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Finite element approximations of bifurcation problem for Marguerre-von Kármán equations

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Abstract In this work, we consider the bifurcation problem for Marguerre-von Kármán equations, which constitute a mathematical model for the buckling of a nonlinearly thin elastic shallow shell, subjected to boundary conditions of von Kármán type on its lateral face. First, we reduce the continuous problem of these equations to a single equation with cubic operator, whose unknowns are the vertical displacement and the intensity of the lateral compression. Then, we solve this equation by adapting a Kikuchi's method [22]. Next, we establish the convergence of a conforming finite element approximations of non trivial solutions bifurcating from the trivial solution at neighborhood of the simple eigenvalues of the linearized problem. Then, we obtain the corresponding error estimates.

1 Introduction

The two-dimensional Marguerre-von Kármán equations for nonlinearly elastic shallow shells were originally proposed by Marguerre [24] in 1938 and von Kármán and Tsien [30] in 1939, they generalize the equations of von Kármán for thin elastic plates proposed by von Kármán [29] in 1910.

In 1986, Ciarlet and Paumier [14] justified the classical Marguerre-von Kármán equations by means of a formal asymptotic analysis. Then, in 2002, Gratie [17] has generalized these equations, where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type, the remaining portion being free. She showed that the leading term of the asymptotic expansion is characterized by a two-dimensional boundary value problem called generalized Marguerre-von Kármán equations. In 2006, Ciarlet and Gratie [11] have established an existence theorem for these equations.

The questions of existence, uniqueness, regularity and stability for these equations. We quote the works carried out by Kesavan and Srikanth [21], Kavian and Rao [18], Rao [26, 27], Léger and Miara [23], Devdariani, Janjgava and Gulua [15].

In the same way but for the dynamical case, we quote the previous works [7, 8], when we identified the dynamical equations for generalized Marguerre-von Kármán shallow shells and we established the existence of solutions to these equations. In the same way but for the unilateral contact case, we quote our work [1] for justification of the generalized Marguerre-von Kármán equations with Signorini conditions.

For numerical approximations, some studies have been done for the von Kármán equations. Miyoshi [25] studied the mixed finite element method for these equations. Kesavan [19, 20] proposed an iterative finite element method of the bifurcation problem for von Kármán equations, by adapting a Kikuchi's method [22], and mixed finite element method for the same problem. Brezzi [4] and Brezzi et al. [5, 6] analyzed a finite element approximations of von Kármán equations and studied a Hellan-Herrmann-Johnson mixed finite element scheme for the von Kármán equations. Reinhart [28] proposed an approximation of the von Kármán equations using a Hermann-Miyoshi finite element scheme. Ciarlet et al. [13] studied the finite element method for the generalized von Kármán equations. Recently in [16], we established the convergence of a conforming finite element approximations to the generalized Marguerre-von Kármán equations.

The objective of this study is to extend the results which studied by Kesavan [19] for the von

Kármán plate to the Marguerre-von Kármán shallow shell.

2 Classical Marguerre-von Kármán equations

Let ω be a bounded and simply-connected open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ. As shown in [14], the classical Marguerre-von Kármán equations are written as

$$\frac{2E}{3(1-\nu^2)}\triangle^2\zeta_3 = 2[\varphi,\zeta_3+\theta]+p_3 \text{ in } \omega,$$

$$\triangle^2\varphi = -\frac{E}{2}[\zeta_3,\zeta_3+2\theta] \text{ in } \omega,$$
$$\zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma,$$
$$\varphi = \varphi_0 \text{ and } \partial_\nu\varphi = \varphi_1 \text{ on } \gamma,$$

where

$$\varphi_0(y) = -\gamma_1\int_{\gamma(y)} h_2 d\gamma + \gamma_2\int_{\gamma(y)} h_1 d\gamma + \int_{\gamma(y)} (x_1h_2 - x_2h_1)d\gamma,\ y\in\gamma,$$
$$\varphi_1(y) = -\nu_1\int_{\gamma(y)} h_2 d\gamma + \nu_2\int_{\gamma(y)} h_1 d\gamma,\ y\in\gamma,$$
$$[\eta,\xi] = \partial_{11}\eta\partial_{22}\xi + \partial_{22}\eta\partial_{11}\xi - 2\partial_{12}\eta\partial_{12}\xi.$$

The known functions θ and p_3 are, the function that defines the middle surface of the shell and the resultant of the vertical forces acting on the shell respectively. The functions φ_0 and φ_1 are known functions of the appropriately density $(h_\alpha):\gamma\to\mathbb{R}^2$ of the resultant of the horizontal forces acting on the lateral face of the shell. The constants E and ν are respectively the Young modulus and the Poisson coefficient of the elastic material constituting of the shell. The unknown $\zeta_3:\bar\omega\to\mathbb{R}$ is the vertical component of the displacement field of the middle surface of the shell and the unknown $\varphi:\bar\omega\to\mathbb{R}$ is the Airy function.

