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CERTAIN SPECIAL SUBCLASSES OF ANALYTIC FUNCTION ASSOCIATED WITH BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we have established and studied two new subclasses of bi-univalent functions defined in the open unit disc U. Furthermore, we find Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for these new subclasses .

1 Introduction

Let A denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of A, which consists of functions of the form (1.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.

A function $f \in S$ is said to be starlike of order α $(0 \leq \alpha < 1)$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U$$

and is convex of order α $(0 \leq \alpha < 1)$ if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U.$$

Denote these classes respectively by $S^*(\alpha)$ and $K(\alpha)$.

It is well known by the Koebe one quarter theorem [4] that the image of U under every function $f \in S$ contains a disc of radius $\frac{1}{4}$. Thus every univalent function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z, \ \ z \in U$ and $f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f); \ \ r_0(f) \geq \frac{1}{4}\right)$.
The inverse of $f(z)$ has a series expansion in some disc about the origin of the form

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \dots \ . \tag{1.2}$$

A function $f(z)$ univalent in a neighborhood of the origin and its inverse satisfy the condition $f\left(f^{-1}(w)\right) = w$.
Using (1.1), we have

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \dots \ . \tag{1.3}$$

Now using (1.2), we get following result

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \ . \tag{1.4}$$

A function $f \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let $\sum$ denote the class of bi-univalent functions in U given by (1.1).
Some examples of functions in the class $\sum$ are

$$\frac{z}{1-z}, \; -log(1-z), \; \frac{1}{2}log\left(\frac{1+z}{1-z}\right) \; and \; so \; on.$$

However, the familiar Koebe function is not bi-univalent. Also functions in S such as $\frac{2z-z^2}{2}$ and $\frac{z}{1-z^2}$ are not bi-univalent functions (see[10]).

In [6] Lewin first investigated the class $\sum$ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [7], on the other hand showed that $\max_{f\in\sum} |a_2| = \frac{4}{3}$.

The coefficient estimate problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n \geq 3;\ n \in \mathbb{N})$ for each $f \in \sum$ given by (1.1) is still an open problem.

In [3] Brannan and Taha introduced certain subclasses of bi-univalent function class $\sum$ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of the univalent function class S. Thus following Brannan and Taha [3], a function $f \in A$ of the form (1.1) is in the class $S^*_{\sum}(\alpha)$ $(0 < \alpha \leq 1)$ of strongly bi-starlike functions of order α if it satisfies following conditions :

$$f \in \sum \; and \; \left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U; 0 < \alpha \leq 1)$$

$$and \; \left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in U; 0 < \alpha \leq 1),$$

where g is extension of f^{-1} to U. The classes $S^*_{\sum}(\alpha)$ and $K_{\sum}(\alpha)$ of bi-starlike function of order α and bi-convex function of order α respectively, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_{\sum}(\alpha)$ and $K_{\sum}(\alpha)$, they found non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ (for details see [3]).

In [9] Srivastava et al. introduced two new subclasses of analytic and bi-univalent functions as follows :
Definition 1.1 A function $f(z)$ given by (1.1) is said to be in the class $H_{\sum}(\alpha)$ if the following conditions are satisfied :

$$f \in \sum \; and \; |arg(f'(z))| < \frac{\alpha\pi}{2} \qquad (z \in U)$$

$$and \; |arg(g'(w))| < \frac{\alpha\pi}{2} \qquad (w \in U),$$

where $0 < \alpha \leq 1$ and the function g is extension of f^{-1} to U and is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \; .$$

Definition 1.2 A function $f(z)$ given by (1.1) is said to be in the class $H_{\sum}(\beta)$ if the following conditions are satisfied :

$$f \in \sum \; and \; Re\,(f'(z)) > \beta \quad (z \in U)$$

$$and \; Re\,(g'(w)) > \beta \quad (w \in U),$$

where $0 \leq \beta < 1$, and the function g is extension of f^{-1} to U and is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \; .$$

In [1] Babalola defined the class $\mathfrak{L}_\lambda(\beta)$ of λ-pseudo starlike functions of order β as follows.
Definition 1.3 Let $f \in A$, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real then $f(z) \in \mathfrak{L}_\lambda(\beta)$ in the unit

disc U if and only if

$$Re\left(\frac{z[f'(z)]^{\lambda}}{f(z)}\right) > \beta \qquad (z \in U).$$

Also in [1] Babalola proved that all pseudo-starlike functions are Bazilevic of type $\left(1-\frac{1}{\lambda}\right)$, order $\beta^{\frac{1}{\lambda}}$ and univalent in open disc U. Recently, Joshi et al. [5] introduced and investigated the subclasses of bi-univalent functions associated with pseudo starlike functions.
Motivated by aforementioned work of Babalola [1], we introduce two new subclasses of bi-univalent function classes $H_{\sum}^{\alpha,\lambda}$ and $H_{\sum}^{\lambda}(\beta)$ which is similar type to λ-pseudo starlike functions. We estimates on the initial coefficients $|a_2|$ and $|a_3|$ for these two new subclasses of bi-univalent functions. The techniques used are same as Srivastava et al.[10].

