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New nonlinear inequalities and application to control systems

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Abstract We give some continuous integral inequalities of Gronwall type which can be used in the stability analysis of various problems in the theory of the nonlinear differential equations and control systems.

1 Introduction

It is well known that Gronwall-Bellman type integral inequalities play important roles in the study of existence, uniqueness, continuation, boundedness, oscillation and stability properties to the solutions of differential and integral equations. In 1919, Gronwall [4] introduced the famous Gronwall inequality in the study of the solution of the differential equation. Since then, a lot of contributions have been achieved by many researchers (see [1, 3]). The original Gronwall inequality has been extended to the more general case, including the generalized linear and nonlinear Gronwall type inequalities. These inequalities can be used in the analysis of various problems in the theory of the nonlinear differential equations and control systems (see [5] and references therein). In this paper we give a new generalization of this Lemma. An application to control system in point of view stabilization by output feedback is given to illustrate the main result of this paper.

2 Some Gronwall-Bellman type integral inequalities

We recall some Gronwall-Bellman type integral inequalities [3], which we will use in our work.

Proposition 2.1. Let ψ a constant, x and χ be real continuous functions defined in $[a, b]$, $\chi(t) \geq 0$ for $t \in [a, b]$. We suppose that on $[a, b]$, we have the inequality

$$x(t) \leq \psi + \int_a^t \chi(s)x(s)ds,$$

then

$$x(t) \leq \psi exp(\int_a^t \chi(u)du).$$

Proposition 2.2. Let $z(t)$ be a positive differentiable function satisfying the inequality

$$z(t) \leq c + \int_a^t (f(s)z(s) + g(s)z^n(s))ds, \; t \in I = [a, b]$$

where $c \geq 0$, the functions $f(t)$ et $g(t)$ are continuous in I and $n > 1$ is a constant.

Then,

$$z(t) \le \frac{cexp(\int_a^t f(s)ds)}{(1-(n-1)c^{n-1}\int_a^t g(s)exp((n-1)\int_a^s f(u)du)ds)^{\frac{1}{n-1}}}$$

Under the assumption, $t, s \in [a, b]$,

$$1-(n-1)c^{n-1}\int_a^t g(s)exp((n-1)\int_a^s f(u)du)ds > 0.$$

Proposition 2.3. Let $z(t)$ be a positive differentiable function satisfying the inequality,

$$z(t) \le c + \int_a^t \sum_{i=1}^n (f_i(s)z^i(s))ds, t \in I = [a, b],$$

where $c \ge 0$, the functions $f_i(t)$ for $i = 1, ..., n$ are continuous in I and $n > 1$ is a constant.

Then,

$$z(t) \le \frac{cexp(\int_a^t f_1(s)ds)}{(1-(n-1)\int_a^t \sum_{i=2}^n c^{i-1} f_i(s)exp((n-1)\int_a^s f_1(u)du)ds)^{\frac{1}{n-1}}}.$$

Under the assumption, $t, s \in [a, b]$,

$$1-(n-1)\int_a^t \sum_{i=2}^n c^{i-1} f_i(s)exp((n-1)\int_a^s f_1(u)du)ds > 0.$$

3 New nonlinear inequalities

In this section we point out some new inequalities of the results presented above.

Proposition 3.1. Let $z(t)$ be a positive differentiable function satisfying the inequality,

$$z(t) \le c(t) + \int_a^t (f(s)z(s) + g(s)z^n(s))ds, t \in I = [a, b]$$

where $c(t)$ is a non negative increasing function, the functions $f(t)$ and $g(t)$ are continuous on I and $n > 1$ is a constant.