We consider here, the buckling of a nonlinearly thin elastic shallow shell under the compressive forces of von Kármán's type applied on its lateral face, such that, before deformation this forces is collinear to the normal of γ, and λ is a parameter measuring the magnitude of this forces, denotes the intensity of the lateral compression. In this case, the Airy function be given by $\varphi+\lambda\theta_0$, where the function $\theta_0\in H_0^2(\omega)$ is the unique solution of the boundary value problem:

$$\triangle^2\theta_0 = 0 \text{ in } \omega,$$

$$\theta_0 = \varphi_0 \text{ and } \partial_\nu\theta_0 = \varphi_1 \text{ on } \gamma,$$

such that $\varphi_0\in H^{\frac{5}{2}}(\omega)$ and $\varphi_1\in H^{\frac{3}{2}}(\omega)$.

The classical Marguerre-von Kármán equations becomes

$$\frac{2E}{3(1-\nu^2)}\triangle^2\zeta_3 = 2[\varphi,\zeta_3+\theta]+2\lambda[\theta_0,\zeta_3+\theta]+p_3 \text{ in } \omega,$$

$$\triangle^2\varphi = -\frac{E}{2}[\zeta_3,\zeta_3+2\theta] \text{ in } \omega,$$
$$\zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma,$$
$$\varphi = \partial_\nu\varphi = 0 \text{ on } \gamma.$$

Next, we write the classical Marguerre-von Kármán equations in a simpler form, using the following relations:

$\zeta_3 = (\frac{2}{3(1-\nu^2)})^{\frac{1}{2}}\xi$, $\varphi = \frac{E}{3(1-\nu^2)}\Phi$, $\theta = (\frac{2}{3(1-\nu^2)})^{\frac{1}{2}}\tilde\theta$, $\theta_0 = \frac{E}{3(1-\nu^2)}\tilde\theta_0$ and $p_3 = (\frac{2}{3(1-\nu^2)})^{\frac{3}{2}}f$.

We find that the unknowns (ξ,Φ,λ) satisfy the canonical Marguerre-von Kármán equations:

$$\triangle^2\xi = [\Phi,\xi+\tilde\theta]+\lambda[\tilde\theta_0,\xi+\tilde\theta]+f \text{ in } \omega,$$

$$\triangle^2\Phi = -[\xi, \xi + 2\tilde{\theta}] \text{ in } \omega,$$
$$\xi = \partial_\nu \xi = 0 \text{ on } \gamma,$$
$$\Phi = \partial_\nu \Phi = 0 \text{ on } \gamma.$$

Following Ciarlet [10], we obtain

$$\tilde{\theta}_0 = -\frac{3(1-\nu^2)}{2E}(x_1^2 + x_2^2). \tag{2.1}$$

3 The continuous problem

3.1 The continuous operator equation

We assume without loss of generality that $\tilde{\theta} \in H_0^2(\omega)$ and $(\Delta^2)^{-1} f = -\tilde{\theta}$, where $(\Delta^2)^{-1}$ is the inverse of Δ^2 with homogenous Dirichlet boundary condition in ω.

First, we let the bilinear continuous operator:

$$B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega),$$

be defined as follows: for each pair $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B(\xi, \eta) \in H_0^2(\omega)$ is the unique solution of the boundary value problem:

$$\Delta^2 B(\xi, \eta) = [\xi, \eta] \ \text{ in } \omega, \tag{3.1}$$

$$B(\xi, \eta) = \partial_\nu B(\xi, \eta) = 0 \text{ on } \gamma. \tag{3.2}$$

Next, we denote the cubic nonlinear operator $C : H_0^2(\omega) \to H_0^2(\omega)$ be defined by

$$C(\eta) = B(B(\eta, \eta), \eta), \tag{3.3}$$

and the linear operator $L_1 : H_0^2(\omega) \to H_0^2(\omega)$ be defined by

$$\begin{aligned} L_1\eta &= B(\tilde{\chi}, \eta) \\ &= B(B(\tilde{\theta}, \eta), \tilde{\theta}), \end{aligned}$$

where $\tilde{\chi} = B(\tilde{\theta}, \tilde{\theta})$.

