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DUALITY PRINCIPLE IN g-FRAMES

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Abstract. The concept of Riesz dual sequences (R-dual sequences) was introduced by Casazza et al. in 2004. Recently, for generalizing this concept to g-frames the concept of g-Riesz dual sequences has been introduced and various definitions of R-duals for frames are in the literature. In this paper, we generalize these concepts for g-frame and introduce g-Riesz duals (g-R-duals) of type II, III and IV. Since the g-R-dual of type IV is the most general g-R-dual, we focus on the g-R-dual of type IV. We give characterizations of g-frames and g-Riesz bases in terms of their g-R-dual of type IV. We characterize all dual g-frames of a g-frame in terms of its g-R-dual of type IV which can be considered as Wexler-Raz biorthogonality relations for g-frames. Also, we present a generalization of Ron-Shen duality principle to g-frames. In addition, we investigate the construction of dual g-frames in more details and we give another characterization of dual g-frames with respect to its g-R-dual sequence.

1 Introduction and preliminaries

The concept of R-duality of a Bessel sequence in a separable Hilbert space was introduced by Casazza, Kutyniok and Lammers in [1], in order to obtain a generalization of duality principles in Gabor frames to abstract frame theory.
Let $(e_i)_{i\in\mathcal{I}}, (h_i)_{i\in\mathcal{I}}$ be orthonormal bases and $(f_i)_{i\in\mathcal{I}}$ be a Bessel sequence. The R-dual sequence of $(f_i)_{i\in\mathcal{I}}$ with respect to the orthonormal bases $(e_i)_{i\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ is the sequence $(w_j^f)_{j\in\mathcal{I}}$, such that for every $j\in\mathcal{I}$

$$w_j^f = \sum_{i\in\mathcal{I}} \langle f_i, e_j\rangle h_i.$$

The R-duality has been favored by many authors. The R-duality with respect to orthonormal bases has been discussed in [2] and [3]. In [8], the authors introduced various alternative R-duals and showed their relation with Gabor frames. In [11], the authors generalized the R-duality in Banach spaces. In [4] the authors, proved that the duality principle extends to any pair of projective unitary representation of countable groups. Recently, for generalizing this concept to g-frames the concept of g-Riesz dual sequences has been introduced [7]. Various definitions of R-duals for frames are in the literature.

In this paper, we generalize these concepts to g-frame and introduce g-Riesz duals (g-R-duals) of type II, III and IV. Since the g-R-dual of type IV is the most general g-R-dual, we focus on the g-R-dual of type IV. We give characterizations of g-frames and g-Riesz bases in terms of their g-R-dual of type IV. We characterize all dual g-frames of a g-frame in terms of its g-R-dual of type IV, which can be considered as Wexler-Raz biorthogonality relations for g-frames. Also, we present a generalization of Ron-Shen duality principle to g-frames. In addition, we investigate the construction of dual g-frames in more details and we give another characterization of dual g-frames with respect to its g-R-dual sequence.
Throughout this paper H denotes a separable Hilbert space with inner product $\langle ., .\rangle$ and $\mathcal{I}$ is a subset of $\mathbb{Z}$, and $\{H_i : i\in\mathcal{I}\}$ is a sequence of separable Hilbert spaces . Also, for every $i\in\mathcal{I}$, $L(H, H_i)$ is the set of all bounded, linear operators from H to H_i.

In the rest of this section we review several well-known definitions and results. The new results are stated in Section 2.
For every sequence $\{H_i\}_{i\in\mathcal{I}}$, the space

$$(\sum_{i\in\mathcal{I}}\bigoplus H_i)_{\ell^2}=\{(f_i)_{i\in\mathcal{I}}:f_i\in H_i,i\in\mathcal{I},\sum_{i\in\mathcal{I}}\|f_i\|^2<\infty\}$$

with pointwise operations and the following inner product is a Hilbert space

$$\langle(f_i)_{i\in\mathcal{I}},(g_i)_{i\in\mathcal{I}}\rangle=\sum_{i\in\mathcal{I}}\langle f_i,g_i\rangle.$$

