DUALITY PRINCIPLE IN g-FRAMES

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Abstract. The concept of Riesz dual sequences (R-dual sequences) was introduced by Casazza et al. in 2004. Recently, for generalizing this concept to g-frames the concept of g-Riesz dual sequences has been introduced and various definitions of R-duals for frames are in the literature. In this paper, we generalize these concepts for g-frame and introduce g-Riesz duals (g-R-duals) of type II, III and IV. Since the g-R-dual of type IV is the most general g-R-dual, we focus on the g-R-dual of type IV. We give characterizations of g-frames and g-Riesz bases in terms of their g-R-dual of type IV. We characterize all dual g-frames of a g-frame in terms of its g-R-dual of type IV which can be considered as Wexler-Raz biorthogonality relations for g-frames. Also, we present a generalization of Ron-Shen duality principle to g-frames. In addition, we investigate the construction of dual g-frames in more details and we give another characterization of dual g-frames with respect to its g-R-dual sequence.

1 Introduction and preliminaries

The concept of R-duality of a Bessel sequence in a separable Hilbert space was introduced by Casazza, Kutyniok and Lammers in [1], in order to obtain a generalization of duality principles in Gabor frames to abstract frame theory. Let \((e_i)_{i\in I}, (h_i)_{i\in I}\) be orthonormal bases and \((f_i)_{i\in I}\) be a Bessel sequence. The R-dual sequence of \((f_i)_{i\in I}\) with respect to the orthonormal bases \((e_i)_{i\in I}\) and \((h_i)_{i\in I}\) is the sequence \((w^f_i)_{i\in I}\), such that for every \(j \in I\)

\[ w^f_i = \sum_{i\in I} \langle f_i, e_j \rangle h_i. \]

The R-duality has been favored by many authors. The R-duality with respect to orthonormal bases has been discussed in [2] and [3]. In [8], the authors introduced various alternative R-duals and showed their relation with Gabor frames. In [11], the authors generalized the R-duality in Banach spaces. In [4] the authors, proved that the duality principle extends to any pair of projective unitary representation of countable groups. Recently, for generalizing this concept to g-frames the concept of g-Riesz dual sequences has been introduced [7]. Various definitions of R-duals for frames are in the literature.

In this paper, we generalize these concepts to g-frame and introduce g-Riesz duals (g-R-duals) of type II, III and IV. Since the g-R-dual of type IV is the most general g-R-dual, we focus on the g-R-dual of type IV. We give characterizations of g-frames and g-Riesz bases in terms of their g-R-dual of type IV. We characterize all dual g-frames of a g-frame in terms of its g-R-dual of type IV, which can be considered as Wexler-Raz biorthogonality relations for g-frames. Also, we present a generalization of Ron-Shen duality principle to g-frames. In addition, we investigate the construction of dual g-frames in more details and we give another characterization of dual g-frames with respect to its g-R-dual sequence.

Throughout this paper \(H\) denotes a separable Hilbert space with inner product \(\langle \cdot, \cdot \rangle\) and \(I\) is a subset of \(\mathbb{Z}\), and \(\{H_i : i \in I\}\) is a sequence of separable Hilbert spaces. Also, for every \(i \in I\), \(L(H, H_i)\) is the set of all bounded, linear operators from \(H\) to \(H_i\).
In the rest of this section we review several well-known definitions and results. The new results are stated in Section 2.

For every sequence \( \{H_i\}_{i \in \mathcal{I}} \), the space

\[
(\sum_{i \in \mathcal{I}} \bigoplus H_i)_{\geq} = \{ (f_i)_{i \in \mathcal{I}} : f_i \in H_i, \ i \in \mathcal{I}, \ \sum_{i \in \mathcal{I}} \|f_i\|^2 < \infty \}
\]

with pointwise operations and the following inner product is a Hilbert space

\[
\langle (f_i)_{i \in \mathcal{I}}, (g_i)_{i \in \mathcal{I}} \rangle = \sum_{i \in \mathcal{I}} \langle f_i, g_i \rangle.
\]

A sequence \( \Lambda = \{ \Lambda_i \in L(H, H_i) : i \in \mathcal{I} \} \) is called a g-frame for \( H \) with respect to \( \{H_i : i \in \mathcal{I} \} \) if there exist \( A, B > 0 \) such that for every \( f \in H \)

\[
A\|f\|^2 \leq \sum_{i \in \mathcal{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2,
\]

\( A, B \) are called g-frame bounds. We call \( \Lambda \) a tight g-frame if \( A = B \) and a Parseval g-frame if \( A = B = 1 \). If only the right hand side inequality is required, \( \Lambda \) is called a g-Bessel sequence. If \( \Lambda \) is a g-Bessel sequence, then the synthesis operator for \( \Lambda \) is the linear operator,

\[
T_\Lambda : \left( \sum_{i \in \mathcal{I}} \bigoplus H_i \right)_{\geq} \mapsto H \quad T_\Lambda (f_i)_{i \in \mathcal{I}} = \sum_{i \in \mathcal{I}} \Lambda_i^* f_i.
\]

We call the adjoint of the synthesis operator, the analysis operator. The analysis operator is the linear operator,

\[
T_\Lambda^* : H \mapsto \left( \sum_{i \in \mathcal{I}} \bigoplus H_i \right)_{\geq} \quad T_\Lambda^* f = (\Lambda_i f)_{i \in \mathcal{I}}.
\]

We call \( S_\Lambda = T_\Lambda T_\Lambda^* \) the g-frame operator of \( \Lambda \) and \( S_\Lambda f = \sum_{i \in \mathcal{I}} \Lambda_i^* \Lambda_i f \). \( f \in H \).

