if R is a ring, R' denotes the integral closure of R (in $\text{tq}(R)$). Also, if E is an R-module and $P \in \text{Spec}(R)$, then $E_P := E_{R \setminus P}$; in particular, if $R \subseteq T$ are rings and $P \in \text{Spec}(R)$, then $T_P := T_{R \setminus P}$. Following [17, page 28], we let INC denote the incomparability property of ring extensions. Given an idealization $A = R(+)E$ arising from a ring R and an R-module E, it will be convenient to let $U(+)V$ denote the subset $U \times V$ of A, for any sets $U \subseteq R$ and $V \subseteq E$ (without any presumption as to whether the set $U(+)V$ may have any additional algebraic structure). As usual, $\subset$ denotes proper inclusion. Any unexplained material is standard, as in [17].

2 On the overrings of an idealization

We begin by collecting some results of J. A. Huckaba concerning idealizations that will be useful below. The recent literature has seen some variants of the traditional meaning of "torsion-free", but we will adopt the following definition of this term (while stressing that it is still widely, but no longer universally, adopted). If R is a ring and E is an R-module, we say that E is a *torsion-free R-module* if $Z(E) \subseteq Z(R)$.

Proposition 2.1. (J. A. Huckaba). Let R be a ring and E an R-module. Put $S := R \setminus (Z(R) \cup Z(E))$, and consider the idealization $A := R(+)E$. Then:

(a) ([16, Theorem 25.1 (1), (2)]) J is an ideal of A if and only if $J = I(+)C$ for some ideal I of R and some R-submodule C of E such that $IE \subseteq C$. If these conditions hold, then $I = \{r \in R \mid (r, e) \in J \text{ for some } e \in E\}$ and $C = \{c \in E \mid (r, c) \in J \text{ for some } r \in R\}$.

(b) ([16, Theorem 25.1 (3)]) $\text{Spec}(A) = \{P(+)E \mid P \in \text{Spec}(R)\}$.

(c) ([16, Theorem 25.3]) Let $r \in R$ and $e \in E$. Then $(r, e) \in Z(A)$ if and only if $r \in R \setminus S$ (that is, if and only if $r \in Z(R) \cup Z(E)$).

(d) ([16, Lemma 25.4 and Theorem 25.5 (2)]) If $P \in \text{Spec}(R)$, then an R-algebra isomorphism $A_{P(+)E} \to R_P(+)E_P$ can be given by $(r, e)/(s, f) \mapsto (r/s, (se - rf)/s^2)$ for all $r \in R$, $s \in R \setminus P$ and $e, f \in E$.

(e) ([16, Lemma 25.4 and Theorem 25.5 (1)]) The assignment in (d), but this time for $s \in S$, induces an R-algebra isomorphism $\text{tq}(A) \to R_S(+)E_S$.

(f) ([16, Lemma 25.4 and Theorem 25.5 (3)]) If E is a torsion-free R-module, then (e) gives an R-algebra isomorphism $\text{tq}(A) \to \text{tq}(R)(+)E_S$.

(g) ([16, Theorem 25.6]) $A' = (R' \cap R_S)(+)E_S$. In particular, if E is a torsion-free R-module, then $A' = R'(+)E_S$.

(h) ([16, Corollary 25.7]) If R is an integrally closed ring, then $A' = R(+)E_S$.

(i) ([16, Corollary 25.8]) If E is a torsion-free R-module, $R(+)E_S$ is integrally closed if and only if R is integrally closed.

We pause to record how Proposition 2.1 (g) leads to a characterization of integrally closed idealizations.

Corollary 2.2. Let R be a ring and E an R-module. Put $A := R(+)E$ and $S := R \setminus (Z(R) \cup Z(E))$. Then:

(a) $R' \cap R_S = R$ if and only if R is integrally closed in R_S.

(b) $E_S = E$ if and only if E is a module over R_S.

(c) The following conditions are equivalent:

(1) A is integrally closed;

(2) $R' \cap R_S = R$ and $E_S = E$;

(3) R is integrally closed in R_S and E is a module over R_S.

Proof. The proof of (a) is easy and omitted. Before proving (b), we will explicate the conditions in its statement. In general, we can view $E \subseteq E_S$ via the injective R-module homomorphism $i : E \to E_S$, $e \mapsto e/1$. The condition "$E_S = E$" means that i is surjective. The condition "E is a module over R_S" means that the abelian group E (under addition) is an R_S-submodule of E_S. Now, the "only if" assertion in (b) is clear since E_S is an R_S-module. For the "if" assertion in (b), it suffices to prove that if E is a module over R_S, with $e \in E$ and $s \in S$, then $e/s \in E$. This, in turn, holds, since $e/s = (1/s) \cdot (e/1) \in R_S \cdot E = E$, thus completing the proof of (b). Finally, to prove (c), note that Proposition 2.1 (g) leads to (1) $\Leftrightarrow$ (2); and (2) $\Leftrightarrow$ (3) follows by combining (a) and (b). $\square$

We next determine the integrally closed overrings of any idealization.

Theorem 2.3. Let R be a ring and E an R-module. Put $A := R(+)E$ and $S := R \setminus (Z(R) \cup Z(E))$. Then:

(a) Let T be an overring of R. Put $\Delta := \{\delta \in R_S \mid (\delta, f) \in T \text{ for some } f \in E_S\}$ and $F := \{f \in E_S \mid (\delta, f) \in T \text{ for some } \delta \in R_S\}$. Then Δ is an overring of R (and $\Delta \subseteq R_S$), $T \subseteq \Delta \oplus F$ as abelian groups (under addition), and T is an R-submodule of $\Delta(+)E_S$. Moreover,

$$T' = (\Delta(+)E_S)' = (\Delta' \cap R_S)(+)E_S.$$

(b) The integrally closed overrings of A are the rings of the form $\mathcal{D}(+)E_S$, where $\mathcal{D}$ is a ring such that $R \subseteq \mathcal{D} \subseteq R_S$ and $\mathcal{D}$ is integrally closed in R_S.

(c) Suppose, in addition, that E is torsion-free as an R-module. Then the integrally closed overrings of A are the rings of the form $\mathcal{D}(+)E_S$, where $\mathcal{D}$ is an integrally closed overring of R.

Proof. (a) Using the isomorphism prescribed in Proposition 2.1 (e), we can identify $\text{tq}(A) = R_S(+)E_S$. The definitions of Δ and F are now clear. One can easily verify that Δ is an overring of R which is contained in R_S, $T \subseteq \Delta \oplus F$ as abelian groups (under addition), and T is an R-submodule of $\Delta(+)E_S$. The rest of the proof of (a) will be devoted to establishing the stated descriptions of T'. That will be done by an argument some of which has been inspired by the proof of [16, Theorem 25.6].

Since $T \subseteq \Delta \oplus F \subseteq \Delta(+)E_S$, we have $T' \subseteq (\Delta(+)E_S)'$. We claim that $(\Delta(+)E_S)' \subseteq (\Delta' \cap R_S)(+)E_S$. Proving this claim amounts to showing that if $(\eta, f) \in R_S(+)E_S$ is integral over $\Delta(+)E_S$, then $\eta \in \Delta'$ $(\cap R_S)$. This, in turn, can be shown by reworking the calculation (adapted to the present context) from lines 3-7 in [16, page 166]. This proves the above claim.