Then,

$$z(t) \le \frac{c(t)exp(\int_a^t f(s)ds)}{(1-(n-1)\int_a^t g(s)c^{n-1}(s)exp((n-1)\int_a^s f(u)du)ds)^{\frac{1}{n-1}}}.$$

Under the assumption, $t, s \in [a, b]$,

$$1-(n-1)\int_a^t g(s)c^{n-1}(s)exp((n-1)\int_a^s f(u)du)ds > 0.$$

Proof. The expression

$$z(t) \le c(t) + \int_a^t (f(s)z(s) + g(s)z^n(s))ds$$

can be written equivalently

$$\frac{z(t)}{c(t)} \le 1 + \frac{1}{c(t)}\int_a^t (f(s)z(s) + g(s)z^n(s))ds.$$

Thus,

$$\frac{z(t)}{c(t)} \leq 1 + \int_a^t (f(s)\frac{z(s)}{c(s)} + g(s)\frac{z^n(s)}{c(s)})ds.$$

So,

$$\frac{z(t)}{c(t)} \leq 1 + \int_a^t (f(s)\frac{z(s)}{c(s)} + g(s)c^{n-1}(s)\frac{z^n(s)}{c^n(s)})ds,$$

if we set

$$u(t) = \frac{z(t)}{c(t)}$$

it comes that

$$u(t) \leq 1 + \int_a^t (f(s)u(s) + g(s)c^{n-1}(s)u^n(s))ds.$$

Hence, we get

$$u(t) \leq \frac{exp(\int_a^t f(s)ds)}{(1-(n-1)\int_a^t g(s)c^{n-1}(s)exp((n-1)\int_a^s f(u)du)ds)^{\frac{1}{n-1}}}.$$

It follows that,

$$z(t) \leq \frac{c(t)exp(\int_a^t f(s)ds)}{(1-(n-1)\int_a^t g(s)c^{n-1}(s)exp((n-1)\int_a^s f(u)du)ds)^{\frac{1}{n-1}}}.$$

Proposition 3.2. Let $z(t)$ be a positive differentiable function satisfying the inequality,

$$z(t) \leq c(t) + \int_a^t \sum_{i=1}^{n} (f_i(s)z^i(s))ds, t \in I = [a,b].$$

where $c(t)$ is a non negative increasing function, the functions $f_i(t)$ for $i = 1, ..., n$ are continuous on I and $n > 1$ is a constant.

Then,

$$z(t) \leq \frac{c(t)exp(\int_a^t f_1(s)ds)}{(1-(n-1)\int_a^t \sum_{i=2}^{n} c^{i-1}(s)f_i(s)exp((n-1)\int_a^s f_1(u)du)ds)^{\frac{1}{n-1}}}.$$

Under the assumption, $t, s \in [a,b]$,

$$1-(n-1)\int_a^t \sum_{i=2}^{n} c^{i-1}(s)f_i(s)exp((n-1)\int_a^s f_1(u)du)ds > 0.$$

Proof. The expression

$$z(t) \leq c(t) + \int_a^t \sum_{i=1}^{n} (f_i(s)z^i(s))ds$$

can be written equivalently,

$$\frac{z(t)}{c(t)} \leq 1 + \frac{1}{c(t)}\int_a^t \sum_{i=1}^{n} (f_i(s)z^i(s))ds.$$

Thus,

$$\frac{z(t)}{c(t)} \leq 1 + \int_a^t \sum_{i=1}^{n} (f_i(s)\frac{z^i(s)}{c(s)})ds.$$

So,

$$\frac{z(t)}{c(t)} \leq 1+\int_a^t \sum_{i=1}^n (f_i(s)c^{i-1}(s)\frac{z^i(s)}{c^i(s)})ds$$

if we set

$$u(t)=\frac{z(t)}{c(t)}$$

it comes that

$$u(t) \leq 1+\int_a^t \sum_{i=1}^n (f_i(s)c^{i-1}(s)u^i(s))ds.$$

Thus, we get

$$u(t) \leq \frac{exp(\int_a^t f_1(s)ds)}{(1-(n-1)\int_a^t \sum_{i=2}^n f_i(s)c^{i-1}(s)exp((n-1)\int_a^s f_1(u)du)ds)^{\frac{1}{n-1}}}$$

and finally, it comes that

$$z(t) \leq \frac{c(t)exp(\int_a^t f_1(s)ds)}{(1-(n-1)\int_a^t \sum_{i=2}^n f_i(s)c^{i-1}(s)exp((n-1)\int_a^s f_1(u)du)ds)^{\frac{1}{n-1}}}.$$