Also, we denote the linear operator $L_2 : H_0^2(\omega) \to H_0^2(\omega)$ be defined by

$$L_2\eta = B(\tilde{\theta}_0, \eta). \tag{3.4}$$

Taking into account (2.1), we deduce that

$$\Delta^2 L_2\eta = -\frac{3(1-\nu^2)}{E}\Delta\eta. \tag{3.5}$$

Finally, we denote $\tilde{\xi} = \xi + \tilde{\theta}$, then, the canonical Marguerre-von Kármán equations are reduced to a cubic operator equation, such that the pair $(\tilde{\xi}, \lambda) \in H_0^2(\omega) \times \mathbb{R}$ satisfies the continuous operator equation:

$$\tilde{\xi} - \lambda L_2\tilde{\xi} + C(\tilde{\xi}) - L_1\tilde{\xi} = 0, \tag{3.6}$$

and the Airy function $\Phi \in H_0^2(\omega)$ is given by

$$\Phi = \tilde{\chi} - B(\tilde{\xi}, \tilde{\xi}). \tag{3.7}$$

For more details about this operators, see [10].

Noting that, finding the solution $\tilde{\xi}$ of the above operator equation (3.6) is equivalent to solving the following variational problem:

$$\begin{cases} \text{Find } (\tilde{\xi}, \lambda) \in H_0^2(\omega) \times \mathbb{R} \text{ such that,} \\ (\tilde{\xi} - \lambda L_2\tilde{\xi} + C(\tilde{\xi}) - L_1\tilde{\xi}, \eta)_\Delta = 0 \text{ for all } \eta \in H_0^2(\omega), \end{cases}$$

where $(.,.)_\Delta$ is the inner-product on $H_0^2(\omega)$ defined by $(\zeta,\eta)_\Delta = \int_\omega \Delta\zeta\Delta\eta d\omega$ and let $\|.\|_\Delta$ denote the associated norm.

The cubic operator equation (3.6) generalizes an operator equation originally introduced by Berger [2] and Berger and Fife [3], then used by Kesavan [19], Ciarlet, Gratie and Sabu [12], Ciarlet and Gratie [11], Ciarlet, Gratie and Kesavan [13] for analyzing the generalized von Kármán and Marguerre-von Kármán equations.

The linearized problem for the canonical Marguerre-von Kármán equations consists in finding the pair
$(\phi,\lambda) \in H_0^2(\omega)\times\mathbb{R}$ such that

$$\phi = \lambda L_2\phi,$$

or

$$\triangle^2\phi = \lambda[\tilde{\theta}_0,\phi] \text{ in } \omega,$$

$$\phi = \partial_\nu\phi = 0 \text{ on } \gamma.$$

Since the linear operator L_2 is a compact, self-adjoint and positive definite (see [10]), we conclude that L_2 has an infinite number of distinct eigenvalues $\lambda_k > 0$, each of finite multiplicity, such that

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_k < ... \to +\infty,$$

where

$$\lambda_1 = \inf_{\substack{\eta\in H_0^2(\omega)\\ \eta\neq 0}} \frac{\|\eta\|_\Delta^2}{(L_2\eta,\eta)_\Delta}.$$

3.2 Existence and regularity results

Let ϕ_0 be an normalized eigenfunction of the operator L_2 corresponding to the simple eigenvalue λ_0, in the sense

$$(L_2\phi_i,\phi_j)_\Delta = \delta_{ij}, \tag{3.8}$$

where ϕ_i be an eigenfunction of the operator L_2 corresponding to the eigenvalue λ_i.

Here, we adapt a Kikuchi's method used in [22] to the operator equation (3.6), consists in finding the solution (ζ,λ) of (3.6), with λ in the neighborhood of λ_0 and ζ of small norm, in the sense that

$$\zeta = \varepsilon\phi_0 + \upsilon, \tag{3.9}$$

where $\varepsilon > 0$ designates a parameter approaches zero.

Theorem 3.1. Let $\varepsilon > 0$ and $(\zeta,\lambda) \in H_0^2(\omega)\times\mathbb{R}$, with

$$\zeta = \varepsilon\phi_0 + \upsilon,\ \upsilon \in \{\phi_0\}^\perp. \tag{3.10}$$

Then (ζ,λ) be solution of the operator equation (3.6) if and only if

$$\upsilon = QS_\varepsilon(\zeta) \tag{3.11}$$

and

$$\lambda = \lambda_0 + \frac{1}{\varepsilon}(C(\zeta) - L_1\zeta,\phi_0)_\Delta, \tag{3.12}$$

where

$$S_\varepsilon\eta = \frac{1}{\varepsilon}(C(\eta) - L_1\eta,\phi_0)_\Delta L_2\eta - C(\eta) + L_1\eta, \tag{3.13}$$

and the mapping $Q : H_0^2(\omega) \to \{\phi_0\}^\perp$ be defined by

$$(I - \lambda_0 L_2)Q\eta = P_0\eta, \tag{3.14}$$

such that P_0 is the orthogonal projection in $H_0^2(\omega)$ onto $\{\phi_0\}^\perp$.