In order to derive our main results, we have to recall here the following Lemma.
Lemma 1.1 [8] Let $h \in P$ the family of all functions h analytic in U for which $Re\{h(z)\} > 0$ and have the form

$$h(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \;\; for \;\; z \in U.$$

Then $|p_n| \leq 2$ for each n.

2 Coefficient bounds for the function class $H_{\sum}^{\alpha,\lambda}$.

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $H_{\sum}^{\alpha,\lambda}$ if the following conditions are satisfied :

$$f \in \sum \;\; and \;\; \left|arg(f'(z))^{\lambda}\right| < \frac{\alpha\pi}{2} \quad (z \in U) \tag{2.1}$$

$$and \;\; |arg\,(g'(w))^{\lambda}| < \frac{\alpha\pi}{2} \quad (w \in U), \tag{2.2}$$

where $0 < \alpha \leq 1$, $\lambda > 0$ and the function g is extension of f^{-1} to U and is given by

$$g(w) = w - a_2 w^2 + [2a_2^2 - a_3]w^3 + ... \;\; . \tag{2.3}$$

We state and prove the following results.

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $H_{\sum}^{\alpha,\lambda}$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\lambda(2\lambda+\alpha)}} \tag{2.4}$$

and

$$|a_3| \leq \frac{\alpha(2\lambda+3\alpha)}{3\lambda^2}. \tag{2.5}$$

Proof. We can write the argument inequality in (2.1) and (2.2) as

$$[f'(z)]^{\lambda} = [p(z)]^{\alpha} \tag{2.6}$$

and

$$[g'(w)]^{\lambda} = [q(w)]^{\alpha} \tag{2.7}$$

respectively.
Where $p(z)$ and $q(w)$ satisfy the inequalities $Re(p(z)) > 0 \;\; (z \in U)$ and $Re(q(w)) > 0 \;\; (w \in U)$. Furthermore the functions $p(z), q(w) \in P$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots \ .$$

Clearly,

$$[p(z)]^\alpha = 1 + \alpha p_1 z + \left(\alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2\right) z^2 + \dots \tag{2.8}$$

and

$$[q(w)]^\alpha = 1 + \alpha q_1 w + \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2\right) w^2 + \dots \ . \tag{2.9}$$

Also

$$[f'(z)]^\lambda = 1 + 2\lambda a_2 z + [3\lambda a_3 + 2\lambda(\lambda-1)a_2^2]z^2 + \dots \tag{2.10}$$

and

$$[g'(w)]^\lambda = 1 - 2\lambda a_2 w + [(2\lambda^2 + 4\lambda)a_2^2 - 3\lambda a_3]w^2 + \dots \ . \tag{2.11}$$

Now equating the coefficients in (2.6) and (2.7) we get

$$2\lambda a_2 = \alpha p_1 \ , \tag{2.12}$$

$$3\lambda a_3 + 2\lambda(\lambda-1)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2 \ , \tag{2.13}$$

$$-2\lambda a_2 = \alpha q_1 \ , \tag{2.14}$$

$$(2\lambda^2 + 4\lambda)a_2^2 - 3\lambda a_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2 \ . \tag{2.15}$$

From equations (2.12) and (2.14) we get

$$p_1 = -q_1 \tag{2.16}$$

and

$$8\lambda^2 a_2^2 = \alpha^2(p_1^2 + q_1^2) \ . \tag{2.17}$$

Now by adding equations (2.13) and (2.15), we get

$$(4\lambda^2 + 2\lambda)a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 + q_1^2\right),$$

by using (2.17), we get

$$(4\lambda^2 + 2\lambda)a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}\left(\frac{8\lambda^2 a_2^2}{\alpha^2}\right)$$

$$\Rightarrow a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\lambda(2\lambda + \alpha)}.$$

Applying Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2| \le \frac{2\alpha}{\sqrt{2\lambda(2\lambda + \alpha)}}.$$

This gives the bound on $|a_2|$ as given in (2.4).
Next, in order to find the bound on $|a_3|$, by subtracting (2.15) from (2.13) we get

$$6\lambda a_3 - 6\lambda a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 - q_1^2\right).$$

From (2.16) we get $p_1^2 = q_1^2$ and also using (2.17) we have

$$6\lambda a_3 = \frac{3\alpha^2 p_1^2}{2\lambda} + \alpha(p_2 - q_2)$$

$$a_3 = \frac{\alpha^2 p_1^2}{4\lambda^2} + \frac{\alpha(p_2 - q_2)}{6\lambda}.$$

Applying Lemma 1 for the coefficients p_1, p_2 and q_2, we get

$$|a_3| \leq \frac{\alpha(2\lambda + 3\alpha)}{3\lambda^2}.$$

This completes the proof of Theorem 1.