Proposition 3.3. Let $x(t)$ be real,continuous,non negative function and $c(t)$ be a positive differentiable increasing function such that for $t>t_0$,

$$x(t) \leq c(t)+\int_{t_0}^t k(t,s)x(s)ds$$

where $k(t,s)$ is a continuously differentiable function in t and continuous in s with $k(t,s) \geq 0$ for $t \geq s \geq t_0$.

Then,

$$x(t) \leq c(t)exp(\int_{t_0}^t (k(s,s)+\int_{t_0}^s \partial_1 k(s,r)dr)ds).$$

Proof. Let $y(t)=\int_{t_0}^t k(t,s)x(s)ds$, by differentiation we get

$$y'(t)=k(t,t)x(t)+\int_{t_0}^t \partial_1 k(t,s)x(s)ds,$$

it follows that

$$y'(t) \leq k(t,t)(y(t)+c(t))+\int_{t_0}^t \partial_1 k(t,s)(y(s)+c(s))ds.$$

Then,

$$y'(t)+c'(t) \leq c'(t)+k(t,t)(y(t)+c(t))+\int_{t_0}^t \partial_1 k(t,s)(y(s)+c(s))ds.$$

So,

$$\frac{y'(t)+c'(t)}{y(t)+c(t)} \leq \frac{c'(t)}{y(t)+c(t)}+k(t,t)+\int_{t_0}^t \partial_1 k(t,s)\frac{y(s)+c(s)}{y(t)+c(t)}ds.$$

Thus,

$$\frac{y'(t)+c'(t)}{y(t)+c(t)} \leq \frac{c'(t)}{c(t)}+k(t,t)+\int_{t_0}^t \partial_1 k(t,s)ds,$$

and by integration, we get

$$\ln(y(t)+c(t)) \leq \ln c(t) + \int_{t_0}^{t}(k(s,s)+\int_{t_0}^{s}\partial_1 k(s,r)dr)ds.$$

Hence,

$$\ln x(t) \leq \ln c(t) + \int_{t_0}^{t}(k(s,s)+\int_{t_0}^{s}\partial_1 k(s,r)dr)ds,$$

and finally it comes that

$$x(t) \leq \ c(t)exp(\int_{t_0}^{t}(k(s,s)+\int_{t_0}^{s}\partial_1 k(s,r)dr)ds).$$

Theorem 3.4. Let $x:[a,b]\to\mathbb{R}_+$ be a continuous function that satisfies the inequality,

$$x(t) \leq c(t) + \int_a^t \psi(s)\omega(x(s))ds$$

where $c:[a,b]\to\mathbb{R}_+$ is a differentiable function, $\psi:[a,b]\to\mathbb{R}_+$ continuous and $\omega:[a,b]\to\mathbb{R}_+^*$ is continuous and monotone increasing.

Then, the estimation

$$x(t) \leq G^{-1}(\int_a^t \psi(s)ds + G(c(t)))$$

holds, where

$$G(\delta) = \int_{\varepsilon}^{\delta}\frac{ds}{\omega(s)}, \varepsilon>0, \delta>0.$$

Proof. Let $y(t)=\int_a^t \psi(s)\omega(x(s))ds$, by differentiation we get

$$y'(t) = \psi(t)\omega(x(t)),$$

it comes that

$$y'(t) \leq \psi(t)\omega(y(t)+c(t)).$$

So,

$$y'(t)+c'(t) \leq c'(t)+\psi(t)\omega(y(t)+c(t)).$$

Thus,

$$\frac{y'(t)+c'(t)}{\omega(y(t)+c(t))} \leq \psi(t) + \frac{c'(t)}{\omega(y(t)+c(t))}.$$