For give the existence results, we use the following lemma

Lemma 3.2. We let the operator $\Lambda_\varepsilon : H_0^2(\Omega) \to \mathbb{R}$ be defined by

$$\Lambda_\varepsilon\eta = \lambda_0 + \frac{1}{\varepsilon}(C(\eta) - L_1\eta, \phi_0)_\Delta, \tag{3.15}$$

such that

$$S_\varepsilon\eta = (\Lambda_\varepsilon\eta - \lambda_0)L_2\eta - C(\eta) + L_1\eta,\ \forall\eta \in H_0^2(\omega). \tag{3.16}$$

and we assume here that $\|\tilde{\theta}\|_\Delta \leq c\varepsilon$. Then, there exists a constant c independent of ε, such that

$$\|Q\eta\|_\Delta \leq c\|\eta\|_\Delta,\ \forall\eta \in H_0^2(\omega), \tag{3.17}$$

$$\|L_1\zeta_1 - L_1\zeta_2\|_\Delta \leq c\varepsilon^2\|\zeta_1 - \zeta_2\|_\Delta,\ \forall\zeta_i \in U_\varepsilon, \tag{3.18}$$

$$\|C(\zeta_1) - C(\zeta_2)\|_\Delta \leq c\varepsilon^2\|\zeta_1 - \zeta_2\|_\Delta,\ \forall\zeta_i \in U_\varepsilon, \tag{3.19}$$

$$|\Lambda_\varepsilon\zeta - \lambda_0| \leq c\varepsilon^2,\ \forall\zeta \in U_\varepsilon, \tag{3.20}$$

$$|\Lambda_\varepsilon\zeta_1 - \Lambda_\varepsilon\zeta_2| \leq c\varepsilon\|\zeta_1 - \zeta_2\|_\Delta,\ \forall\zeta_i \in U_\varepsilon, \tag{3.21}$$

$$\|S_\varepsilon\zeta\|_\Delta \leq c\varepsilon^3,\ \forall\zeta \in U_\varepsilon, \tag{3.22}$$

$$\|S_\varepsilon\zeta_1 - S_\varepsilon\zeta_2\|_\Delta \leq c\varepsilon^2\|\zeta_1 - \zeta_2\|_\Delta,\ \forall\zeta_i \in U_\varepsilon, \tag{3.23}$$

where

$$U_\varepsilon = \{\varepsilon\phi_0 + v;\ v \in V_\varepsilon\}, \tag{3.24}$$

$$V_\varepsilon = \{v \in H_0^2(\omega); \|v\|_\Delta \leq \varepsilon\}. \tag{3.25}$$

Theorem 3.3. For ε is small enough, then there exists a unique solution (ζ, λ) to the operator equation (3.6), with ζ is of the form $\varepsilon\phi_0 + v,\ v \in \{\phi_0\}^\perp \cap V_\varepsilon$, such that

$$\zeta \neq 0,\ \|\zeta\|_\Delta = O(\varepsilon),\ |\lambda - \lambda_0| = O(\varepsilon^2). \tag{3.26}$$

For obtaining the regularity results, we use the same arguments of Kesavan [19].

First, we assume, that ω is sufficiently smooth and if $g \in L^2(\omega)$, then the solution of the following problem

$$\Delta^2 u = g, \tag{3.27}$$

satisfies $u \in H^3(\omega) \cap H_0^2(\omega)$, with $\|u\|_{3,\omega} \leq c|g|_{0,\omega}$.