□

3 Coefficient bounds for the function class $H_{\sum}^{\lambda}(\beta)$.

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $H_{\sum}^{\lambda}(\beta)$ if the following conditions are satisfied :

$$f \in \sum \ \ and \ \ Re[(f'(z))^{\lambda}] > \beta \tag{3.1}$$

$$and \ \ Re[(g'(w))^{\lambda}] > \beta. \tag{3.2}$$

where $z \in U$, $w \in U$, $0 \leq \beta < 1$, $\lambda > 0$ and the function g is defined in (2.3). For functions in the class $H_{\sum}^{\lambda}(\beta)$ the following coefficient estimates hold.

Theorem 3.1. *Let $f(z)$ given by (1.1) be in the class $H_{\sum}^{\lambda}(\beta)$. Then*

$$| \, a_2 \, | \leq \sqrt{\frac{2(1-\beta)}{\lambda(2\lambda+1)}} \tag{3.3}$$

and

$$| \, a_3 \, | \leq \frac{(1-\beta)(2\lambda - 3\beta + 3)}{3\lambda^2}. \tag{3.4}$$

Proof. First of all, the argument inequalities in (3.1) and (3.2) can be written in their equivalent forms as :

$$(f'(z))^{\lambda} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$(g'(w))^{\lambda} = \beta + (1-\beta)q(w) \tag{3.6}$$

respectively.
Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ... \ .$$

Clearly,

$$\beta + (1-\beta)p(z) = 1 + (1-\beta)p_1 z + (1-\beta)p_2 z^2 + ...$$

and

$$\beta + (1-\beta)q(w) = 1 + (1-\beta)q_1 w + (1-\beta)q_2 w^2 + \ldots \ .$$

Also

$$(f'(z))^\lambda = 1 + 2\lambda a_2 z + [3\lambda a_3 + 2\lambda(\lambda-1)a_2^2]z^2 + \ldots$$

and

$$(g'(w))^\lambda = 1 - 2\lambda a_2 w + [(2\lambda^2 + 4\lambda)a_2^2 - 3\lambda a_3]w^2 + \ldots \ .$$

Now, equating the coefficients in (3.5) and (3.6), we get

$$2\lambda a_2 = (1-\beta)p_1 \ , \tag{3.7}$$

$$3\lambda a_3 + 2(\lambda^2 - \lambda)a_2^2 = (1-\beta)p_2 \ , \tag{3.8}$$

$$-2\lambda a_2 = (1-\beta)q_1 \ , \tag{3.9}$$

$$(2\lambda^2 + 4\lambda)a_2^2 - 3\lambda a_3 = (1-\beta)q_2. \tag{3.10}$$

From equations (3.7) and (3.9), we have

$$p_1 = -q_1 \tag{3.11}$$

and

$$8\lambda^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2) \ . \tag{3.12}$$

Now, by adding equations (3.8) and (3.10), we get

$$(4\lambda^2 + 2\lambda)a_2^2 = (1-\beta)(p_2 + q_2)$$

$$\Rightarrow |a_2^2| \le \frac{(1-\beta)(|p_2|+|q_2|)}{(4\lambda^2 + 2\lambda)} \ .$$

Applying Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{\lambda(2\lambda+1)}}.$$

Which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$6\lambda a_3 - 6\lambda a_2^2 = (1-\beta)(p_2 - q_2).$$
$$6\lambda a_3 = 6\lambda a_2^2 + (1-\beta)(p_2 - q_2).$$

From (3.11), we get $p_1^2 = q_1^2$ and also using (3.12) we have

$$a_3 = \frac{(1-\beta)^2(p_1^2)}{4\lambda^2} + \frac{(1-\beta)(p_2 - q_2)}{6\lambda}.$$

Applying Lemma 1 for the coefficients p_1, p_2 and q_2, we get

$$|\ a_3\ | \le \frac{(1-\beta)(2\lambda - 3\beta + 3)}{3\lambda^2}.$$

This completes the proof of Theorem 2.

□

By specializing the parameter in this work we get result studied by earlier author.

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