If \( \Lambda = (\Lambda_i)_{i \in \mathcal{I}} \) is a g-frame with lower and upper g-frame bounds \( A, B \), respectively, then the g-frame operator of \( \Lambda \) is a bounded, positive and invertible operator on \( H \) and

\[
A(f, f) \leq \langle S_\Lambda f, f \rangle \leq B(f, f) \quad (f \in H)
\]

so

\[
A I \leq S_\Lambda \leq B I.
\]

The canonical dual g-frame for \( \Lambda = (\Lambda_i)_{i \in \mathcal{I}} \) is defined by \( \widetilde{\Lambda} = (\widetilde{\Lambda_i})_{i \in \mathcal{I}} \), where \( \widetilde{\Lambda_i} = \Lambda_i S_\Lambda^{-1} \) which is also a g-frame for \( H \) with lower and upper g-frame bounds \( \frac{1}{B} \) and \( \frac{1}{A} \), respectively. Also for every \( f \in H \), we have

\[
f = \sum_{i \in \mathcal{I}} \Lambda_i^* \widetilde{\Lambda_i} f = \sum_{i \in \mathcal{I}} \Lambda_i^* \Lambda_i f.
\]

We say \( \Lambda = \{ \Lambda_i \in L(H, H_i) : i \in \mathcal{I} \} \) is a g-frame sequence if it is a g-frame for \( \overline{\operatorname{span}} \{ \Lambda_i^* (H_i) \}_{i \in \mathcal{I}} \).

A sequence \( \Lambda = \{ \Lambda_i \in L(H, H_i) : i \in \mathcal{I} \} \) is g-complete if \( \{ f : \Lambda_i f = 0, \ \forall i \in \mathcal{I} \} = \{0 \} \). We note that the g-Bessel sequence \( \Lambda \) is g-complete if and only if \( T_\Lambda^* \) is injective. We say that \( \Lambda \) is a g-orthonormal basis for \( H \), if

\[
\langle \Lambda_i^* f_i, \Lambda_j^* f_j \rangle = \delta_{i,j} \langle f_i, f_j \rangle, \quad \forall f_i \in H_i, f_j \in H_j, i, j \in \mathcal{I}
\]

and

\[
\sum_{i \in \mathcal{I}} \|\Lambda_i f\|^2 = \|f\|^2 \quad (f \in H).
\]

A sequence \( \Lambda = \{ \Lambda_i \in L(H, H_i) : i \in \mathcal{I} \} \) is a g-Riesz sequence if there exist \( A, B > 0 \) such that for every finite subset \( F \subset \mathcal{I} \) and \( g_i \in H_i, i \in F \)

\[
A \sum_{i \in F} \|g_i\|^2 \leq \sum_{i \in F} \Lambda_i^* g_i \|f\|^2 \leq B \sum_{i \in F} \|g_i\|^2.
\]

(1.1)
G-Riesz sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ is called a g-Riesz basis if it is g-complete, too. So $\Lambda$ is a g-Riesz basis if and only if $T_\Lambda$ is a bounded invertible operator. Clearly, every g-orthonormal basis is a g-Riesz basis.

Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\Theta = \{\Theta_i \in L(H, H_i) : i \in I\}$ be g-Bessel sequences with g-Bessel bounds $B$ and $C$, respectively. The operator $S_{\Theta \Lambda} : H \mapsto H$ defined by

$$S_{\Theta \Lambda} f = \sum_{i \in I} \Lambda_i^* \Theta_i f, \quad (f \in H)$$

is a bounded operator, $\|S_{\Theta \Lambda}\| \leq \sqrt{BC}$, $S_{\Theta \Lambda}^* = S_{\Theta \Lambda}$ and $S_{\Theta \Lambda} = S_{\Lambda \Theta}$. Two g-Bessel sequences $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\Theta = \{\Theta_i \in L(H, H_i) : i \in I\}$ are called dual g-frames if

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i f = \sum_{i \in I} \Theta_i^* \Lambda_i f, \quad (f \in H).$$

For more details about g-frames, see [6, 9].

## 2 Main results

In this section, first we consider the g-Riesz dual (g-R-dual) with respect to g-orthonormal bases as the g-R dual of type I in [7] and we introduce alternative definitions of g-R-duals.