A sequence $\Lambda=\{\Lambda_i\in L(H,H_i):i\in\mathcal{I}\}$ is called a *g-frame* for H with respect to $\{H_i:i\in\mathcal{I}\}$ if there exist $A,B>0$ such that for every $f\in H$

$$A\|f\|^2\le\sum_{i\in\mathcal{I}}\|\Lambda_i f\|^2\le B\|f\|^2,$$

A,B are called g-frame bounds. We call Λ a *tight g-frame* if $A=B$ and a *Parseval g-frame* if $A=B=1$. If only the right hand side inequality is required, Λ is called a *g-Bessel sequence*.
If Λ is a g-Bessel sequence, then *the synthesis operator* for Λ is the linear operator,

$$T_\Lambda:(\sum_{i\in\mathcal{I}}\bigoplus H_i)_{\ell^2}\mapsto H\qquad T_\Lambda(f_i)_{i\in\mathcal{I}}=\sum_{i\in\mathcal{I}}\Lambda_i^*f_i.$$

We call the adjoint of the synthesis operator, *the analysis operator*. The analysis operator is the linear operator,

$$T_\Lambda^*:H\mapsto(\sum_{i\in\mathcal{I}}\bigoplus H_i)_{\ell^2}\qquad T_\Lambda^*f=(\Lambda_i f)_{i\in\mathcal{I}}.$$

We call $S_\Lambda=T_\Lambda T_\Lambda^*$ the *g-frame operator* of Λ and $S_\Lambda f=\sum_{i\in\mathcal{I}}\Lambda_i^*\Lambda_i f,\quad(f\in H)$.
If $\Lambda=(\Lambda_i)_{i\in\mathcal{I}}$ is a g-frame with lower and upper g-frame bounds A,B, respectively, then the g-frame operator of Λ is a bounded, positive and invertible operator on H and

$$A\langle f,f\rangle\le\langle S_\Lambda f,f\rangle\le B\langle f,f\rangle\quad(f\in H)$$

so

$$A.I\le S_\Lambda\le B.I.$$

The canonical dual g-frame for $\Lambda=(\Lambda_i)_{i\in\mathcal{I}}$ is defined by $\widetilde{\Lambda}=(\widetilde{\Lambda_i})_{i\in\mathcal{I}}$, where $\widetilde{\Lambda_i}=\Lambda_i S_\Lambda^{-1}$ which is also a g-frame for H with lower and upper g-frame bounds $\frac{1}{B}$ and $\frac{1}{A}$, respectively. Also for every $f\in H$, we have

$$f=\sum_{i\in\mathcal{I}}\Lambda_i^*\widetilde{\Lambda_i}f=\sum_{i\in\mathcal{I}}\widetilde{\Lambda_i}^*\Lambda_i f.$$

We say $\Lambda=\{\Lambda_i\in L(H,H_i):i\in\mathcal{I}\}$ is a *g-frame sequence* if it is a g-frame for $\overline{span\{\Lambda_i^*(H_i)\}_{i\in\mathcal{I}}}$. A sequence $\Lambda=\{\Lambda_i\in L(H,H_i):i\in\mathcal{I}\}$ is *g-complete* if $\{f:\Lambda_i f=0,\forall i\in\mathcal{I}\}=\{0\}$. We note that the g-Bessel sequence Λ is g-complete if and only if T_Λ^* is injective. We say that Λ is a *g-orthonormal basis* for H, if

$$\langle\Lambda_i^*f_i,\Lambda_j^*f_j\rangle=\delta_{i,j}\langle f_i,f_j\rangle,\quad\forall f_i\in H_i,f_j\in H_j,i,j\in\mathcal{I}$$

and

$$\sum_{i\in\mathcal{I}}\|\Lambda_i f\|^2=\|f\|^2\quad(f\in H).$$

A sequence $\Lambda=\{\Lambda_i\in L(H,H_i):i\in\mathcal{I}\}$ is a *g-Riesz sequence* if there exist $A,B>0$ such that for every finite subset $F\subset\mathcal{I}$ and $g_i\in H_i,i\in F$

$$A\sum_{i\in F}\|g_i\|^2\le\|\sum_{i\in F}\Lambda_i^*g_i\|^2\le B\sum_{i\in F}\|g_i\|^2.\tag{1.1}$$

G-Riesz sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is called a *g-Riesz basis* if it is g-complete, too. So Λ is a g-Riesz basis if and only if T_Λ is a bounded invertible operator. Clearly, every g-orthonormal basis is a g-Riesz basis.

Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Theta = \{\Theta_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-Bessel sequences with g-Bessel bounds B and C, respectively. The operator $S_{\Lambda\Theta} : H \mapsto H$ defined by

$$S_{\Lambda\Theta} f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Theta_i f, \quad (f \in H)$$

is a bounded operator, $\|S_{\Lambda\Theta}\| \leq \sqrt{BC}$, $S_{\Lambda\Theta}^* = S_{\Theta\Lambda}$ and $S_{\Lambda\Lambda} = S_\Lambda$.
Two g-Bessel sequences $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Theta = \{\Theta_i \in L(H, H_i) : i \in \mathcal{I}\}$ are called *dual g-frames* if

$$f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Theta_i f = \sum_{i\in\mathcal{I}} \Theta_i^* \Lambda_i f, \quad (f \in H).$$

For more details about g-frames, see [6, 9].