It now suffices to show that $(\Delta' \cap R_S)(+)E_S \subseteq T'$. As each element of $0(+)E_S$ is nilpotent (and hence integral over T), it will be enough to show that if $\xi \in \Delta' \cap R_S$, then $\rho := (\xi, 0)$ is integral over T. We have

$$\xi^n + \delta_{n-1}\xi^{n-1} + \cdots + \delta_1\xi + \delta_0 = 0 \in R_S,$$

for some finitely many elements $\delta_i \in \Delta$ (and $n \geq 1$). For each i, pick $f_i \in F$ such that $t_i := (\delta_i, f_i) \in T$. Then

$$\rho^n + t_{n-1}\rho^{n-1} + \cdots + t_1\rho + t_0 = (\xi^n + \delta_{n-1}\xi^{n-1} + \cdots + \delta_1\xi + \delta_0, g) = (0, g),$$

where $g := \xi^{n-1}f_{n-1} + \cdots + \xi f_1 + f_0 \in E_S$. As $(0, g)^2 = (0, 0)$, squaring both sides of the preceding display leads to

$$(\rho^n + t_{n-1}\rho^{n-1} + \cdots + t_1\rho + t_0)^2 = (0, g)^2 = (0, 0) \in R_S(+)E_S,$$

and expansion reveals that ρ is indeed integral over T, thus completing the proof of (a).

(b) Suppose first that T is an integrally closed overring of A. Let Δ be as in the statement of (a). Put $\mathfrak{D} := \Delta' \cap R_S$. It is clear that $\mathfrak{D}$ is a ring such that $R \subseteq \Delta \subseteq \mathfrak{D} \subseteq R_S$, and it is easy to see that $\mathfrak{D}$ is integrally closed in R_S (cf. Corollary 2.2 (a)). As T is integrally closed, $T = T'$, which, by (1), coincides with $\mathfrak{D}(+)E_S$. Thus, T can be expressed in the asserted form.

Next, suppose that $\mathcal{D}$ is a ring such that $R \subseteq \mathcal{D} \subseteq R_S$ and $\mathcal{D}$ is integrally closed in R_S, and consider the ring $T := \mathcal{D}(+)E_S$. By Proposition 2.1 (e), T is an overring of A. By the first assertion in (a), $T' = (\mathcal{D}' \cap R_S)(+)E_S$. The assumptions on the ring $\mathcal{D}$ are equivalent to $\mathcal{D}' \cap R_S = \mathcal{D}$, and so $T' = \mathcal{D}(+)E_S = T$. Thus, T is integrally closed.

(c) The assertion is trivial if R is the zero ring, and so, without loss of generality, $R \neq 0$. Then, since E is a torsion-free R-module, we have $Z(E) = 0 \subseteq Z(R)$, and so $S = Z(R)$ and $R_S = \text{tq}(R)$. The assertion is now immediate as a special case of (b). □

In the next section, we will apply the fact (from Theorem 2.3 (b)) that any integrally closed overring of an idealization is an idealization. It seems to be an open question to find an equally explicit description of an arbitrary overring of an idealization.

Proposition 2.4 will give a companion for Corollary 2.2 (b) that will be of use in the next section. First, recall that if R is a ring and E is an R-module, we say that E is a *divisible R-module* if, for all $e \in E$ and $r \in R \setminus Z(R)$, there exists $f \in E$ such that $rf = e$.

Proposition 2.4. Let R be a ring and E an R-module. Put $S := R \setminus (Z(R) \cup Z(E))$. Then:

(a) If E is a divisible R-module, then E is a module over R_S.

(b) If E is torsion-free as an R-module and E is a module over R_S, then E is a divisible R-module.

(c) If E is torsion-free as an R-module, then E is a module over R_S if and only if E is a divisible R-module.

(d) If E is a module over $\text{tq}(R)$, then E is a divisible R-module.

Proof. (a) Since E is an R-module, there is a ring homomorphism $g : R \to \text{Hom}_{\mathbb{Z}}(E, E)$ (given by $g(r)(e) = re$ for all $r \in R$ and $e \in E$). Our task is to show that g induces an R-algebra homomorphism $R_S \to \text{Hom}_{\mathbb{Z}}(E, E)$; equivalently, to show that if $r \in S$, then $g(r)$ is a bijection. Note that $g(r)$ is an injection (for, if, on the contrary, there exists a nonzero element $e \in \ker(g(r))$, then $re = 0$, whence $r \in Z(E)$, contradicting $r \in S$). Finally, if $e \in E$, the hypothesis that E is a divisible R-module supplies $f \in E$ such that $rf = e$ (since $r \in R \setminus Z(R)$), whence $g(r)(f) = rf = e$, so that $g(r)$ is also a surjection.

(b) We will show that if $e \in E$ and $r \in R \setminus Z(R)$, then there exists $f \in E$ such that $rf = e$. Note that $r \notin Z(E)$, since the hypothesis that E is a torsion-free R-module means that $Z(E) \subseteq Z(R)$. Therefore, $r \in S$. Hence, the hypothesis that E is a module over R_S yields an element $f := (1/r) \cdot e$. It is clear that $rf = e$.

(c) Combine (a) and (b).

(d) As in the proof of (a), we need to show that the ring homomorphism $g : R \to \text{Hom}_{\mathbb{Z}}(E, E)$ induces an R-algebra homomorphism $R_{R \setminus Z(R)} \to \text{Hom}_{\mathbb{Z}}(E, E)$; equivalently, to show that if $r \in R \setminus Z(R)$, then $g(r)$ is a bijection. To see that $g(r)$ is an injection, note that if $e \in \ker(g(r))$, then $e = (1/r) \cdot (re) = (1/r) \cdot g(r)(e) = (1/r) \cdot 0 = 0$. Finally, to show that $g(r)$ is a surjection, observe that if $e \in E$, then $g(r)$ sends $f := (1/r) \cdot e$ to e, since $g(r)(f) = rf = e$. □

Remark 2.5. Let R be a ring and E an R-module. As usual, put $S := R \setminus (Z(R) \cup Z(E))$. The above result leads to the question whether the “torsion-free" hypothesis in Proposition 2.4 (c) can be deleted (equivalently, whether the converse of Proposition 2.4 (a) is valid). In other words, is it the case that E is a divisible R-module if (and only if) E is an R_S-module? By Proposition 2.2 (b), an equivalent question is the following. Is it the case that E is a divisible R-module if (and only if) $E = E_S$? The answer is in the negative. To see this, we will make the first of several uses of the following construction of J. A. Huckaba [16, page 166].

Let R be any nonzero ring and take the R-module E to be $\oplus R/P_\alpha$, where P_α ranges over the set of prime ideals of R. As above, take $S := R\setminus(Z(R)\cup Z(E))$ and $A := R(+)E$. As Huckaba notes, $Z(E)$ is the set of nonunits of R, so that (by Proposition 2.1 (e)) tq(A) can be identified with $R(+)E_S$. In fact, since S is the set of units of R, we have (not only that $R_S = R$ but also) that $E_S = R_S \cdot E = R \cdot E = E$. Thus, tq$(A) = R(+)E = A$; that is, A is a total quotient ring.

To answer the above question in the negative, we now apply the above construction to the case where (R, m) is any quasi-local domain of (Krull) dimension 1. Then Spec$(R) = \{0, m\}$ and the above construction leads (up to R-module isomorphism) to $E = R \oplus R/m$. Pick any nonzero element $r \in m$. Put $e := (0, 1 + m) \in E$. As we saw above that $E = E_S$, it remains only to show that E is not a divisible R-module. To do so, observe that $r \in R \setminus Z(R)$ and each element $f = (\rho, \xi) \in E$ satisfies $rf = (r\rho, r\xi) = (r\rho, 0) \neq (0, 1 + m) = e$.

3 Testing the Two Definitions

We begin this section by seeking an idealization-theoretic analogue of the following result [4, Corollary 3]: a quasi-local integrally closed domain R is a strongly divided domain if and only if each proper simple overring of R is treed. Using Proposition 2.1 (b), one sees easily that an idealization $A := R(+)E$ is treed if and only if R is treed. Also because of Proposition 2.1 (b), we henceforth restrict attention to quasi-local rings (although, as in Proposition 3.1, that restriction is occasionally irrelevant).