**Definition 2.1.** Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-frame for $H$ with g-frame operator $S$.

(i) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ be g-orthonormal bases. The g-R-dual of type I of $\Lambda$ with respect to $\Gamma$ and $\Gamma$ is $\Phi^\Lambda = (\Phi^\Lambda_j)_{j \in I}$ defined by

$$\Phi^\Lambda_j f = \Gamma_j S_{\Lambda \Gamma j^*} f \quad (f \in H).$$

(ii) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ be g-orthonormal bases. The g-R-dual of type II of $\Lambda$ with respect to $\Gamma$ and $\Gamma$ is $\Phi^\Lambda = (\Phi^\Lambda_j)_{j \in I}$ defined by

$$\Phi^\Lambda_j f = \Gamma_j S_{\Gamma \Gamma} f \quad (f \in H).$$

(iii) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ be g-orthonormal bases and $M : H \mapsto H$ be a bounded invertible operator with $\|M\| \leq \sqrt{\|S\|}$ and $\|M^{-1}\| \leq \sqrt{\|S^{-1}\|}$. The g-R-dual of type III of $\Lambda$ with respect to triplet $(\Gamma, \Gamma, M)$ is $\Phi^\Lambda = (\Phi^\Lambda_j)_{j \in I}$ defined by

$$\Phi^\Lambda_j f = \Gamma_j S_{\Lambda \Gamma j^* (\Gamma M)} f \quad (f \in H).$$

(iv) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ be g-Riesz bases. The g-R-dual of type IV of $\Lambda$ with respect to $\Gamma$ and $\Gamma$ is $\Phi^\Lambda = (\Phi^\Lambda_j)_{j \in I}$ defined by

$$\Phi^\Lambda_j f = \Gamma_j S_{\Lambda \Gamma j^*} f \quad (f \in H).$$

In all of the above cases, it is obvious that $\Phi^\Lambda_j$ is well-defined and $\Phi^\Lambda_j \in L(H, H_j)$, for every $j \in I$.

Clearly, the g-R-duals of type II are contained in the class of g-R-duals of type III and the g-R-duals of type III are contained in the class of g-R-duals of type IV. Moreover, the g-R-duals of type I, II, and III are contained in the class of g-R-duals of type IV.

In the following proposition, we show that for tight g-frames the g-R-duals of type I, II and III coincide.
Proposition 2.2. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a tight g-frame. Then the g-R-duals of type I, II and III coincide.

Proof. Denote the g-frame operator for $\Lambda$ by $S$. Since $\Lambda$ is a tight g-frame, then $S = AI$ for some $A > 0$.
For every $j \in \mathcal{I}$, $\Gamma_j S_{\Lambda} = \Gamma_j A^{-\frac{1}{2}} S_{\Lambda(\Sigma\Lambda^{-\frac{1}{2}})} = \Gamma_j S^{-\frac{1}{2}} S_{\Lambda^\dagger(\Sigma\Lambda^{-\frac{1}{2}})}$. Therefore the g-R-duals of type I and II coincide.

Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\mathcal{Y} = \{\mathcal{Y}_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-orthonormal bases. Take the bounded invertible operator $M : H \to H$ such that $\|M\| \leq \|S\| = \sqrt{A}$ and $\|M^{-1}\| = \sqrt{\|S^{-1}\|} = \frac{1}{\sqrt{A}}$. Suppose that $(g_i)_{i \in \mathcal{I}} \in (\sum_{i \in \mathcal{I}} \mathcal{Y}_i) \mathcal{E}$, then we have

$$||\sum_{i \in \mathcal{I}} (\mathcal{Y}_i M)^* g_i||^2 = ||\sum_{i \in \mathcal{I}} M^* \mathcal{Y}_i^* g_i||^2 \leq ||M^*||^2 ||\sum_{i \in \mathcal{I}} \mathcal{Y}_i^* g_i||^2 \leq A \sum_{i \in \mathcal{I}} ||g_i||^2,$$

and

$$||\sum_{i \in \mathcal{I}} (\mathcal{Y}_i M)^* g_i||^2 = ||\sum_{i \in \mathcal{I}} M^* \mathcal{Y}_i^* g_i||^2 \geq \frac{1}{||M^*||^2 ||\sum_{i \in \mathcal{I}} \mathcal{Y}_i^* g_i||^2} \sum_{i \in \mathcal{I}} ||g_i||^2.$$

Therefore $(\mathcal{Y}_i M)_{i \in \mathcal{I}}$ is a tight g-Riesz basis with bound $A$. We can see that $M^* A M$ is a unitary operator. Since $\mathcal{Y}$ is a g-orthonormal basis, then $(\frac{1}{\sqrt{A}} \mathcal{Y}_i M)_{i \in \mathcal{I}}$ is a g-orthonormal basis, denote it by $(\Psi_i)_{i \in \mathcal{I}}$. Hence $(\sqrt{A} \Psi_i)_{i \in \mathcal{I}} = (\mathcal{Y}_i M)_{i \in \mathcal{I}}$. Now, the g-R-dual of $\Lambda$ of type III with respect to $(\Gamma, \mathcal{Y}, M)$ is

$$\Phi_j = \Gamma_j S_{\Lambda(\Sigma\Lambda^{-\frac{1}{2}})}(\mathcal{Y}_j M) = \Gamma_j S_{\Lambda(\Sigma\Lambda^{-\frac{1}{2}})}(\sqrt{A} \Psi_j) = \Gamma_j S_{\Lambda(\Psi_j)},$$

which is a g-R-dual of type I of $\Lambda$.