2 Main results

In this section, first we consider the g-Riesz dual(g-R-dual) with respect to g-orthonormal bases as the g-R dual of type I in [7] and we introduce alternative definitions of g-R-duals.

Definition 2.1. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-frame for H with g-frame operator S.

(i) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-orthonormal bases. The g-R-dual of type I of Λ with respect to Γ and Υ is $\Phi^\Lambda = (\Phi_j^\Lambda)_{j\in\mathcal{I}}$ defined by

$$\Phi_j^\Lambda f = \Gamma_j S_{\Lambda\Upsilon} f \quad (f \in H).$$

(ii) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-orthonormal bases. The g-R-dual of type II of Λ with respect to Γ and Υ is $\Phi^\Lambda = (\Phi_j^\Lambda)_{j\in\mathcal{I}}$ defined by

$$\Phi_j^\Lambda f = \Gamma_j S^{-\frac{1}{2}} S_{\Lambda(\Upsilon S^{\frac{1}{2}})} f \quad (f \in H).$$

(iii) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-orthonormal bases and $M : H \to H$ be a bounded invertible operator with $||M|| \leq \sqrt{||S||}$ and $||M^{-1}|| \leq \sqrt{||S^{-1}||}$. The g-R-dual of type III of Λ with respect to triplet (Γ, Υ, M) is $\Phi^\Lambda = (\Phi_j^\Lambda)_{j\in\mathcal{I}}$ defined by

$$\Phi_j^\Lambda f = \Gamma_j S_{(\Lambda S^{-\frac{1}{2}})(\Upsilon M)} f \quad (f \in H).$$

(iv) Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-Riesz bases. The g-R-dual of type IV of Λ with respect to Γ and Υ is $\Phi^\Lambda = (\Phi_j^\Lambda)_{j\in\mathcal{I}}$ defined by

$$\Phi_j^\Lambda f = \Gamma_j S_{\Lambda\Upsilon} f \quad (f \in H).$$

In all of the above cases, it is obvious that Φ_j^Λ is well-defined and $\Phi_j^\Lambda \in L(H, H_j)$, for every $j \in \mathcal{I}$.

Clearly, the g-R-duals of type II are contained in the class of g-R-duals of type III and the g-R-duals of type III are contained in the class of g-R-duals of type IV. Moreover, the g-R-duals of type I, II, and III are contained in the class of g-R-duals of type IV.

In the following proposition, we show that for tight g-frames the g-R-duals of type I, II and III coincide.

Proposition 2.2. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a tight g-frame. Then the g-R-duals of type I , II and III coincide.*

Proof. Denote the g-frame operator for Λ by S. Since Λ is a tight g-frame, then $S = AI$ for some $A > 0$.
For every $j \in \mathcal{I}$, $\Gamma_j S_{\Lambda\Upsilon} = \Gamma_j A^{-\frac{1}{2}} S_{\Lambda(\Upsilon A^{\frac{1}{2}})} = \Gamma_j S^{-\frac{1}{2}} S_{\Lambda(\Upsilon S^{\frac{1}{2}})}$. Therefore the g-R-duals of type I and II coincide.

Let $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-orthonormal bases. Take the bounded invertible operator $M : H \to H$ such that $||M|| \leq \sqrt{||S||} = \sqrt{A}$ and $||M^{-1}|| \leq \sqrt{||S^{-1}||} = \frac{1}{\sqrt{A}}$. Suppose that $(g_i)_{i\in\mathcal{I}} \in (\sum_{i\in\mathcal{I}} \oplus H_i)_{\ell^2}$, then we have

$$||\sum_{i\in\mathcal{I}}(\Upsilon_i M)^* g_i||^2 = ||\sum_{i\in\mathcal{I}} M^*\Upsilon_i^* g_i||^2 \leq ||M^*||^2||\sum_{i\in\mathcal{I}}\Upsilon_i^* g_i||^2 \leq A\sum_{i\in\mathcal{I}}||g_i||^2.$$

and

$$||\sum_{i\in\mathcal{I}}(\Upsilon_i M)^* g_i||^2 = ||\sum_{i\in\mathcal{I}} M^*\Upsilon_i^* g_i||^2 \geq \frac{1}{||(M^*)^{-1}||^2}||\sum_{i\in\mathcal{I}}\Upsilon_i^* g_i||^2 \geq A\sum_{i\in\mathcal{I}}||g_i||^2.$$