Then,

$$\frac{y'(t)+c'(t)}{\omega(y(t)+c(t))} \leq \psi(t) + \frac{c'(t)}{\omega(c(t))},$$

and by integration, we get

$$\int_a^t \frac{y'(s)+c'(s)}{\omega(y(s)+c(s))}ds \leq \int_a^t (\psi(s)+\frac{c'(s)}{\omega(c(s))})ds.$$

$$\int_{c(a)}^{c(t)+y(t)}\frac{du}{\omega(u)} \leq \int_a^t \psi(s)ds + \int_{c(a)}^{c(t)}\frac{du}{\omega(u)}.$$

Thus, it comes that

$$G(c(t)+y(t)) \leq \int_a^t \psi(s)ds + G(c(t)).$$

Hence,

$$G(x(t)) \leq \int_a^t \psi(s)ds + G(c(t)).$$

We obtain,
$$x(t) \leq G^{-1}(\int_a^t \psi(s)ds + G(c(t))).$$

Theorem 3.5. Let $u(t)$, $v(t)$ and $h(t,r)$ be nonnegative continuous functions for $t \geq r \geq a$, $c(t)$ a continuously differentiable nonnegative increasing function. If
$$u(t) \leq c(t) + \int_a^t (v(s)u(s) + \int_a^s h(s,r)u(r)dr)ds$$
for $t \geq a$, then
$$u(t) \leq c(t)exp \int_a^t (v(s) + \int_a^s h(s,r)dr)ds.$$

Proof. Let
$$b(t) = c(t) + \int_a^t (v(s)u(s) + \int_a^s h(s,r)u(r)dr)ds,$$
by differentiation, we get
$$b'(t) = c'(t) + v(t)u(t) + \int_a^t h(t,r)u(r)dr$$
then,
$$\frac{b'(t)}{b(t)} = \frac{c'(t)}{b(t)} + v(t)\frac{u(t)}{b(t)} + \frac{1}{b(t)} \int_a^t h(t,r)u(r)dr.$$
It follows that,
$$\frac{b'(t)}{b(t)} \leq \frac{c'(t)}{c(t)} + v(t) + \int_a^t h(t,r)u(r)dr.$$
By integration, we get
$$\int_a^t \frac{b'(s)}{b(s)}ds \leq \int_a^t \frac{c'(s)}{c(s)}ds + \int_a^t (v(s) + \int_a^s h(s,r)u(r)dr)ds.$$
We obtain,
$$\ln b(t) \leq \ln c(t) + \int_a^t (v(s) + \int_a^s h(s,r)u(r)dr)ds.$$
So,
$$b(t) \leq c(t)exp(\int_a^t (v(s) + \int_a^s h(s,r)u(r)dr)ds).$$

Theorem 3.6. Let the nonnegative function $u(t)$ defined on $[t_0, \infty)$ satisfy the inequality
$$u(t) \leq c(t) + \int_{t_0}^t k(t,s)u(s)ds + \int_{t_0}^t \int_{t_0}^s G(t,s,\sigma)u(\sigma)d\sigma ds$$
$c(t)$ is a continuously differentiable nonnegative increasing function, $k(t,s)$ and $G(t,s,\sigma)$ are differentiable nonnegative functions for $t \geq s \geq \sigma$.

Then,
$$u(t) \leq c(t)exp \int_{t_0}^t (k(r,r) + \int_{t_0}^r \partial_1 k(r,s)ds + \int_{t_0}^r G(r,r,\sigma)d\sigma + \int_{t_0}^r \int_{t_0}^s \partial_1 G(r,s,\sigma)d\sigma ds)dr.$$