Proposition 3.1. Let (R, m) be a quasi-local ring and E an R-module such that the idealization $A := R(+)E$ is integrally closed. Put $S := R \setminus (Z(R) \cup Z(E))$. Then:

(a) The following two conditions are equivalent:

(1) Each proper simple overring of A is treed;

(2) $R[\rho]$ is treed for each $\rho \in R_S \setminus R$.

(b) Suppose, in addition, that E is torsion-free as an R-module. Then the following two conditions are equivalent:

(1) Each proper simple overring of A is treed;

(2) Each proper simple overring of R is treed.

Proof. By Proposition 2.1 (b), A is quasi-local, with unique maximal ideal $m(+)E$. Recall from Proposition 2.1 (e) that we can identify tq$(A) = R_S(+)E_S$.

(a) By Corollary 2.2 (c), the “integrally closed" hypothesis on A means that $R' \cap R_S = R$ and $E_S = E$ (equivalently, that R is integrally closed in R_S and E is a module over R_S). In other words, $A = R(+)E$ where E is a module over R_S and the quasi-local ring R is integrally closed in R_S. A proper simple overring of A is any ring T of the form $T = A[(\rho, f)]$ where $(\rho, f) \in$ tq$(A) \setminus A$; that is (since $E = E_S$), a ring of the form $T = A[(\rho, f)]$ where $\rho \in R_S \setminus R$ and $f \in E$. As $(0, f) \in A \subset T$, we have $T = A[(\rho, 0)] = R[\rho](+)E$. By an earlier comment, such a ring T is treed if and only if $R[\rho]$ is treed, and so (a) follows at once.

(b) The assertion is trivial if R is the zero ring, and so, without loss of generality, $R \neq 0$. Then, as in the proof of Theorem 2.3 (c), the “torsion-free" hypothesis leads to tq$(R) = R_S$, and so (b) follows as a special case of (a). □

Before making a connection with Proposition 3.1, we give a characterization of “strongly divided ring in the first sense" which is new even in the case of domains.

Proposition 3.2. Let (R, m) be a quasi-local ring (resp., quasi-local domain). Then R is a strongly divided ring in the first sense (resp., strongly divided domain) if (and only if), whenever T is an integrally closed overring of R and Q is a prime ideal of T such that $Q \cap R \neq m$, then Q is a prime ideal of R (that is, $Q \subseteq R$).

Proof. The "only if" assertion is trivial. As for the "if" assertion, suppose that T is an overring of R and P is a prime ideal of T such that $P \cap R \neq m$. Our task is to prove that P is a prime ideal of R (that is, $P \subseteq R$). Since the ring extension $T \subseteq T'$ is integral, it follows from the Lying-over Theorem (cf. [17, Theorem 44]) that there exists $Q \in \mathrm{Spec}(T')$ such that $Q \cap T = P$. Then $Q \cap R = Q \cap (T \cap R) = P \cap R \neq m$. Therefore, by the hypothesis of the "if" assertion, $Q \subseteq R$ and so, *a fortiori*, $P \subseteq R$. Finally, the parenthetical assertion characterizing strongly divided domains now follows because a domain is a strongly divided domain if and only if it is a strongly divided ring in the first sense. □

We pause to isolate an easy, but surprisingly useful, result.

Proposition 3.3. Let R be a ring. If $\mathrm{tq}(R) = R$, then R is a strongly divided ring in the first sense. In particular, if $\dim(R) = 0$, then R is a strongly divided ring in the first sense.

Proof. The first assertion is trivial. The second assertion is a special case of the first, since every zero-dimensional ring is its own total quotient ring (by [17, Theorem 84]). □

Theorem 3.4. Let (R, m) be a quasi-local ring and E an R-module. Put $A := R(+)E$ and $S := R \setminus (Z(R) \cup Z(E))$. Then:

(a) The following two conditions are equivalent:

(1) A is a strongly divided ring in the first sense;

(2) $E = E_S$ and, whenever $\mathcal{D}$ is a ring such that $R \subseteq \mathcal{D} \subseteq R_S$ and $\mathcal{D}$ is integrally closed in R_S and $Q \in \mathrm{Spec}(\mathcal{D})$ such that $Q \cap R \neq m$, then $Q \in \mathrm{Spec}(R)$ (that is, $Q \subseteq R$).

(b) Suppose, in addition, that E is torsion-free as an R-module. Then the following three conditions are equivalent:

(i) A is a strongly divided ring in the first sense;

(ii) $E = E_S$ and R is a strongly divided ring in the first sense;

(iii) E is a module over $\mathrm{tq}(R)$ and R is a strongly divided ring in the first sense.

Proof. (a) Suppose first that $\dim(R) = 0$. We will show that both (1) and (2) hold. By Proposition 2.1 (b), $\dim(A) = 0$, and so Proposition 3.3 shows that (1) holds. To prove (2), it suffices to prove that $E = E_S$. This, in turn, will follow from Corollary 2.2 (b) if we show that $R_S = R$. To that end, observe that since $\dim(R) = 0$, it follows from [17, Theorem 84] that each non-zero-divisor in R is a unit of R, whence S must be the set of units of R, and so $R_S = R$. Therefore, we may assume henceforth that $\dim(R) > 0$.

Applying Proposition 3.2 to the ring A, we see (in view of Theorem 2.3 (b) and parts (e) and (b) of Proposition 2.1) that (1) is equivalent to the following condition: $\mathcal{D}(+)E_S$, whenever $\mathcal{D}$ is a ring such that $R \subseteq \mathcal{D} \subseteq R_S$ and $\mathcal{D}$ is integrally closed in R_S and $Q \in \mathrm{Spec}(\mathcal{D})$ such that $(Q(+)E_S) \cap A \subset m(+)E$ (that is, such that $Q \cap R \subset m$), then $Q(+)E_S \subseteq A$ (that is, $Q \subseteq R$ and $E = E_S$). It therefore suffices to prove that if no such $\mathcal{D}$ and Q exist with the property that $Q \cap R \subset m$, then $E = E_S$. In fact, since $\dim(R) > 0$, such $\mathcal{D}$ and Q *must* exist, because every minimal prime ideal of a base ring is lain over in any ring extension (cf. [17, Exercise 1, page 41]).

(b) The "torsion-free" hypothesis ensures that $R_S = \mathrm{tq}(R)$. Under these conditions, we see via Proposition 3.2 that condition (2) in the statement of (a) becomes equivalent to (ii). It now follows via Corollary 2.2 (b) that (b) is a special case of (a). □

Remark 3.5. It is important to note that one cannot delete the condition "$E = E_S$" from the statements of conditions (a) (2) and (b) (ii) in Theorem 3.4; nor can one delete the condition "E is a module over $\mathrm{tq}(R)$" from the statement of condition (b) (iii) in Theorem 3.4. Perhaps the easiest example showing this fact is given as follows. Let (R, m) be a quasi-local one-dimensional domain. Then R is a strongly divided domain (and hence a strongly divided ring in the first sense) by [4, Proposition 1 (b)]. However, $A := R(+)R$ is not a strongly divided ring in the first sense, by Theorem 3.4 (a) (even though $E = R$ is a torsion-free R-module), since E $(= R)$ is not a module (that is, is not a vector space) over $\mathrm{qf}(R) = E_S$.

The domain-theoretic case of Theorem 3.4 (b) deserves to be isolated. We give it next and follow it with several applications.

Corollary 3.6. Let R be a quasi-local domain with quotient field K and let E be a torsion-free R-module. Put $A := R(+)E$. Then A is a strongly divided ring in the first sense $\Leftrightarrow$ R is a strongly divided domain and $KE = E \Leftrightarrow R$ is a strongly divided domain and E is a vector space over K.