Since the g-R-duals of type I are contained in the class of g-R-duals of type III and for tight g-frames the g-R-duals of type I and II coincide, then for tight g-frames, g-R-duals of type I, II and III coincide.

Since the g-R-dual of type IV is the most general g-R-dual, we focus on the g-R-dual of type IV and give some characterizations of it. Note that all results about the g-R-dual of type IV hold for the g-R-duals of type I, II and III.

In the following proposition, we present an algorithm which invert the process of mapping $\Lambda$ to its g-R-dual of type IV ($\Phi^\Lambda$).

Proposition 2.3. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-Bessel sequence with g-Bessel bound $A$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\mathcal{Y} = \{\mathcal{Y}_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-Riesz bases. Let $\Phi = (\Phi_j^\Lambda)_{j \in \mathcal{I}}$ be the g-R-dual sequence of type IV of $\Lambda$ with respect to $\Gamma$ and $\mathcal{Y}$. Then $\Phi$ is a g-Bessel sequence and $\Lambda$ is the g-R-dual sequence of type IV of $\Phi^\Lambda$ with respect to $\Gamma$ and $\mathcal{Y}$.

In the sense that for every $i \in \mathcal{I}$ we have

$$\Lambda_i f = \sum_{j \in \mathcal{I}} \overline{\mathcal{Y}_i} \Phi_j^\Lambda \mathcal{Y}_j f = \overline{\mathcal{Y}_i} S_{\Phi^\Lambda} f \quad (f \in H).$$

Proof. Let $B$ and $C$ be upper g-Riesz bounds for $\Gamma$ and $\mathcal{Y}$, respectively. Since $\mathcal{Y}$ is a g-Riesz basis with upper g-Riesz bound $C$, then it is a g-frame with upper g-frame bound $C$, too. On the other hand, $\Lambda$ is a g-Bessel sequence with g-Bessel bound $A$. Therefore $||S_{\mathcal{Y}}|| \leq \sqrt{AC}$, see [6]. Hence for every $f \in H$ we have

$$\sum_{j \in \mathcal{I}} ||\Phi_j^\Lambda f||^2 = \sum_{j \in \mathcal{I}} ||\Gamma_j S_{\Lambda} f||^2 \leq B ||S_{\Lambda} f||^2 \leq ABC ||f||^2.$$

Therefore $\Phi^\Lambda$ is a g-Bessel sequence in $H$. 
For every $f \in H$ and $g_i \in H_i$, we have
\[
\langle \tilde{\gamma}_i s_{\Phi^* \Gamma} f, g_i \rangle = \sum_{j \in I} \langle \Phi_j^{\ast} \tilde{\gamma}_j f, \tilde{\gamma}_j^* g_i \rangle = \sum_{j \in I} \langle S_{\gamma_j \Gamma_j} \tilde{\gamma}_j f, \tilde{\gamma}_j^* g_i \rangle = \langle S_{\gamma_j \Gamma_j} \sum_{j \in I} \tilde{\gamma}_j f, \tilde{\gamma}_j^* g_i \rangle = \langle f, \sum_{k \in \mathbb{F}} \Lambda_k^* \gamma_k \tilde{\gamma}_k^* g_i \rangle = \langle f, \Lambda_i^* g_i \rangle = \langle \Lambda_i f, g_i \rangle.
\]
Thus for every $i \in I$
\[
\Lambda_i f = \sum_{j \in I} \tilde{\gamma}_j \Phi_j^{\ast} \tilde{\gamma}_j = \tilde{\gamma}_i s_{\Phi^* \Gamma} f \quad (f \in H).
\]

\[\square\]

**Corollary 2.4.** Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-frame with g-frame operator $S$ and $(\Phi_j)_{j \in \mathbb{F}}$ be the g-R-dual of type III of $\Lambda$ with respect to g-orthonormal bases $\Gamma = \{\Gamma_j \in L(H, H_i) : i \in I\}$ and invertible operator $M$. Then for every $i \in I$ and $f \in H$,
\[
\Lambda_i f = Y_i S^{-1} S_{\Phi(\Gamma S)^{-1}} f.
\]

Also, if $(\Phi_j)_{j \in \mathbb{F}}$ is the g-R-dual of type II of $\Lambda$ with respect to g-orthonormal bases $\Gamma$ and $Y$, then for every $i \in I$ and $f \in H$,
\[
\Lambda_i f = Y_i S^{-1} S_{\Phi(\Gamma S)^{-1}} f.
\]

**Proof.** Since $(\Phi_j)_{j \in \mathbb{F}}$ is the g-R-dual of type III of $\Lambda$ with respect to g-orthonormal bases $\Gamma$ and $Y$ and the bounded invertible operator $M$, then $(\Phi_j)_{j \in \mathbb{F}}$ is the g-R-dual of type IV of $\Lambda$ with respect to g-Riesz bases $(\Gamma_j S^{-\frac{1}{2}})_{j \in \mathbb{F}}$ and $(\Gamma_j M)_{j \in \mathbb{F}}$. By Proposition 2.3, we have
\[
\Lambda_i f = (\Gamma_i M)_{j \in \mathbb{F}} S_{\Lambda_j \Gamma_j S^{-\frac{1}{2}}}_{j \in \mathbb{F}} f.
\]

It is easy to check that $(\Gamma_i M)_{j \in \mathbb{F}} = (\Gamma_i (M^* S)^{-1}\Gamma_j S^{-\frac{1}{2}})_{j \in \mathbb{F}}$ and $(\Gamma_j S^{-\frac{1}{2}})_{j \in \mathbb{F}} = (\Gamma_j S^\frac{1}{2})_{j \in \mathbb{F}}$. Therefore, for every $i \in I$ and $f \in H$, $\Lambda_i f = Y_i S^{-1} S_{\Phi(\Gamma S)^{-1}} f$.