Proof. Let $y(t) = \int_{t_0}^t k(t,s)u(s)ds + \int_{t_0}^t \int_{t_0}^s G(t,s,\sigma)d\sigma ds$

by differentiation, we get
$$y'(t) = k(t,t)u(t) + \int_{t_0}^t \partial_1 k(t,s)u(s)ds + \int_{t_0}^t G(t,t,\sigma)u(\sigma)d\sigma + \int_{t_0}^t \int_{t_0}^s \partial_1 G(t,s,\sigma)u(\sigma)d\sigma ds$$

then,

$$y'(t)+c'(t)=c'(t)+k(t,t)u(t)+\int_{t_0}^{t}\partial_1 k(t,s)u(s)ds+\int_{t_0}^{t}G(t,t,\sigma)u(\sigma)d\sigma$$
$$+\int_{t_0}^{t}\int_{t_0}^{s}\partial_1 G(t,s,\sigma)u(\sigma)d\sigma ds.$$

Thus,

$$y'(t)+c'(t)\leq c'(t)+k(t,t)(y(t)+c(t))+\int_{t_0}^{t}\partial_1 k(t,s)(y(s)+c(s))ds+$$

$$\int_{t_0}^{t}G(t,t,\sigma)(y(\sigma)+c(\sigma))d\sigma+\int_{t_0}^{t}\int_{t_0}^{s}\partial_1 G(t,s,\sigma)(y(\sigma)+c(\sigma))d\sigma ds$$

and

$$\frac{y'(t)+c'(t)}{y(t)+c(t)}\leq\frac{c'(t)}{c(t)}+k(t,t)+\int_{t_0}^{t}\partial_1 k(t,s)ds+\int_{t_0}^{t}G(t,t,\sigma)d\sigma+\int_{t_0}^{t}\int_{t_0}^{s}\partial_1 G(t,s,\sigma)d\sigma ds.$$

By integration, we obtain

$$\ln(y(t)+c(t))\leq\ln(c(t))+\int_{t_0}^{t}(k(r,r)+\int_{t_0}^{r}\partial_1 k(r,s)ds+\int_{t_0}^{r}G(r,r,\sigma)d\sigma$$
$$+\int_{t_0}^{r}\int_{t_0}^{s}\partial_1 G(r,s,\sigma)d\sigma ds)dr.$$

Finally,

$$\ln(u(t))\leq\ln(c(t))+\int_{t_0}^{t}(k(r,r)+\int_{t_0}^{r}\partial_1 k(r,s)ds+\int_{t_0}^{r}G(r,r,\sigma)d\sigma+\int_{t_0}^{r}\int_{t_0}^{s}\partial_1 G(r,s,\sigma)d\sigma ds)dr.$$

4 Application to control systems

We consider as application the output feedback stabilization of nonlinear dynamical systems that model many phenomena from various disciplines.

Now consider the following nonlinear system:

$$\begin{cases}\dot{x}(t) &= Ax(t)+\sum_{i=1}^{m}g_i(x(t))u_i(t)+Bu(t)+\delta(t)\\ y(t) &= Cx(t)\\ x(0) &= x_0\end{cases}\tag{4.1}$$

where $x(t)\in(\mathbb{R}^+)^n$ is the vector of state, $x_0\in(\mathbb{R}^+)^n$ is the initial condition, $y(t)\in\mathbb{R}^m$ is output, $u(t)\in\mathbb{R}^p$ is the vector controls and A, B, C are constant matrices of appropriate size such that (A,B) is controllable and (A,C) is observable. The function $\delta(t)$ is a continuous function which satisfies $\forall t,\ \int_0^t\|\delta(s)\|ds<+\infty$.

The nonlinear function $g_i(x(t))$ is measurable with $g_i(0)=0$ and satisfies the following hypothesis: for all $i=1,...,m$, there exists an integer $q>1$ as:

$$\|g_i(x(t))\|\leq\alpha_i\|x(t)\|^q\tag{4.2}$$

where α_i are positive constants.

We assume that the gain L exists because that the fact that (A, B) is conrollable and (A, C) is observable provide necessary conditions but not sufficient for the existence of the gain K such that all eigenvalues of the matrix $(A - BKC)$ either negative real part.

This implies that there exists $M > 0$ and $\omega < 0$, such that

$$\|e^{(A-BKC)t}\| \leq Me^{\omega t}, \quad \forall t > 0. \tag{4.3}$$

Exponential stabilization by output feedback static system is an extension of proposition 3.1, it is given by the following result.