Therefore $(\Upsilon_i M)_{i\in\mathcal{I}}$ is a tight g-Riesz basis with bound A. We can see that $\frac{M}{\sqrt{A}}$ is a unitary operator. Since Υ is a g-orthonormal basis, then $(\frac{1}{\sqrt{A}}\Upsilon_i M)_{i\in\mathcal{I}}$ is a g-orthonormal basis, denote it by $(\Psi_i)_{i\in\mathcal{I}}$. Hence $(\sqrt{A}\Psi_i)_{i\in\mathcal{I}} = (\Upsilon_i M)_{i\in\mathcal{I}}$. Now, the g-R-dual of Λ of type III with respect to (Γ, Υ, M) is

$$\Phi_j = \Gamma_j S_{(\Lambda S^{-\frac{1}{2}})(\Upsilon M)} = \Gamma_j S_{(\frac{1}{\sqrt{A}}\Lambda)(\sqrt{A}\Psi)} = \Gamma_j S_{(\Lambda)(\Psi)}$$

which is a g-R-dual of type I of Λ.

Since the g-R-duals of type II are contained in the class of g-R-duals of type III and for tight g-frames the g-R-duals of type I and II coincide, then for tight g-frames, g-R-duals of type I, II and III coincide. □

Since the g-R-dual of type IV is the most general g-R-dual, we focus on the g-R-dual of type IV and we give some characterizations of it. Note that all results about the g-R-dual of type IV hold for the g-R-duals of type I, II and III.

In the following proposition, we present an algorithm which invert the process of mapping Λ to its g-R dual of type IV (Φ^Λ).

Proposition 2.3. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-Bessel sequence with g-Bessel bound A and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ be g-Riesz bases. Let $\Phi^\Lambda = (\Phi_j^\Lambda)_{j\in\mathcal{I}}$ be the g-R dual sequence of type IV of Λ with respect to Γ and Υ. Then Φ^Λ is a g-Bessel sequence and Λ is the g-R-dual sequence of type IV of Φ^Λ with respect to g-Riesz bases $(\widetilde{\Upsilon}_i)_{i\in\mathcal{I}}$ and $(\widetilde{\Gamma}_i)_{i\in\mathcal{I}}$, where $(\widetilde{\Upsilon}_i)_{i\in\mathcal{I}}$ and $(\widetilde{\Gamma}_i)_{i\in\mathcal{I}}$ are dual g-Riesz bases of Υ and Γ, respectively. In the sense that for every $i \in \mathcal{I}$ we have*

$$\Lambda_i f = \sum_{j\in\mathcal{I}}\widetilde{\Upsilon}_i\Phi_j^{\Lambda*}\widetilde{\Gamma}_j f = \widetilde{\Upsilon}_i S_{\Phi^\Lambda\widetilde{\Gamma}} f \quad (f \in H).$$

Proof. Let B and C be upper g-Riesz bounds for Γ and Υ, respectively. Since Υ is a g-Riesz basis with upper g-Riesz bound C, then it is a g-frame with upper g-frame bound C, too. On the other hand, Λ is a g-Bessel sequence with g-Bessel bound A. Therefore $||S_{\Lambda\Upsilon}|| \leq \sqrt{AC}$, see [6]. Hence for every $f \in H$ we have

$$\sum_{j\in\mathcal{I}}\|\Phi_j^\Lambda f\|^2 \quad = \quad \sum_{j\in\mathcal{I}}\|\Gamma_j S_{\Lambda\Upsilon} f\|^2 \leq B\|S_{\Lambda\Upsilon} f\|^2 \leq ABC\|f\|^2.$$

Therefore Φ^Λ is a g-Bessel sequence in H.