Since the class of g-R-duals of type III is contained in the class of g-R-duals of type IV, by substituting $M = S^\frac{1}{2}$ in the above equation, we have $\Lambda_i f = Y_i S^{-1} S_{\Phi(\Gamma S)^{-1}} f$.

**Theorem 2.5.** Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-Bessel sequence in $H$ and $\Phi^\Lambda = \{\Phi_j \in L(H, H_i) : j \in I\}$ be the g-R-dual sequence of type IV of $\Lambda$ with respect to g-Riesz bases $\Gamma = \{\Gamma_j \in L(H, H_i) : i \in I\}$ and $Y = \{Y_i \in L(H, H_i) : i \in I\}$. Then $\Lambda$ is a g-frame if and only if $\Phi^\Lambda$ is a g-Riesz sequence.

**Proof.** Let $0 < B_1 < B_2$ and $0 < C_1 < C_2$ be g-Riesz bounds for $\Gamma$ and $Y$, respectively.

Suppose that $\Lambda$ is a g-frame with bounds $0 < A_1 \leq A_2$. For every finite subset $F \subset I$ we have
\[
\| \sum_{j \in F} \Phi_j^\ast g_j \|^2 = \| \sum_{j \in F} S_{\gamma_j \Gamma_j} g_j \|^2 = \| S_{\gamma_j \Gamma_j} (\sum_{j \in F} \Gamma_j g_j) \|^2 \leq A_2 C_2 \| \sum_{j \in F} \Gamma_j g_j \|^2 \leq A_2 B_2 C_2 \sum_{j \in F} \| g_j \|^2.
\]

Similarly, we can get the following result
\[
\| \sum_{j \in F} \Phi_j^\ast g_j \|^2 \geq A_1 B_1 C_1 \sum_{j \in F} \| g_j \|^2.
\]
Therefore \( \{\Phi_j^\alpha\}_{j \in \mathbb{Z}} \) is a g-Riesz sequence in \( H \).

Conversely, let \( \{\Phi_j^\alpha\}_{j \in \mathbb{Z}} \) be a g-Riesz sequence with g-Riesz bounds \( 0 < D_1 \leq D_2 \) in \( H \). Suppose that \( f \in \text{span}_{j \in \mathbb{Z}}(\Gamma_{j} H_j) \), then there is a finite set \( F \subset I \) and \( \{g_j \in H_j : j \in F\} \) such that \( f = \sum_{j \in F} \Gamma_j^* g_j \). We have

\[
\sum_{i \in I} \|A_i f\|^2 = \sum_{i \in I} \|A_i(\sum_{j \in F} \Gamma_j^* g_j)\|^2 = \sum_{i \in F} \|A_i \Gamma_j^* g_j\|^2 \leq \frac{1}{C_1} \|\sum_{i \in F} \sum_{j \in F} \Gamma_i^* A_i \Gamma_j^* g_j\|^2 = \frac{1}{C_1} \|\sum_{j \in F} \sum_{i \in F} \Gamma_i^* A_i \Gamma_j^* g_j\|^2 \\
= \frac{1}{C_1} \|\sum_{j \in F} \Phi_j^\alpha g_j\|^2 \leq \frac{D_2}{C_1} \sum_{j \in F} \|g_j\|^2 \leq \frac{D_2}{B_1 C_1} \|f\|^2.
\]

Similarly, we can get the following result

\[
\sum_{i \in I} \|A_i f\|^2 \geq \frac{D_1}{B_2 C_2} \|f\|^2.
\]

Since \( \text{span}_{j \in \mathbb{Z}}(\Gamma_{j} H_j) = H \), then \( \Lambda \) is a g-frame in \( H \).

In the following theorem, we give a characterization of g-Riesz bases in terms of their g-Rdual of type IV.

**Theorem 2.6.** Let \( \Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\} \) be a g-Bessel sequence for \( H \) and \( \Phi^\lambda = \{\Phi_j^\lambda \in L(H, H_j) : j \in I\} \) be the g-R-dual sequence of type IV of \( \Lambda \) with respect to g-Riesz bases \( \Gamma \) = \{\Gamma_i \in L(H, H_i) : i \in I\} and \( \Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in I\} \). Then \( \Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\} \) is a g-Riesz basis if and only if \( \Phi^\lambda \) is a g-Riesz basis.