Proof. The assertions follow from Theorem 3.4 (b) once we notice that $R_S = K$ and $KE = R_S E = E_S$. □

Corollary 3.7. Let R be a quasi-local domain with quotient field K and let E be a vector space over K. Put $A := R(+)E$. Then A is a strongly divided ring in the first sense if and only if R is a strongly divided domain.

Proof. The assertion follows from Corollary 3.6 once we notice that every vector space over K must be a torsion-free R-module. □

Remark 3.8. (a) The conditions in Corollaries 3.6 and 3.7 are tractable and will form the setting for a number of applications in the rest of this paper. However, one should note that even if R is a quasi-local domain (and E is an R-module), the equivalence of those conditions can fail in the absence of the “torsion-free" hypothesis. For instance, let us revisit the construction of Huckaba that was recalled in Remark 2.5. Specialize to the context where (R, m) is a quasi-local domain with quotient field K and, as before, take the R-module E to be $\oplus R/P_\alpha$, where P_α ranges over the set of prime ideals of R. Recall from Remark 2.5 that $A := R(+)E$ is its own total quotient ring, and so A is trivially a strongly divided ring in the first sense. However, the other conditions from the statements of Theorem 3.4 (b) and Corollary 3.6 need not hold in this setting. Indeed, although $E = E_S$ (where, as usual, we define $S := R \setminus (Z(R) \cup Z(E))$), E need not be a module over tq(R) $(= K)$ and R need not be a strongly divided ring in the first sense. To see this, one need only take R to be a quasi-local domain which is not a strongly divided domain (for instance, take R to be the divided domain in [4, Example 3]). To avoid such “pathology", we will often impose a "torsion-free" hypothesis from now on.

(b) We next pursue the role of the “torsion-free" condition. Consider further the construction in (a). Using the notation from (a), observe that S is the set of units of R, whence $E = E_S$ canonically. Thus, unless each nonunit of R is a zero-divisor (in R), E is not a torsion-free R-module. In particular, if R is a domain but not a field, then E is not a torsion-free R-module. On the other hand, if R is a field, call it K, then $A = K(+)K = $ tq(A) and, in this case, $E = K$ is a torsion-free module over $K = R$ and R is trivially a strongly divided domain.

While the next result could have been given in the preceding section, it is also motivated by the above discussion concerning torsion-freeness and it will be used in the proof of Corollary 3.10.

Proposition 3.9. Let R be a quasi-local ring and E a torsion-free R-module. Put $A := R(+)E$ and $S := R \setminus (Z(R) \cup Z(E))$. If A is an integrally closed ring, then R is integrally closed, E is a divisible R-module and $E = E_S$.

Proof. The “torsion-free" hypothesis ensures that $R_S = $ tq(R). The assertions then follow by combining parts (c) and (b) of Corollary 2.2 with Proposition 2.4 (b). □

Corollary 3.10. Let (R, m) be a quasi-local domain and E a torsion-free R-module such that the idealization $A := R(+)E$ is integrally closed. Then the following conditions are equivalent:
(1) Each proper simple overring of A is treed;
(2) R is a strongly divided domain;
(3) Each proper simple overring of R is treed.

Proof. Since R is integrally closed by Corollary 2.2 (c) (or by Proposition 3.9), we see via [4, Corollary 3] that (2) $\Leftrightarrow$ (3); and (1) $\Leftrightarrow$ (3) by Proposition 3.1 (b). □

Corollary 3.11. Let (R, m) be a quasi-local domain with quotient field K and E a vector space over K such that the idealization $A := R(+)E$ is integrally closed. Then the following conditions are equivalent:
(1) A is a strongly divided ring in the first sense;
(2) Each proper simple overring of A is treed;
(3) R is a strongly divided domain;
(4) Each proper simple overring of R is treed.

Proof. As E is a vector space over K, we have that E is a torsion-free R-module. The assertion now follows by combining Corollary 3.7 and 3.10. □

Remark 3.12. Corollary 3.11 is the kind of desirable result that was mentioned in the first paragraph of this section. It is natural to ask if the "torsion-free" hypothesis in Corollary 3.10 (which played a major role in the proof of Corollary 3.11) can be deleted. The answer is in the negative. To see this, one need only rework the example in Remark 3.8 (a).

It also seems natural to ask if the concept of "a strongly divided ring in the second sense" admits results with some of the flavor of the earlier results in this section. We begin the analysis of this concept by answering the most clearly relevant question.

Proposition 3.13. Let R be a quasi-local domain. Then the following conditions are equivalent:
(1) R is a strongly divided ring in the first sense;
(2) R is a strongly divided ring in the second sense;
(3) R is a strongly divided domain.

Proof. (1) ⇔ (3) since, by definition, a domain is a strongly divided ring in the first sense if and only if it is a strongly divided domain. Also, (2) ⇒ (3) since $R/0 \cong R$. It remains to prove that (3) ⇒ (2); in other words, that if R is a strongly divided domain and $P \in \mathrm{Spec}(R)$, then R/P is a strongly divided domain. This assertion holds if P is a maximal ideal of R since any field is trivially a strongly divided domain; and it also holds if P is a non-maximal prime ideal of R, by [4, Proposition 2 (a)]. □

Next, we show that the analogue of Theorem 3.4 for the "strongly divided ring in the second sense" concept is surprisingly easy.

Proposition 3.14. Let (R, m) be a quasi-local ring and E an R-module. Put $A := R(+)E$. Then A is a strongly divided ring in the second sense if and only if R is a strongly divided ring in the second sense.

Proof. By Proposition 2.1 (b), A is a strongly divided ring in the second sense if and only if $A/(P(+)E)$ is a strongly divided domain for each $P \in \mathrm{Spec}(R)$. The assertion now follows because, for any such P, we have $A/(P(+)E) \cong R/P$. □

The domain-theoretic case of Proposition 3.14 deserves to be isolated.

Corollary 3.15. Let (R, m) be a quasi-local domain and E an R-module. Put $A := R(+)E$. Then A is a strongly divided ring in the second sense if and only if R is a strongly divided domain.

Proof. Combine Propositions 3.13 and 3.14. □

Recall from the proof of Proposition 3.13 that each factor domain of a strongly divided domain must be a strongly divided domain. Finding a generalization or even an analogue of this assertion for the "strongly divided ring in the first sense" concept is problematic because the total quotient ring of a factor ring can behave erratically (especially for non-divided rings). However, Proposition 3.16 (a) will easily give a generalization for the "strongly divided ring in the second sense" concept. In addition, Proposition 3.16 (b) goes beyond finding an analogue for the second assertion in Proposition 3.3. In view of Proposition 3.16 (a), it seems natural to ask if the class of strongly divided rings in the second sense is stable under the formation of rings of fractions. In part (c) of the next result, we give an affirmative answer to this question, and then in part (d), we give the domain-theoretic special case of (c) which seems to have been overlooked in [4].

Proposition 3.16. (a) Let R be a ring. Then R is a strongly divided ring in the second sense (if and) only if R/I is a strongly divided ring in the second sense for each ideal I of R.

(b) If (R, m) is a quasi-local ring and $\dim(R) \leq 1$, then R is a strongly divided ring in the second sense.

(c) If R is a strongly divided ring in the second sense and S is a multiplicatively closed subset of R, then R_S is a strongly divided ring in the second sense.

(d) If R is a strongly divided domain and S is a multiplicatively closed subset of R, then R_S is a strongly divided domain.

Proof. (a) The "if" assertion is clear because $R/0 \cong R$. Conversely, suppose that R is a strongly divided ring in the second sense, and let I be an ideal of R. Our task is to show that if P is any prime ideal of R that contains I, then $(R/I)/(P/I)$ is a strongly divided domain. The assertion now follows since $(R/I)/(P/I) \cong R/P$.