Let $\alpha = \sum_{i=1}^{m} \alpha_i, R = \frac{1}{M}\left[\left(\frac{-\omega}{\alpha M\|KC\|}\right)^{\frac{1}{q}} - r\right], \lambda = \left(1 + \frac{\alpha M\|KC\|\left[M\|x_0\| + r\right]^q}{\omega}\right)^{\frac{1}{q}},$

where $r > 0$ such that $\left(\frac{-\omega}{\alpha M\|KC\|}\right)^{\frac{1}{q}} > r$, satisfying $\int_0^{+\infty} \|\delta(s)\|ds = r$.

For $\|x_0\| < R$, the system (4.1) controlled by the linear state output feedback $u(t) = -Ky(t)$, satisfies:

$$\|x(t)\| \leq \frac{M}{\lambda}\|x_0\|e^{\omega t} + \frac{r}{\lambda}.$$

This can be seen as follows, if $C \neq I_n$ and $p < n$, the solution of system (4.1) is controlled by the state output feedback $u(t) = -Ky(t)$ is given by:

$$x(t) = e^{(A-BKC)t}x_0 + \int_0^t e^{(A-BKC)(t-s)}\left[\delta(s) + \sum_{i=1}^{m} g_i(x(s))(KCx(s))_i\right] ds,$$

where $(KCx(s))_i$ is the i^{ieme} component of the vector $KCx(s)$.

while taking into account (4.2) and (4.3) with $\alpha = \sum_{i=1}^{m} \alpha_i$, we have:

$$\|x(t)\| \leq M\|x_0\|e^{\omega t} + e^{\omega t}\int_0^t \left[\alpha M\|KC\|\|x(s)\|^{q+1} + \|\delta(s)\|\right] ds,$$

then,

$$\|x(t)e^{-\omega t}\| \leq M\|x_0\| + \int_0^t \|\delta(s)\|ds + \int_0^t \alpha M\|KC\|e^{\omega qs}\|x(s)e^{-\omega s}\|^{q+1}ds$$

the application of proposition with $c(t) = M\|x_0\| + \int_0^t \|\delta(s)\|ds$, $f(s) = 0$, $g(s) = \alpha M\|KC\|e^{\omega qs}$,

and $n = q + 1$, yields

$$\|x(t)e^{-\omega t}\| \leq \frac{M\|x_0\| + \int_0^t \|\delta(s)\|ds}{\left(1 - q\int_0^t \alpha M\|KC\|e^{\omega qs}c^q(s)ds\right)^{\frac{1}{q}}}.$$

Thus,

$$\|x(t)\| \leq \frac{M\|x_0\|e^{\omega t} + e^{\omega t}\int_0^t \|\delta(s)\|ds}{\left(1 - q\int_0^t \alpha M\|KC\|e^{\omega qs}c^q(s)ds\right)^{\frac{1}{q}}}.$$

So,

$$\|x(t)\| \leq \frac{M\|x_0\|e^{\omega t} + r}{\left(1 + \frac{\alpha M\|KC\|\left[M\|x_0\| + r\right]^q}{\omega}\right)^{\frac{1}{q}}},$$

this proves that, for all $t \geq 0$

$$\|x(t)\| \leq \frac{M}{\lambda}\|x_0\|e^{\omega t} + \frac{r}{\lambda},$$

under the assumption of proposition 3.1,

$$1 - q\int_0^t \alpha M\|KC\|e^{\omega qs}c^q(s)ds > 0.$$

$$1 + \frac{\alpha M\|KC\|\,[M\|x_0\| + r]^q}{\omega} > 0,$$

and

$$\|x_0\| < \frac{1}{M}\left[\left(\frac{-\omega}{\alpha M\|KC\|}\right)^{\frac{1}{q}} - r\right]$$

Conclusion In this paper, some new nonlinear integral inequalities are established. These inequalities can be used to the study of the output stabilization problem of control systems.

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