Then, we show the following results

Lemma 3.4. We assume that $\tilde{\theta}_0 \in W^{2,\infty}(\omega)$, then, if (ζ, λ) be solution of the operator equation (3.6), we have

$$\zeta \in H^3(\omega) \cap H_0^2(\omega). \tag{3.28}$$

Lemma 3.5. If $\zeta \in H^3(\omega) \cap H_0^2(\omega)$ and $\eta \in H^2(\omega)$, then we have

$$B(\zeta, \eta) \in H^{\frac{5}{2}}(\omega), \tag{3.29}$$

$$C(\zeta) \in H^3(\omega), \tag{3.30}$$

where

$$\|B(\zeta, \eta)\|_{\frac{5}{2},\omega} \leq c\|\zeta\|_{3,\omega}\|\eta\|_{2,\omega}, \tag{3.31}$$

$$\|C(\zeta)\|_{3,\omega} \leq c\|\zeta\|_{3,\omega}^2\|\zeta\|_\Delta. \tag{3.32}$$

Lemma 3.6. We assume that $\tilde{\chi} \in W^{2,\infty}(\omega)$, then, if $\zeta \in H_0^2(\omega)$, we have

$$L_1\zeta \in H^3(\omega), \tag{3.33}$$

where

$$\|L_1\zeta\|_{3,\omega} \leq c\|\zeta\|_\Delta. \tag{3.34}$$

Lemma 3.7. We assume that $\tilde{\theta}_0 \in W^{2,\infty}(\omega)$, then, if $\zeta \in H_0^2(\omega)$, we have

$$L_2\zeta \in H^3(\omega), \tag{3.35}$$

where

$$\|L_2\zeta\|_{3,\omega} \leq c\|\zeta\|_\Delta. \tag{3.36}$$

Finally, we give the following consequence of the lemmas 3.5, 3.6, 3.7, if $\zeta \in H^3(\omega) \cap U_\varepsilon$, we have

$$S_\varepsilon\zeta \in H^3(\omega), \tag{3.37}$$

where

$$\|S_\varepsilon\zeta\|_{3,\omega} \leq c\varepsilon^3 + c\varepsilon\|\zeta\|^2_{3,\omega}. \tag{3.38}$$

4 The discrete problem

Let ω is a convex, polygonal domain and $V_h \subset H_0^2(\omega)$ be standard conforming finite element space (see, e.g., [9]).

First, we assume that there exist a linear operator $r_h : H^3(\omega) \cap H_0^2(\omega) \to V_h$, such that

$$\|\eta - r_h\eta\|_\Delta \leq ch^{m-1}\|\eta\|_{m+1,\omega}, \tag{4.1}$$

for all $\eta \in H^{m+1}(\omega) \cap H_0^2(\omega)$, where $2 \leq m \leq l$ and $c > 0$ is a constant independent of h.

In particular, this assumption imply that

$$\lim_{h\to 0}\left(\inf_{\eta_h \in V_h} \|\eta - \eta_h\|_\Delta\right) = 0, \forall \eta \in H_0^2(\omega). \tag{4.2}$$

Next, we give the following property is due to Kesavan [19].

Lemma 4.1. If $\eta \in H^{\frac{5}{2}}(\omega)$, then there exist a constant $c > 0$ independent of h, such that

$$\inf_{\eta_h \in V_h} \|\eta - \eta_h\|_\Delta \leq ch^{\frac{1}{2}}\|\eta\|_{\frac{5}{2},\omega}. \tag{4.3}$$

4.1 The Linearized discrete problem

The linearized discrete problem for the Marguerre-von Kármán equations consists in finding the pair
$(\phi_h, \lambda_h) \in V_h \times \mathbb{R}$ such that

$$\int_\omega \Delta\phi_h \Delta\eta_h d\omega = \lambda_h \int_\omega [\tilde{\theta}_0, \phi_h]\eta_h d\omega,\ \forall \eta_h \in V_h, \tag{4.4}$$

or

$$(\phi_h, \eta_h)_\Delta = \lambda_h (L_2\phi_h, \eta_h)_\Delta,\ \forall \eta_h \in V_h. \tag{4.5}$$

If h is small enough, then there exists a simple eigenvalues λ_{oh} who approaches to λ_0 and there exists an eigenfunction ϕ_{0h}, corresponding to λ_{oh} who approaches to ϕ_0, such that

$$(L_2\phi_{0h}, \phi_{0h})_\Delta = 1. \tag{4.6}$$

For $\phi_0 \in H^3(\omega) \cap H_0^2(\omega)$, we have

$$\|\phi_0 - \phi_{0h}\|_\Delta \leq ch\|\phi_0\|_{3,\omega}, \tag{4.7}$$

$$|\lambda_0 - \lambda_{0h}| \leq ch^2 \|\phi_0\|_{3,\omega}. \tag{4.8}$$

Let $\varsigma \in H_0^2(\omega)$, then $Q_h\varsigma$ is the unique solution of the following problem

$$(Q_h\varsigma, \eta_h)_\Delta - \lambda_{0h}(L_2 Q_h\varsigma, \eta_h)_\Delta = (P_{0h}\varsigma, \eta_h)_\Delta, \ \forall \eta_h \in V_h, \tag{4.9}$$

$$(Q_h\varsigma, \phi_{0h})_\Delta = 0, \tag{4.10}$$

such that P_{0h} is the orthogonal projection in $H_0^2(\omega)$ onto $\{\phi_{0h}\}^\perp \cap V_h$.