(b) We will prove that if $P \in \text{Spec}(R)$, then R/P is a strongly divided domain. This holds if $P = m$ since any field is a strongly divided domain. Moreover, this also holds if $P \neq m$ since any quasi-local one-dimensional domain is strongly divided [4, Proposition 1 (b)].

(c), (d) We begin the proof of (c) by showing that it can be reduced to proving (d). Consider $Q \in \text{Spec}(R_S)$; that is, $Q = PR_S$ where $P \in \text{Spec}(R)$ satisfies $P \cap S = \emptyset$. To prove (c), one must show that R_S/Q is a strongly divided domain. By combining (a) with Proposition 3.13, we see that $D := R/P$ is a strongly divided domain. Note that $S^* := \{s + P \in D \mid s \in S\}$ is a multiplicatively closed subset of D. It is well known (and easy to check) that the canonical ring homomorphism $j : D_{S^*} \to R_S/Q$ (given by $(r + P)/(s + P) \mapsto r/s + PR_S$ for all $r \in R$ and $s \in S$) is an isomorphism. Therefore, to complete the reduction of (c) to (d), it will be enough to show that D_{S^*} is quasi-local. This, in turn, follows from the more general consequence of [17, Theorem 9] that if A is any quasi-local treed ring, then so is any nonzero ring of fractions of A. (Of course, this fact applies to the present situation since the quasi-local domain D is treed, as can be seen by combining [4, Proposition 1 (a)] with the fact that any divided domain must be treed [1, Theorem 1.3], [9, Lemma 2.2 (c)].)

In view of the above isomorphism (j), we may now assume (after a harmless change of notation) that R is a domain. In other words, it remains only to prove (d), so that (R, m) is assumed to be a strongly divided domain. By the above comments, R_S is a quasi-local treed domain, say with unique maximal ideal QR_S for some prime ideal Q ($\subseteq m$) of R such that $Q \cap S = \emptyset$. Our task is to show that if T is an overring of R_S (and hence of R) and P is a prime ideal of T such that $P \cap R_S \subset QR_S$, then $P \subseteq R_S$. In fact, since $P \cap R = (P \cap R_S) \cap R \subset QR_S \cap R = Q \subseteq m$, the fact that R is a strongly divided domain yields that $P \subseteq R$, whence $P \subseteq R_S$, as desired. □

It seems natural to ask if idealizations admit a result with at least some of the flavor of Proposition 3.13. We show next that the answer is in the affirmative if we add some hypotheses. However, Examples 3.19 and 3.20 will show that the answer is in the negative in general, as these examples will show that the two senses of "strongly divided ring" are not equivalent, even for idealizations.

Corollary 3.17. Let R be a quasi-local domain with quotient field K, let E be a vector space over K, put $A := R(+)E$. Then A is a strongly divided ring in the first sense if and only if A is a strongly divided ring in the second sense (if and only if R is a strongly divided domain).

Proof. Combine Corollaries 3.7 and 3.15. □

Remark 3.18. Corollary 3.17 allows us to expand the statement of Corollary 3.11 by adding a fifth equivalent condition, namely, (5): A is a strongly divided ring in the second sense.

Example 3.19. Let R be a (quasi-local) strongly divided domain which is distinct from $K :=$ qf(R). Let E be a torsion-free R-module which is not a vector space over K (for instance, take E to be R). Put $A := R(+)E$. Then A is a strongly divided ring in the second sense, but A is not a strongly divided ring in the first sense.

Proof. The first assertion follows from Corollary 3.15; and the second assertion follows from Theorem 3.4 (b). □

Example 3.20. Let R be a quasi-local domain which is not a strongly divided domain. Take the R-module E to be $\oplus R/P_\alpha$, where P_α ranges over the set of prime ideals of R. Put $A := R(+)E$. Then A is a strongly divided ring in the first sense, but A is not a strongly divided ring in the second sense.

Proof. Notice that A has been built with the oft-used construction of Huckaba. As shown in Remark 2.5, tq$(A) = A$, and so it follows from the first assertion in Proposition 3.3 that A is a strongly divided ring in the first sense. However, since R is not a strongly divided domain, it follows from Corollary 3.15 that A is not a strongly divided ring in the second sense. □

We next expand upon the inequivalence of the two "strongly divided ring" concepts by showing that the second assertion in Proposition 3.3 cannot be generalized so as to have the full flavor of Proposition 3.16 (b). While the ring R in Example 3.21 will be shown to admit the same conclusion as the idealization in Corollary 3.19, R is not overtly an idealization.

Example 3.21. There exists a one-dimensional (quasi-) local Noetherian ring which is (a strongly divided ring in the second sense, but) not a strongly divided ring in the first sense. One way to construct such an R is as follows. Let X and Y be commuting algebraically independent indeterminates over a field k, put $B := k[X,Y]/(XY)$, let $x := X + (XY)$ and $y := Y + (XY)$ (in B), and set $R := B_{(x,y)}$.

Proof. The class of Noetherian rings is stable under the formation of polynomial rings in finitely many variables, the formation of factor rings and the formation of rings of fractions. Therefore, R is Noetherian. Moreover, since (x, y) is a maximal ideal of B, we see that R is quasi-local. Consider the elements $u := x/1$ and $v := y/1$ in R. As $\dim(k[X, Y]) = 2$ (cf. [17, Theorem 149]), it is easy to see that the only prime ideals of R are $p := Ru$, $q := Rv$, and $Ru + Rv$ ($= p + q =: m$); and that these ideals are pairwise distinct. Hence, $\dim(R) = 1$. Therefore, by Proposition 3.16 (b), R is a strongly divided ring in the second sense.

It remains to show that R is not a strongly divided ring in the first sense. To do so, we need to identify $T := \mathrm{tq}(R)$. First, note that since X and Y are non-associated prime elements of the unique factorization domain $k[X, Y]$, it follows easily that $Bx \cap By = 0$. Then, since localization commutes with finite intersections, we have $Ru \cap Rv = 0$. Thus, R is a reduced ring. Therefore, $Z(R)$ is the union of the minimal prime ideals of R, whence $T = R_{R\setminus(p\cup q)}$.

Observe that $pT \in \mathrm{Spec}(T)$ and $pT \cap R = p \subset m$. Hence, to prove that R is not a strongly divided ring in the first sense, it suffices to show that $pT \neq p$ (equivalently, that $pT \not\subseteq R$). Suppose, on the contrary, that $pT \subseteq R$. Pick an element $\lambda \in m \setminus (p \cup q)$. (To find λ, one could appeal to the Prime Avoidance Lemma [17, Theorem 81], but in this case, it is just as easy to take $\lambda := u + v$.) For each $n \geq 1$, $\lambda^n \in m \setminus (p \cup q)$. As $pT = p$ has been assumed, it follows that $u/\lambda^n \in p \subseteq R$ for each $n \geq 1$. Then, by cross-multiplication,

$$u \in \cap_{n=1}^{\infty} \lambda^n R \subseteq \cap_{n=1}^{\infty} m^n.$$

The last-displayed intersection is 0, by the Krull Intersection Theorem (cf. [17, Theorem 79]). But $u \neq 0$ (since, for instance, $Ru = p \not\subseteq q$). This (desired) contradiction completes the proof. □

Remark 3.22. In the Introduction, motivated by [11, Theorems 2.4 and 2.5], we raised the question whether a ring A that is strongly divided in the first (resp., second) sense and satisfies $Z(A) = \mathrm{Nil}(A)$ must also be strongly divided in the second (resp., first) sense. We can answer one of these questions in the negative (leaving the other question open). Indeed, the idealization in Example 3.19 does satisfy $Z(A) = \mathrm{Nil}(A)$ because of the following general fact. If R is a domain and E is a torsion-free R-module, then every zero-divisor of the ring $R(+)E$ is nilpotent. This methodology cannot be applied to the construction in Example 3.20 (where the ambient module is not torsion-free).