**Proof.** We know that \( \Lambda \) is a g-Bessel sequence if and only if \( \Phi^\lambda \) is a g-Bessel sequence. For every \( f \in H \), we have

\[
S_{\Lambda^*} f = \sum_{j \in \mathbb{Z}} \Gamma_j^* (S_{\Lambda^*} f) = \sum_{j \in \mathbb{Z}} \Gamma_j^* \Phi_j f = S_{\Phi^\lambda} f.
\]

Therefore \( S_{\Lambda^*} = S_{\Phi^\lambda} \). Since \( S_{\Lambda^*} = T_{\Lambda^*} T_{\Upsilon}^* \) and \( \Upsilon \) is a g-Riesz basis, then \( S_{\Lambda^*} \) is invertible if and only if \( T_{\Lambda^*} \) is invertible which is equivalent to \( \Lambda \) is a g-Riesz basis. Therefore \( \Lambda \) is a g-Riesz basis if and only if \( S_{\Lambda^*} \) is invertible. Similarly \( \Phi^\lambda \) is a g-Riesz basis if and only if \( S_{\Phi^\lambda} \) is invertible. Since \( S_{\Phi^\lambda} = S_{\Phi_{\Gamma}^\lambda} \) by the above relation, \( \Lambda \) is a g-Riesz basis if and only if \( \Phi^\lambda \) is a g-Riesz basis.

We note that, since every g-orthonormal basis is a g-Riesz basis, the above theorem is a generalization of Proposition 3.10 in [7].

In the following theorem, we characterize all dual g-frames of a g-frame in terms of its g-Rdual of type IV which can be considered as a generalization of Wexler-Raz biorthogonality relations to g-frames.

**Theorem 2.7.** Let \( \Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\} \), \( \Psi = \{\Psi_i \in L(H, H_i) : i \in I\} \) be g-frames and \( \Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\} \), \( \Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in I\} \) be g-Riesz bases in \( H \). Let \( \Phi^\psi \) be the g-R-dual of type IV of \( \Psi \) with respect to g-Riesz bases \( \Gamma \) and \( \Upsilon \) and \( \Phi^\lambda \) be the g-R-dual of type IV of \( \Lambda \) with respect to g-Riesz bases \( \Gamma \) and \( \Upsilon \). Then the following statements are equivalent:

(i) \( \Psi \) and \( \Lambda \) are dual g-frames.

(ii) \( S_{\Psi^*} = S_{\Phi^\lambda} = I \).

(iii) \( \langle \Phi_j^\psi g_j, \Phi_k^\lambda g_k \rangle = \delta_{jk} \langle g_j, g_k \rangle \quad \forall g_j \in H_j, g_k \in H_k \quad (j, k \in I) \).
Proof. The equivalence of (1) and (2) is obvious. Since \( Y \) is a g-Riesz basis, Corollary 3.3 in [9], easily implies that \( S_{Q^*}S_{\Gamma^*} = S_{Q^*} \). For every \( g_j \in H_j, g_k \in H_k, j, k \in I \) we have
\[
\langle \Phi_j^* g_j, \Phi_k^* g_k \rangle = \langle S_{\Gamma_j^*} g_j, S_{\Gamma_k^*} \Gamma_k g_k \rangle = (\Gamma_j^* g_j, S_{Q^*} \Gamma_k g_k).
\]
Therefore \( \langle \Phi_j^* g_j, \Phi_k^* g_k \rangle = \delta_{jk} \langle g_j, g_k \rangle \) if and only if \( (\Gamma_j^* g_j, S_{Q^*} \Gamma_k^* g_k) = (\Gamma_j^* g_j, \Gamma_k^* g_k) \) which is equivalent to \( S_{Q^*} \Gamma_k g_k = \Gamma_k^* g_k \) for every \( k \in I \). Since span \( \in I \Gamma_k^* (H_k) = H \) and \( S_{Q^*} \) is continuous, this is equivalent to \( S_{Q^*} = I \). Therefore (2) is equivalent to (3).

In the following lemma we prove that if \( \Lambda \) is a g-Bessel sequence, then there exists a basic relation between its synthesis operator and span \( \in I (\Phi_j^*) \), see [7, Lemma 3.6].

Lemma 2.8. Let \( \Lambda = \{ \Lambda_i \in L(H, H_i) : i \in I \} \) be a g-Bessel sequence with synthesis operator \( T_{\Lambda^*} \) and \( \Phi^* \) is the g-R-dual sequence of type IV of \( \Lambda \) with respect to g-Riesz bases \( \Gamma = \{ \Gamma_i \in L(H, H_i) : i \in I \} \), \( Y = \{ Y_i \in L(H, H_i) : i \in I \} \). Let \( (h_i)_{i \in I} \subseteq \sum_{i \in I} \ker T_{\Lambda^*} = 0 \). Then
(i) \( h \in \ker T_{\Phi^*} \) if and only if \( (Y_i h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \) (equivalently \( S_{\Lambda^*} h = 0 \)).
(ii) \( (h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \) if and only if \( \sum_{i \in I} \Gamma_i^* h_i \subseteq \ker T_{\Phi^*} \).
(iii) \( \Phi^* \) is g-complete if and only if \( T_{\Lambda^*} \) is injective.