Also, we give the following property is due to Kesavan [19].

Lemma 4.2. There exist a constant c independent of h, such that

$$\|Q_h\varsigma\|_\Delta \leq c\|\varsigma\|_\Delta, \ \forall \varsigma \in H_0^2(\omega), \tag{4.11}$$

$$\|Q\varsigma - Q_h\varsigma\|_\Delta \leq ch\|\varsigma\|_{3,\omega}, \ \forall \varsigma \in H^3(\omega) \cap H_0^2(\omega). \tag{4.12}$$

4.2 The discrete operator equation

Let the bilinear mapping: $B_h : H^2(\omega) \times H^2(\omega) \to V_h$ be defined as follows: for each pair $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B_h(\xi, \eta) \in V_h$ is the unique solution of the variational equation

$$(B_h(\xi, \eta), \varsigma_h)_\Delta = \int_\omega [\xi, \eta]\varsigma_h d\omega \text{ for all } \varsigma_h \in V_h, \tag{4.13}$$

hence, $B_h(\xi, \eta)$ be the orthogonal projection of $B(\xi, \eta)$ on V_h.

The discrete cubic operator $C_h : V_h \to V_h$ is defined by

$$C_h(\eta_h) = B_h(B_h(\eta_h, \eta_h), \eta_h) \tag{4.14}$$

The linear operator $L_{1,h} : V_h \to V_h$ be defined by

$$L_{1,h}\eta_h \quad = \quad B_h(\tilde{\chi}_h, \eta_h),$$

where $\tilde{\chi}_h = B_h(\tilde{\theta}, \tilde{\theta})$.

Also, the linear operator $L_{2,h} : V_h \to V_h$ be defined by

$$L_{2,h}\eta_h = B_h(\tilde{\theta}_0, \eta_h). \tag{4.15}$$

For each $h > 0$, the discrete problem of canonical Marguerre-von Kármán equations consists in finding $(\tilde{\xi}_h, \lambda_h) \in V_h \times \mathbb{R}$, such that $\tilde{\xi}_h$ satisfies the discrete operator equation:

$$\tilde{\xi}_h - \lambda_h L_2\tilde{\xi}_h + C_h(\tilde{\xi}_h) - L_1\tilde{\xi}_h = 0, \tag{4.16}$$

and Φ_h is given by

$$\Phi_h = \tilde{\chi}_h - B_h(\tilde{\xi}_h, \tilde{\xi}_h), \tag{4.17}$$

where $\tilde{\xi}_h = \xi_h + \tilde{\theta}$.

Theorem 4.3. Let $\varepsilon > 0$ and $(\zeta_h, \lambda_h) \in V_h \times \mathbb{R}$, with

$$\zeta_h = \varepsilon\phi_{0h} + \upsilon_h, \ \upsilon_h \in V_h \cap \{\phi_{0h}\}^\perp. \tag{4.18}$$

Then (ζ_h, λ_h) be solution of the discrete operator equation (4.16) if and only if

$$\upsilon_h = T_{h,\varepsilon}\upsilon_h \tag{4.19}$$

and

$$\lambda_h = \lambda_{0h} + \frac{1}{\varepsilon}(C_h(\zeta_h) - L_{1,h}\zeta_h, \phi_{0h})_\Delta, \tag{4.20}$$

where

$$S_{h,\varepsilon}\eta_h = \frac{1}{\varepsilon}(C_h(\eta_h) - L_{1,h}\eta_h, \phi_{0h})_\Delta L_{2,h}\eta_h - C_h(\eta_h) + L_{1,h}\eta_h, \tag{4.21}$$

$$T_{h,\varepsilon}\upsilon_h = Q_h S_{h,\varepsilon}(\varepsilon\phi_{0h} + \upsilon_h). \tag{4.22}$$

Theorem 4.4. For ε and h are small enough, then there exists a unique solution (ζ_h, λ_h) to the discrete operator equation (4.16), with ζ_h is of the form $\varepsilon\phi_{0h} + \upsilon_h,\ \upsilon_h \in \{\phi_{0h}\}^{\perp} \cap V_{h,\varepsilon}$, such that

$$\zeta_h \neq 0,\ \|\zeta_h\|_\Delta = O(\varepsilon),\ |\lambda_h - \lambda_0| = O(\varepsilon^2), \tag{4.23}$$

where

$$V_{h,\varepsilon} = \{\upsilon_h \in V_h; \|\upsilon_h\|_\Delta \leq \varepsilon\}. \tag{4.24}$$