One can view Propositions 3.3 and 3.16(b) and Example 3.21 as resulting from our search for analogues, amid nontrivial zero-divisors, of the fact [4, Proposition 1 (b)] that each quasi-local domain of (Krull) dimension at most 1 is a strongly divided domain. In the same spirit, one can ask for analogues of the result [4, Proposition 1 (a)] that each strongly divided domain is a divided domain. Recall from [9] (resp., [5]) that a domain (resp., ring) R is said to be a *divided domain* (resp., *divided ring*) if, whenever $P \in \mathrm{Spec}(R)$ and $r \in R \setminus P$, then $P \subseteq Rr$. It is easy to see that any divided ring is a quasi-local treed ring.

We will show in Example 3.23 that the most naturally stated analogues of [4, Proposition 1 (a)] are not valid. To further motivate the next result, note that [6, Example 2.18 (a)] constructed a ring R such that $\mathrm{tq}(R) = R$ (so that R is a strongly divided ring in the first sense, by Proposition 3.3) and R is not a divided ring.

The construction in Example 3.23 will involve a pseudo-valuation ring (PVR). The class of PVRs was introduced in [2] and studied further in [3]. We will recall the definition of a PVR and some related facts after Example 3.23.

Example 3.23. For each integer $d \geq 2$, there exists a d-dimensional pseudo-valuation ring (R, m) such that $A := R(+)R$ is a strongly divided ring in the first sense and a strongly divided ring in the second sense, but A is not a divided ring. It can be also be arranged that $Z(R) = m$ (so that $\mathrm{tq}(R) = R$) and $\mathrm{tq}(A) = A$.

Proof. By [3, Example 3.16 (c)], there exists a d-dimensional pseudo-valuation ring (R, m) such that $Z(R) = m$. As in the statement, $A := R(+)R$. Note that $S := R \setminus Z(R) = R \setminus m$ is the set of units of R, and so $R_S = R$ canonically. Hence, by Proposition 2.1 (e), $\mathrm{tq}(A) = R_S(+)R_S = R(+)R = A$. Thus, by Proposition 3.3, A is a strongly divided ring in the first sense. Moreover, since R is a strongly divided ring in the second sense by Lemma 3.23, it follows from Proposition 3.14 that A is also a strongly divided ring in the second sense. It remains only to prove that A is not a divided ring. We will do so via an argument that is inspired by the final paragraph of the proof of [6, Lemma 2.13].

Note that $P_0 := \mathrm{Nil}(R)$ is the unique minimal prime ideal of R (cf. [6, Remark 2.4 (a)]). Consider $Q := P_0(+)R$. It follows from Proposition 2.1 (b) that $\mathrm{Nil}(A) = \mathrm{Nil}(R)(+)R = P_0(+)R = Q$. To complete the proof, it suffices to find an element $a \in A \setminus Q$ such that $Q \not\subseteq Aa$. Take P to be any non-minimal non-maximal prime ideal of R; that is, $P \in \mathrm{Spec}(R)$ and $P_0 \subset P \subset m$. Pick $r \in P \setminus P_0$ and $e \in m \setminus P$. Observe that $a := (r, 0) \in A \setminus Q$. It now suffices to obtain a contradiction from the assumption that $Q \subseteq Aa$. Under this assumption, there exists $b = (s, f) \in A$ such that $(0, e) = ba$; that is, $(0, e) = (sr, fr)$, whence $e = fr \in RP = P$, the desired contradiction. □

Recall that a ring R is said to be a *pseudo-valuation ring* if, for each $P \in \mathrm{Spec}(R)$ and $a, b \in R$, one has that Pa and Rb are comparable with respect to inclusion. The case $a = 1$ shows that each PVR is a divided ring. However, the converse is false, even if R is a domain [10, Remark 4.10 (b)]. To facilitate the proof of Theorem 3.24, we also record here the facts that the class of PVRs is stable under homomorphic images [2, Corollary 3]; and a domain is a PVR if and only if it is a pseudo-valuation domain (PVD).

In view of Proposition 3.13, one consequence of [4, Example 3] is that a divided ring need not be a strongly divided ring in either sense, even for domains. Thus, the next result shows one way that PVRs behave much more nicely than arbitrary divided rings.

Theorem 3.24. If (R, m) is a PVR, then R is a strongly divided ring in the first sense and a strongly divided ring in the second sense.

Proof. We will take care of the "second sense" assertion first. One must show that if $P \in \mathrm{Spec}(R)$, then R/P is a strongly divided domain. By the above-mentioned facts, R/P is a PVD and hence, by [4, Proposition 1 (c)], a strongly divided domain.

We turn to the (more difficult) proof of the "first sense" assertion. Suppose the assertion fails. Then there exist an overring B of R and $P \in \mathrm{Spec}(B)$ such that $p := P \cap R \subset m$ and $P \not\subseteq R$. Pick $u \in P \setminus R$. One can express $u = a/b$ for some $a \in R$ and $b \in R \setminus Z(R)$. In fact, $a = bu \in P \cap R = p$. Note that $a \notin Rb$, since $u \notin R$. In particular, $p \not\subseteq Rb$. However, p and Rb are comparable (since R is a divided ring), and so $Rb \subseteq p$; that is, $b \in p$. In particular, $p \not\subseteq Z(R)$. Next, since R is a quasi-local treed ring, $Z(R) \in \mathrm{Spec}(R)$. (To see this, recall that $Z(A)$ is a union of prime ideals for any ring A and then apply [17, Theorem 9].) Hence $Z(R) \subset p$. Thus, there is a canonical R-algebra homomorphism $h : R_p \to R_{R \setminus Z(R)}$ $(= \mathrm{tq}(R))$. One checks easily that h is an injection. (Indeed, if $c \in R$ and $d \in R \setminus p$ with $(c/d =)\ h(c/d) = 0/1 \in \mathrm{tq}(R)$, then there exists $\zeta \in R \setminus Z(R)$ such that $\zeta c = 0$, whence $c = 0$ and $c/d = 0/1 \in R_p$.) Thus, we can view $R \subseteq R_p \subseteq B_{R \setminus p} \subseteq \mathrm{tq}(R)$.

For the sake of completeness, we pause to show that $pR_p = p$. It is clearly enough to prove that $pR_p \subseteq R$; in other words, that if $x \in p$ and $y \in R \setminus p$, then $x/y \in R$. In fact, since R is a divided ring, $p \subset Ry$, whence there exists $\gamma \in R$ with $x = \gamma y$ and $x/y = \gamma \in R$, as required.

Since R is a PVR and p is a non-maximal prime ideal of R, it follows from [2, Theorem 12] that R_p is a chained ring (in the sense that the ideals of this ring are linearly ordered by inclusion). Suppose first that $R_p a \subseteq R_p b$. Then

$$u = a/b \in R_p \cap P \subseteq R_p \cap P_{R \setminus p} = (R \cap P)_{R \setminus p} = p_{R \setminus p} = pR_p = p \subseteq R,$$

a contradiction. Therefore, $R_p b \subseteq R_p a$. In particular, $b \in R_p a$, whence $a = bu \in R_p au$. If $a \notin Z(R)$, then $1 \in R_p u \subseteq B_{R\setminus p} u$, whence u is a unit of $B_{R\setminus p}$, a contradiction (since u is a member of the prime ideal $PB_{R\setminus p}$ of $B_{R\setminus p}$). Therefore, $a \in Z(R)$.

Finally, since R is a divided ring and $Z(R) \in \mathrm{Spec}(R)$, we have that $Z(R)$ is comparable to Rb. As $b \in R \setminus Z(R)$, it follows that $Z(R) \subseteq Rb$. In particular, $a \in Rb$, the desired contradiction. □

Apart from Theorem 3.24 and the role of PVRs in the proof of Example 3.23, further consideration of PVRs is warranted here because we wish to give a full analogue of the result [4, Proposition 1 (c)] that each PVD is a strongly divided domain. Corollary 3.26 (a) will do so for certain idealizations $A = R(+)E$. In view of the more technical requirements for A to be a PVR when the ring R is not a domain (for which, see the statement of [7, Theorem 3.2 (b)]), we will, in the interest of simplicity, restrict R to being a domain. The next result gleans some useful facts from [7].