Proof. (1) \( h \in \ker T_{\Phi^*} \) if and only if \( (Y_i h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \) if and only if \( (Y_i h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \) if and only if \( (Y_i h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \). Therefore, \( h \in \ker T_{\Phi^*} \) if and only if \( (Y_i h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \).

(2) \( \sum_{i \in I} \Gamma_i^* h_i \subseteq \ker T_{\Lambda^*} \).

(3) \( (h_i)_{i \in I} \subseteq \ker T_{\Lambda^*} \).

In the following theorem, we give another characterization of dual g-frames.

Theorem 2.9. Let \( \Lambda = \{ \Lambda_i \in L(H, H_i) : i \in I \} \) be a g-frame with g-frame operator \( S_{\Lambda} \) and \( \Phi^* \) is the g-R-dual sequence of type IV of \( \Lambda \) with respect to g-orthonormal bases \( \Gamma = \{ \Gamma_i \in L(H, H_i) : i \in I \} \), \( Y = \{ Y_i \in L(H, H_i) : i \in I \} \). Then the following statements are equivalent:
(i) \( \Theta \) is a dual g-frame of \( \Lambda \).
(ii) There exists a g-Bessel sequence \( \{ \Lambda_i \} \subseteq \sum_{i \in I} \ker T_{\Phi^*} \) such that for every \( g_j \in H_j, j \in I \)
\[
\Phi_j^* g_j - \Phi_j^* g_j = M_j g_j.
\]

Proof. Let \( \Theta = (\Theta_i)_{i \in I} \) be a dual g-frame of \( \Lambda = (\Lambda_i)_{i \in I} \). Then for every \( g_j \in H_j, j \in I \), we have
\[
\Gamma_j^* g_j = S_{\Theta^*} (\Gamma_j^* g_j) = \sum_{i \in I} \Lambda_i^* \Theta_i \Gamma_j^* g_j = \sum_{i \in I} \Lambda_i^* (\Theta_i - \Lambda_i S_{\Lambda}^{-1} + \Lambda_i S_{\Lambda}^{-1}) \Gamma_j^* g_j
\]
\[
= \sum_{i \in I} \Lambda_i^* (\Theta_i - \Lambda_i S_{\Lambda}^{-1}) \Gamma_j^* g_j + \sum_{i \in I} \Lambda_i^* \Lambda_i S_{\Lambda}^{-1} \Gamma_j^* g_j
\]
\[
= \sum_{i \in I} \Lambda_i^* (\Theta_i - \Lambda_i S_{\Lambda}^{-1}) \Gamma_j^* g_j + S_{\Lambda \Lambda_{S_{\Lambda}^{-1}}} \Gamma_j^* g_j.
\]
Since $S_{\Lambda S^{-1}} = I$, then $\sum_{i \in I} \Lambda_i^* (\Theta_i - \Lambda_i S^{-1}) \Gamma_j^* g_j = 0$ and by Lemma 2.8, we have

$$\sum_{i \in I} \Lambda_i^* (\Theta_i - \Lambda_i S^{-1}) \Gamma_j^* g_j \in \{ \text{span}_{i \in I} \Phi_j^* \}(H_j) \}^\perp.$$

Now, define $M_j : H_j \to \{ \text{span}_{i \in I} \Phi_j^* \}(H_j) \}^\perp \subseteq H$ by

$$M_j g_j = \sum_{i \in I} \Lambda_i^* \Theta_i \Gamma_j^* g_j - \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} \Gamma_j^* g_j \quad (g_j \in H_j, j \in I).$$

Then $M_j g_j = \Phi_j^* g_j - \Phi_j^{S^{-1}} g_j$ $(g_j \in H_j, j \in I)$. So $M_j : \{ \text{span}_{i \in I} \Phi_j^* \}(H_j) \}^\perp \to H$ and $(M_j^*)_{j \in I}$ is a g-Bessel sequence for $\{ \text{span}_{i \in I} \Phi_j^* \}(H_j) \}^\perp$ with respect to $\{ H_j ; i \in I \}$. Because, let $A'$ be an upper g-frame bound for $\Theta$. Then for every $f \in \{ \text{span}_{i \in I} \Phi_j^* \}(H_j) \}^\perp$, we have

$$\sum_{j \in I} \| M_j f \|^2 = \sum_{j \in I} \| \Phi_j f - \Phi_j^{S^{-1}} f \|^2 = \sum_{j \in I} \| \Gamma_j S_{\Theta \Gamma} f - \Gamma_j S_{\Lambda S^{-1} \Gamma} f \|^2,$$

$$= \sum_{j \in I} \| \Gamma_j S_{\Theta \Gamma} f - \Gamma_j S_{\Lambda S^{-1} \Gamma} f \| \sum_{i \in I} \Lambda_i^* \| f \|^2,$$

since $f \in \{ \text{span}_{i \in I} \Phi_j^* \}(H_j) \}^\perp$ by Lemma 2.8, $\sum_{i \in I} \Lambda_i^* \| f \|^2 = 0$. Therefore

$$\sum_{j \in I} \| M_j f \|^2 = \sum_{j \in I} \| \Gamma_j S_{\Theta \Gamma} f \| \sum_{i \in I} \Lambda_i^* \| f \|^2 \leq A' \| f \|^2.$$