5 Error estimates

For give the error estimates, we use the following lemmas

Lemma 5.1. We assume that $\zeta \in H_0^2(\omega)$ and $\eta \in H^2(\omega)$, there exists a constant c independent of h, such that

$$\|B_h(\zeta, \eta)\|_\Delta \leq c\|\zeta\|_\Delta \|\eta\|_{2,\omega}, \tag{5.1}$$

$$\|C_h(\zeta)\|_\Delta \leq c\|\zeta\|_\Delta^3. \tag{5.2}$$

Lemma 5.2. We assume that $\zeta \in H^3(\omega) \cap H_0^2(\omega)$and $\eta \in H^2(\omega)$, there exists a constant c independent of h, such that

$$\|B(\zeta, \eta) - B_h(\zeta, \eta)\|_\Delta \leq c\|\zeta\|_{3,\omega}\|\eta\|_{2,\omega}, \tag{5.3}$$

$$\|C(\zeta) - C_h(\zeta)\|_\Delta \leq ch^{\frac{1}{2}}\|\zeta\|_{3,\omega}\|\zeta\|_\Delta^2. \tag{5.4}$$

Lemma 5.3. We assume that $\zeta \in H^3(\omega) \cap H_0^2(\omega)$, there exists a constant c independent of h and ε, such that

$$|\lambda - \lambda_h| \leq ch^2 + ch\varepsilon^2 + c\varepsilon\|\zeta - \zeta_h\|_\Delta, \tag{5.5}$$

$$\|S_\varepsilon(\zeta) - S_{h,\varepsilon}(\zeta_h)\|_\Delta \leq ch^{\frac{1}{2}}\varepsilon^3 + c\varepsilon^2\|\zeta - \zeta_h\|_\Delta. \tag{5.6}$$

Finally, we show the following result

Theorem 5.4. We assume that $\zeta \in H^3(\omega) \cap H_0^2(\omega)$, there exists $\varepsilon_0 > 0$ and a constant $c(\varepsilon_0) > 0$, such that for all h small enough and for all ε with $0 < \varepsilon \leq \varepsilon_0$, we have

$$\|\zeta - \zeta_h\|_\Delta \leq ch\varepsilon + ch^{\frac{1}{2}}\varepsilon^3, \tag{5.7}$$

$$|\lambda - \lambda_h| \leq ch^2 + ch\varepsilon^2 + ch^{\frac{1}{2}}\varepsilon^4. \tag{5.8}$$

6 CONCLUSION AND COMMENTARY

This study is concerned with finite element method for approximating solutions to the bifurcation problem for Marguerre-von Kármán equations, solving these equations amounts to solving a single discrete cubic operator equation. First, we establish the existence and uniqueness of non trivial solutions bifurcating from the trivial solution at neighborhood of the simple eigenvalues of the linearized problem, by using a Kikuchi's method. Next, we establish the convergence of a conforming finite element approximations of this solution and we give the corresponding error estimates.

Note that, in the case $\theta \equiv 0$ in $\bar{\omega}$, we recover the bifurcation problem for von Kármán equations.

References

[1] A. Bensayah, D.A. Chacha and A. Ghezal, Asymptotic modelling of a signorini problem of generalized Marguerre-von Kármán shallow shells, *Applicable Analysis*, **92**(9): 1848–1862, 2013.

[2] M. S. Berger, On von Kármán equations and the buckling of a thin elastic plate. I. The clamped plate, *Comm. Pure Appl. Math.*, **20**: 687–719, 1967.

[3] M. S. Berger and P.C. Fife, On von Kármán equations and the buckling of a thin elastic plate. II. Plate with general edge conditions, *Comm. Pure Appl. Math.*, **21**: 227–241, 1968.

[4] F. Brezzi, Finite element approximations of the von Kármán equations, *R.A.I.R.O., Analyse numérique*, **12**(4): 303–312, 1978.

[5] F. Brezzi, J. Rappaz and P.A. Raviart, Finite dimensional approximation of nonlinear problems, Part I: Branches of nonsingular solutions, *Numer. Math.*, **36**: 1–25, 1980.

[6] F. Brezzi, J. Rappaz and P.A. Raviart, Finite dimensional approximation of nonlinear problems, Part III: Simple bifurcation points, *Numer. Math.*, **38**: 1–30, 1981.