Proposition 3.25. Let R be a domain with quotient field K and let E be an R-module. Put $A := R(+)E$. Then:

(a) The following two conditions are equivalent:

(1) A is a PVR;

(2) R is a PVD and E is a divisible R-module.

(b) Suppose, in addition, that E is an overring of R. Then the following conditions are equivalent:

(i) A is a PVR;

(ii) R is a PVD and $E = K$.

Proof. For (a), apply [7, Theorem 3.1]; for (b), apply [7, Theorem 3.2]. □

The domain-theoretic assumption in Proposition 3.25 has substantially simplified matters. Indeed, note that the ring $A = R(+)R$ in Example 3.23 is not a PVR (it is not even a divided ring), although R is a PVR, $R = \mathrm{tq}(R)$, and R is a divisible R-module.

Corollary 3.26. (a) Let R be a domain and E a torsion-free R-module such that $A := R(+)E$ is a PVR. Then A is a strongly divided ring in the first sense and A is a strongly divided ring in the second sense.

(b) Let R be a domain with quotient field K, and let E be a vector space over K. Put $A := R(+)E$. Then A is a PVR if and only if R is a PVD.

Proof. (a) Theorem 3.24 gives both assertions (regardless of whether the ring R is a domain or the module E is torsion-free). We next give an alternate direct proof of the "first sense" assertion. Note, by Proposition 2.1 (b), that R inherits the "quasi-local" property from A. Therefore, by Corollary 3.6, it suffices to prove that R is a strongly divided domain and E is a vector space over $K := \mathrm{qf}(R)$. As Proposition 3.25 (a) ensures that R is a PVD, it follows from [4, Proposition 1 (c)] that R is a strongly divided domain. Next, note that E is a divisible R-module, by Proposition 3.25 (a). Hence, by Proposition 2.4 (a), E is a module over R_S, where $S := R \setminus (Z(R) \cup Z(E))$. Since E is torsion-free over R, we have $S = R \setminus Z(R)$, whence $R_S = \mathrm{tq}(R) = K$.

(b) By Proposition 3.25 (a), it suffices to prove that E is a divisible R-module. This, in turn, follows directly from Proposition 2.4 (d) since E is a module over K $(= \mathrm{tq}(R))$. (For an alternate final step in this proof, one could apply Proposition 2.4 (c) after noting that E is a torsion-free R-module and, as in the proof of (a), $R_S = K$.) □

In the spirit of Corollary 3.11 and Remark 3.18, Proposition 3.28 will give an analogue of the result [4, Corollary 4] that an integrally closed quasi-local domain is a PVD if and only if each of its proper simple overrings is a going-down domain. For the definition of a going-down domain, see [8], [13]. Recall from [11] that a ring R is called a *going-down ring* if R/P is a going-down domain for each $P \in \mathrm{Spec}(R)$. Also, a domain is a going-down ring if and only if it is a going-down domain, by [11, Proposition 2.1 (a)]; the class of going-down rings is stable under the formation of factor rings [11, Proposition 2.1 (b)]; and each divided ring (in particular, each PVR) is a going-down ring [11, Remark (c), page 4] (cf. also [5, Corollary 3], [11, Proposition 2.1 (d)]).

Lemma 3.27. Let R be a ring and E an R-module. Put $A := R(+)E$. Then A is a going-down ring if and only if R is a going-down ring.

Proof. By Proposition 2.1 (b), A is a going-down ring $\Leftrightarrow A/(P(+)E)$ ($\cong R/P$) is a going-down domain for all $P \in \text{Spec}(R) \Leftrightarrow R/P$ is a going-down domain for all $P \in \text{Spec}(R) \Leftrightarrow R$ is a going-down ring. □

Proposition 3.28. Let (R, m) be a quasi-local domain with quotient field K and E a torsion-free R-module such that the idealization $A := R(+)E$ is integrally closed. Then the following conditions are equivalent:

(1) Each proper simple overring of A is a going-down ring;
(2) A is a pseudo-valuation ring;
(3) Each proper simple overring of R is a going-down domain;
(4) R is a pseudo-valuation domain.

Proof. By Corollary 2.2 (c), R is integrally closed and E is a vector space over K. Hence, by Proposition 2.1 (f), $\text{tq}(A) = K(+)E$. It follows easily that the set of proper simple overrings of A consists of the rings of the form $R[u](+)K$, as u runs over all the elements of $K \setminus R$. This is the set of all the rings of the form $D(+)K$ such that D is a proper simple overring of R. Therefore, by Lemma 3.27, (1) $\Leftrightarrow$ (3). Moreover, (3) $\Leftrightarrow$ (4) by [4, Corollary 4]. Finally, (2) $\Leftrightarrow$ (4) by Corollary 3.26 (b). □

Remark 3.29. By combining Corollary 3.7, Proposition 3.13, Proposition 3.14 and Corollary 3.26 (b), we have the following result. Let R be a strongly divided domain which is not a PVD (for instance, as in [4, Example 1, Example 2, Remark 4]) and let $K := \text{qf}(R)$. Then the ring $R(+)K$ is strongly divided in the first sense and strongly divided in the second sense, but $R(+)K$ is not a PVR.

An overring of a PVD need not be a divided domain. Nevertheless, Proposition 3.30 will establish a sufficient condition for an idealization which is a PVR to have the property that *each* of its overrings is a divided ring.

Proposition 3.30. Let R be a PVD with canonically associated valuation domain V and quotient field K. Suppose also that $R' = V$. Let E be a torsion-free R-module, and put $A := R(+)E$. Then the following conditions are equivalent:

(1) E is a vector space over K;
(2) A is a divided ring;
(3) Each overring of A (including A itself) is a divided ring;
(4) A is a PVR;
(5) Each overring of A (including A itself) is a PVR.

Proof. Any PVD is a divided domain [10, page 560]. In particular, R is a divided ring. Hence, (1) $\Rightarrow$ (2) by [6, Proposition 2.14]. On the other hand, since E is a torsion-free R-module, it follows from [6, Lemma 2.13] that (2) $\Rightarrow$ (1). Since each PVR is a divided ring, we get that (4) $\Rightarrow$ (2); and that (5) $\Rightarrow$ (3). Also, (3) $\Rightarrow$ (2) trivially; and (5) $\Rightarrow$ (4) trivially. It therefore suffices to prove that (2) $\Rightarrow$ (5).

Assume (2). As (2) $\Rightarrow$ (1) and E is a torsion-free R-module, it follows from Proposition 2.1 (f) that $\text{tq}(A) = K(+)E$. Hence, each overring B of A can be expressed as $B = D(+)E$ for some corresponding overring D of R. In view of Corollary 3.26 (b), B is a PVR if (and only if) D is a PVD. Therefore, we need only show that D is a PVD. In fact, *each* overring of R is a PVD because of the hypothesis that $R' = V$. To see this, note that it follows easily from [18, Corollary 2.15] that this hypothesis on the pseudo-valuation domain R is equivalent to R being an i-domain, in the sense of [18]. By definition of "i-domain", this means that the canonical function $\text{Spec}(D) \to \text{Spec}(R)$ is an injection for each overring D of R. Hence, for each such D, the ring extension $R \subseteq D$ satisfies INC. Thus, by [15, Theorem 1.7], each such D inherits the "pseudo-valuation domain" property from R, as desired. □

The proof of Proposition 3.30 used a crucial fact about i-domains. Recall also that each i-domain is a going-down domain [18, Proposition 2.12] (but the converse is false). In view of the role of going-down rings in some of the motivating material in the Introduction, it seems fitting to take our leave of that topic here by collecting three more relevant facts about it.