Conversely, suppose that (2) holds. Since for every $g \in H, j \in I$, $\Gamma_j g \in H_j$, then we have

$$M_j \Gamma_j g = \Phi_j^* \Gamma_j g - \Phi_j^{S^{-1}} \Gamma_j g$$

Therefore by [7, Lemma 3.3], for every $i \in I$

$$(\Theta_i - \Lambda_i S^{-1}) g = \sum_{j \in I} \Lambda_i \Gamma_j^* g_j.$$ 

So for every $g_l \in H_l, l \in I$ we have

$$\sum_{i \in I} \Lambda_i^* \Theta_i \Gamma_i^* g_l = \sum_{i \in I} \Lambda_i^* (\Lambda_i S^{-1} + \Theta_i - \Lambda_i S^{-1}) \Gamma_i^* g_l$$

$$= \Gamma_i^* g_l + \sum_{i \in I} \Lambda_i^* (\Theta_i - \Lambda_i S^{-1}) \Gamma_i^* g_l$$

$$= \Gamma_i^* g_l + \sum_{i \in I} \Lambda_i^* (\sum_{j \in I} \Lambda_j M_j (\Gamma_j \Gamma_i^* g_l))$$

$$= \Gamma_i^* g_l + \sum_{i \in I} \Lambda_i^* \sum_{j \in I} \Lambda_j M_j (\Gamma_j \Gamma_i^* g_l)$$

$$= \Gamma_i^* g_l + \sum_{i \in I} \Lambda_i^* \Lambda_i M_i g_l,$$

since $M_i g_l \in \{ \text{span}_{j \in I} \Phi_j^* \}(H_i) \}^\perp$, then by Lemma 2.8, $\sum_{i \in I} \Lambda_i^* \Lambda_i M_i g_l = 0$. Therefore $\Theta$ is a dual g-frame of $\Lambda$ and this implies (1). \hfill \Box

Now, we present a characterization of the canonical dual g-frames.

**Corollary 2.10.** Let $\Lambda = \{ \Lambda_i \in L(H, H_i) : i \in I \}$ be a g-frame with the canonical dual $\tilde{\Lambda} = \{ \tilde{\Lambda}_i \in L(H, H_i) : i \in I \}$ and $\Phi^* = \{ \Phi_j^* \in L(H, H_j) : j \in I \}$ be the $g$-R-dual sequence of type I of $\Lambda$ with respect to $g$-orthonormal bases $\Gamma = \{ \Gamma_i \in L(H, H_i) : i \in I \}$ and $\tilde{\Gamma} = \{ \tilde{\Gamma}_i \in L(H, H_i) : i \in I \}$. Let $\Theta = \{ \Theta_i \in L(H, H_i) : i \in I \}$ be a dual g-frame of $\Lambda$. Then for every $g_j \in H_j, j \in I$

$$\| \Phi_j^* g_j \| \geq \| \Phi_j^{\tilde{\Gamma}} g_j \|,$$

with equality if and only if $\Theta = \tilde{\Lambda}$. 
Proof. Let $T_{\Lambda}$ and $T_{\widetilde{\Lambda}}$ be the synthesis operators of $\Lambda$ and $\widetilde{\Lambda}$, respectively. Easily we can see that $\ker T_{\Lambda} = \ker T_{\widetilde{\Lambda}}$, then by Lemma 2.8, $\text{span}_{j \in \mathcal{I}} \Phi_{\Lambda}^{\ast}(H_{j}) = \text{span}_{j \in \mathcal{I}} \Phi_{\widetilde{\Lambda}}^{\ast}(H_{j})$, so $\text{Ran} \Phi_{\ast}^{\ast} \subseteq \text{span}_{j \in \mathcal{I}} \Phi_{\ast}^{\ast}(H_{j})$. On the other hand by the above theorem, for every $g_{j} \in H_{j}, j \in \mathcal{I}$ we have $\Phi_{\ast}^{\ast} g_{j} = \Phi_{\ast}^{\ast} g_{j} + M_{j} g_{j}$, where $\text{Ran} M_{j} \subseteq \text{span}_{j \in \mathcal{I}} \Phi_{\ast}^{\ast}(H_{j})$. But $\text{span}_{j \in \mathcal{I}} \Phi_{\ast}^{\ast}(H_{j}) = \text{span}_{j \in \mathcal{I}} \Phi_{\ast}^{\ast}(H_{j})$, so $\text{Ran} M_{j} \subseteq \text{span}_{j \in \mathcal{I}} \Phi_{\ast}^{\ast}(H_{j})$. Then for every $g_{j} \in H_{j}, j \in \mathcal{I}$ we have
\[
\|\Phi_{\ast}^{\ast} g_{j}\|^{2} = \|\Phi_{\ast}^{\ast} g_{j} + M_{j} g_{j}\|^{2} \geq \|\Phi_{\ast}^{\ast} g_{j}\|^{2}.
\]
By the above theorem, the equality holds if and only if $\Theta = \widetilde{\Lambda}$.


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