[7] D.A. Chacha, A. Ghezal and A. Bensayah, Modélisation asymptotique d'une coque peu-profonde de Marguerre-von Kármán généralisée dans le cas dynamique, *Journal ARIMA*, **13**: 63–76, 2010.

[8] D.A. Chacha, A. Ghezal and A. Bensayah, Existence result for a dynamical equations of generalized Marguerre-von Kármán shallow shells, *Journal Elasticity*, **111**: 265–283, 2013.

[9] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.

[10] P.G. Ciarlet, *Mathematical elasticity, Theory of plates*, vol. II., North-Holland, Amsterdam, 1997.

[11] P.G. Ciarlet and L. Gratie, On the existence of solutions to the generalized Marguerre-von Kármán equations, *Math. Mech. Solids*, **11**: 83–100, 2006.

[12] P.G. Ciarlet, L. Gratie and N. Sabu, An existence theorem for generalized von Kármán equations, *Journal of Elasticity*, **62**: 239–248, 2001.

[13] P.G. Ciarlet, L. Gratie and S. Kesavan, On the generalized von Kármán equations and their approximation. *Journal of Mathematical Models and Methods in Applied Sciences*, **4**(17): 617–633, 2007.

[14] P.G. Ciarlet and J.C. Paumier, A justification of the Marguerre-von Kármán equations. *Computational Mech.*, **1**: 177–202, 1986.

[15] G. Devdariani, R. Janjgava and B. Gulua, Dirichlet problem for the Marguerre-von Kármán equations system, *Bulletin of TICMI*, **10**: 23–27, 2006.

[16] A. Ghezal and D. A. Chacha, Convergence of finite element approximations for generalized Marguerre-von Kármán equations, *Advances in Applied Mathematics, Springer Proceedings in Mathematics and Statistics*, **87**: 97–106, 2014.

[17] L. Gratie, Generalized Marguerre-von Kármán equations of a nonlinearly elastic shallow shell, *Appl. Anal.*, **81**: 1107–1126, 2002.

[18] O. Kavian and B.P. Rao, Une remarque sur l'existence de solutions non nulles pour les équations de Marguerre-von Kármán. *C.R. Acad. Sci. Pari Sér.* **1317**: 1137–1142, (1993).

[19] S. Kesavan, La méthode de Kikuchi appliquée aux équations de von Kármán, *Numer. Math.*, **32**: 209–232, 1979.

[20] S. Kesavan, Une méthode d'éléments finis mixte pour les équations de von Kármán, *R.A.I.R.O., Analyse numérique*, **14**(2): 149–173, 1980.

[21] S. Kesavan and P.N. Srikanth, On the Dirichlet problem for the Marguerre equations, *Nonlinear Analysis, Theory, Methods & Applications* **7**(**2**): 209–216, 1983.

[22] F. Kikuchi, *An iterative finite element scheme for bifurcation analysis of semi-linear elliptic equations*, Report N^0 542, Institute of Space and Aeronautical Science, University of Tokyo, Japan, 1976.

[23] A. Léger and B. Miara, On the multiplicity of solutions to Marguerre-von Kármán membrane equations. *J. Math. Pures Appl.*, **84**: 357–374, 2005.

[24] K. Marguerre, Zur Theorie der gekrummten Platte grosser Formanderung. in: *Proceedings, Fifth International Congress for Applied Mechanics*: 93–101, 1938.

[25] T. Miyoshi, A mixed finite element method for the solution of von Kármán equations, *Numer. Math.*, **26**: 255–269, 1976.

[26] B.P. Rao, Marguerre-von Kármán equations and membrane model, *Nonlinear Analysis*, **24**: 1131–1140, 1995.

[27] B.P. Rao, Existence of non-trivial solutions of the Marguerre-von Kármán equations and membrane model, in Asymptotic Methods for Elastic Structures, pp. 161âĂŞ-170, ed. P. G. Ciarlet, L. Trabucho & J. M. Viano, de Gruyter, Berlin, 1995.

[28] L. Reinhart, On the numerical analysis of the von Kármán equations: Mixed finite element approximation and continuation techniques, *Numer. Math.*, **39**: 371–404, 1982.

[29] T. von Kármán, Festigkeitesprobleme in maschinenbau, *encyklopadie der mathematischen wissenschaften*, Taubner, **IV/4**: 311–385, 1910.

[30] T. von Kármán and H.S. Tsien, The buckling of spherical shells by external pressure. *J. Aero. Sci*, **7**: 43–50, 1939.

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