Remark 3.31. (a) Despite expectations that may have been raised by combining Corollary 3.26 and Theorem 3.24, there exists a ring R which is strongly divided in the first sense but is not a going-down ring. Indeed, consider the ring R constructed in [11, Example 2, page 11]. It was shown in [11] that $\mathrm{tq}(R) = R$ and that R is not a going-down ring. The assertion now follows from Proposition 3.3.

(b) If R is a strongly divided ring in the second sense, then R is a going-down ring. (This generalizes, but uses, the fact that each strongly divided domain is a going-down domain.) For a proof, let $P \in \mathrm{Spec}(R)$. One must show that R/P is a going-down domain. This conclusion is however clear since R/P is a strongly divided domain.

(c) It seems natural to ask what can be concluded from the context of Proposition 3.28 if one deletes the hypothesis that the idealization A is integrally closed. By using Lemma 3.27 and reworking some of the proof of Proposition 3.28, one can prove the following. (Note the additional hypothesis on the module E.) Let R be a domain with quotient field K, let E be a vector space over K, and put $A := R(+)E$. Then each proper simple overring (resp., each overring) of A is a going-down ring if and only if each proper simple overring (resp., each overring) of R is a going-down domain.

In closing, we present a miscellanea of relevant facts. The first two of these involve PVRs. The final two parts of Remark 3.32 are motivated by the following special case of [4, Corollary 1]: if R is a strongly divided domain, then each integral overring of R is a locally divided domain. Recall from [9] (resp., [6]) that a domain (resp., ring) R is said to be a *locally divided domain* (resp., *locally divided ring*) if R_P is a divided domain (resp., divided ring) for each $P \in \mathrm{Spec}(R)$ (equivalently, for each maximal ideal P of R).

Remark 3.32. (a) Perhaps the easiest example of a pseudo-valuation domain R satisfying the condition "$R' = V$" that was assumed in Proposition 3.30 is provided by $\mathbb{Q} + X\mathbb{Q}(\sqrt{2})[[X]]$ (where, here and below, X denotes an analytic indeterminate over the ambient base ring). Note that this PVD is not integrally closed. While any valuation domain gives an example of an integrally closed PVD that satisfies the "$R' = V$" condition, it need not be the case that an integrally closed PVD must satisfy this condition. To see this, consider the oft-cited example $k + Xk(Y)[[X]]$, where k is a field and Y is an indeterminate over k (cf. [12, Remark 3.2 (b)]).

(b) Despite Corollary 3.26 (a) and Proposition 3.30, a quasi-local ring A that is strongly divided in the first sense and also strongly divided in the second sense need not be a PVR. This was shown in Remark 3.29, where the relevant ring A was an idealization of positive (Krull) dimension. By way of contrast, note that [12, Proposition 2.3] provides a family of examples (the easiest of which is $\mathbb{Z}/8\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$) of quasi-local zero-dimensional rings (hence, rings that are strongly divided in the first sense and strongly divided in the second sense) that are not PVRs. Note that these rings, being total quotient rings, are also trivially integrally closed. On the other hand, the ring A built in Remark 3.29 by using Example 1 (resp., Example 2) of [4] is not (resp., is) integrally closed.

(c) Let R be a domain with quotient field K, let E be a vector space over K, and put $A := R(+)E$. If A is strongly divided in the second sense, then each integral overring of A is a locally divided ring. For a proof, note first via Proposition 2.1 (g) that $A' = R'(+)E$. Then one can reason as in the proof of Proposition 3.30 to reduce our task to showing that if D is any integral overring of R, then $B = D(+)E$ is a locally divided ring. By [6, Proposition 2.14], this is equivalent to proving that each integral overring D of R is a locally divided ring; that is, that each such D is a locally divided domain. Since A is strongly divided in the second sense, so is R (by Proposition 3.14); that is, R is a strongly divided domain, by Proposition 3.13. Hence, by [4, Corollary 1], each integral overring of R is a locally divided domain, as desired.

(d) Let R be a domain and put $A := R(+)R$. Then (regardless of whether A is strongly divided in the second sense) if R is not a field, then A is not a locally divided ring. For a proof, let $K := \mathrm{qf}(R)$. Suppose, on the contrary, that A is a locally divided ring. Then, by [6, Corollary 2.17 (a)], R_M is a vector space over K, for each maximal ideal M of R. Thus, any such M must be 0, and so R is a field, the desired contradiction. This completes the proof.

For the sake of completeness, we conclude by observing that if R is any strongly divided domain which is not a field (for instance, $\mathbb{Z}_{2\mathbb{Z}}$), then $A := R(+)R$ is a strongly divided ring in the second sense (by Propositions 3.14 and 3.13) and, by the above, A is not a (locally) divided ring.

References

[1] T. Akiba, A note on AV-domains, Bull. Kyoto Univ. Education Ser. B **31** (1967), 1–3.

[2] D. F. Anderson, A. Badawi and D. E. Dobbs, Pseudo-valuation rings, pp. 57–67, in Commutative Ring Theory, II, Lecture Notes Pure Appl. Math. **185**, Dekker, New York, 1997.

[3] D. F. Anderson, A. Badawi and D. E. Dobbs, Pseudo-valuation rings, II, Boll. Un. Mat. **8(3-B)** (2000), 535–545.

[4] A. Ayache and D. E. Dobbs, Strongly divided domains, Ric. Mat., **65**(1) (2016), 127–154.

[5] A. Badawi, On divided commutative rings, Comm. Algebra **27** (1999), 1465–1474.

[6] A. Badawi and D. E. Dobbs, On locally divided rings and going-down rings, Comm. Algebra **29** (2001), 2805–2825.

[7] A. Badawi and D. E. Dobbs, Some examples of locally divided rings, pp. 73–83, in Ideal Theoretic Methods in Commutative Algebra, Lecture Notes Pure Appl. Math. **220**, Dekker, New York, 2001.

[8] D. E. Dobbs, On going-down for simple overrings, II, Comm. Algebra **1** (1974), 439–458.

[9] D. E. Dobbs, Divided rings and going down domains, Pac. J. Math. **67** (1976), 353–363.

[10] D. E. Dobbs, Coherence, ascent of going-down, and pseudo-valuation domains, Houston. J. Math. **4** (1978), 551–567.

[11] D. E. Dobbs, Going-down rings with zero-divisors, Houston. J. Math. **23** (1997), 1–12.

[12] D. E. Dobbs, On analogues involving zero-divisors of a domain-theoretic result of Ayache, Pales. J. Math. **3** (Spec 1) (2014), 400–405.

[13] D. E. Dobbs and I. J. Papick, On going-down for simple overrings, III, Proc. Amer. Math. Soc. **54** (1976), 35–38.

[14] D. E. Dobbs and I. J. Papick, Going-down: a survey, Nieuw Arch. v. Wisk. **26** (1978), 255–291.

[15] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, Pac. J. Math. **75** (1978), 137–147.

[16] J. A. Huckaba, Commutative Rings with Zero Divisors, Dekker, New York, 1988.

[17] I. Kaplansky, Commutative Rings, rev. ed., Univ. Chicago Press, Chicago, 1974.

[18] I. J. Papick, Topologically defined classes of going-down domains, Trans. Amer. Math. Soc. **219** (1976), 1–37.

Author information

Ahmed Ayache, University of Bahrain, Faculty of Science, Department of Mathematics, P. O. Box 32038, Sukhir, Bahrain.
E-mail: aaayache@yahoo.com

David E. Dobbs, Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, U.S.A.
E-mail: dobbs@math.utk.edu

Received: March 16, 2016.
Accepted: September 11, 2016.