For every $f \in H$ and $g_i \in H_i$ we have

$$\begin{aligned}\langle \widetilde{\Upsilon}_i S_{\Phi^\Lambda \widetilde{\Gamma}} f, g_i\rangle &= \sum_{j\in\mathcal{I}} \langle {\Phi_j^\Lambda}^* \widetilde{\Gamma}_j f, \widetilde{\Upsilon}_i^* g_i\rangle = \sum_{j\in\mathcal{I}} \langle S_{\Upsilon\Lambda}\Gamma_j^* \widetilde{\Gamma}_j f, \widetilde{\Upsilon}_i^* g_i\rangle \\ &= \langle S_{\Upsilon\Lambda} \sum_{j\in\mathcal{I}} \Gamma_j^* \widetilde{\Gamma}_j f, \widetilde{\Upsilon}_i^* g_i\rangle = \langle f, S_{\Lambda\Upsilon} \widetilde{\Upsilon}_i^* g_i\rangle \\ &= \langle f, \sum_{k\in\mathcal{I}} \Lambda_k^* \Upsilon_k \widetilde{\Upsilon}_i^* g_i\rangle = \langle f, \Lambda_i^* g_i\rangle = \langle \Lambda_i f, g_i\rangle.\end{aligned}$$

Thus for every $i \in \mathcal{I}$

$$\Lambda_i f = \sum_{j\in\mathcal{I}} \widetilde{\Upsilon}_i {\Phi_j^\Lambda}^* \widetilde{\Gamma}_j = \widetilde{\Upsilon}_i S_{\Phi^\Lambda \widetilde{\Gamma}} f \qquad (f \in H).$$

□

Corollary 2.4. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-frame with g-frame operator S and $(\Phi_j)_{j\in\mathcal{I}}$ be the g-R-dual of type III of Λ with respect to g-orthonormal bases $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$, $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$ and invertible operator M. Then for every $i \in \mathcal{I}$ and $f \in H$,*

$$\Lambda_i f = \Upsilon_i (M^*)^{-1} S_{(\Phi)(\Gamma S^{\frac{1}{2}})} f.$$

Also, if $(\Phi_j)_{j\in\mathcal{I}}$ is the g-R-dual of type II of Λ with respect to g-orthonormal bases Γ and Υ, then for every $i \in \mathcal{I}$ and $f \in H$,

$$\Lambda_i f = \Upsilon_i S^{-\frac{1}{2}} S_{\Phi(\Gamma S^{\frac{1}{2}})} f.$$

Proof. Since $(\Phi_j)_{j\in\mathcal{I}}$ is the g-R-dual of type III of Λ with respect to g-orthonormal bases Γ and Υ and the bounded invertible operator M, then $(\Phi_j)_{j\in\mathcal{I}}$ is the g-R-dual of type IV of Λ with respect to g-Riesz bases $(\Gamma_j S^{-\frac{1}{2}})_{j\in\mathcal{I}}$ and $(\Upsilon_i M)_{i\in\mathcal{I}}$. By Proposition 2.3, we have $\Lambda_i f = (\widetilde{\Upsilon_i M})_{i\in\mathcal{I}} S_{\Lambda \widetilde{(\Gamma_j S^{-\frac{1}{2}})_{j\in\mathcal{I}}}} f$.

It is easy to check that that $(\widetilde{\Upsilon_i M})_{i\in\mathcal{I}} = (\Upsilon_i (M^*)^{-1})_{i\in\mathcal{I}}$ and $(\widetilde{\Gamma_j S^{-\frac{1}{2}}})_{j\in\mathcal{I}} = (\Gamma_j S^{\frac{1}{2}})_{j\in\mathcal{I}}$. Therefore for every $i \in \mathcal{I}$ and $f \in H$, $\Lambda_i f = \Upsilon_i (M^*)^{-1} S_{(\Phi)(\Gamma S^{\frac{1}{2}})} f$.

Since the class of g-R-duals of type II is contained in the class of g-R-dual of type III, by substituting $M = S^{\frac{1}{2}}$ in the above equation, we have $\Lambda_i f = \Upsilon_i S^{-\frac{1}{2}} S_{\Phi(\Gamma S^{\frac{1}{2}})} f$. □

In the following theorem, we present an equivalent condition for the sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ to be a g-frame, which can be regarded as a generalization of Ron-Shen duality principle to g-frames.

Theorem 2.5. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-Bessel sequence in H and $\Phi^\Lambda = \{\Phi_j^\Lambda \in L(H, H_j) : j \in \mathcal{I}\}$ be the g-R-dual sequence of type IV of Λ with respect to g-Riesz bases $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$. Then Λ is a g-frame if and only if Φ^Λ is a g-Riesz sequence.*

Proof. Let $0 < B_1 \leq B_2$ and $0 < C_1 \leq C_2$ be g-Riesz bounds for Γ and Υ, respectively.
Suppose that Λ is a g-frame with bounds $0 < A_1 \leq A_2$. For every finite subset $F \subset \mathcal{I}$ we have

$$\begin{aligned}\|\sum_{j\in F} {\Phi_j^\Lambda}^* g_j\|^2 &= \|\sum_{j\in F} S_{\Upsilon\Lambda}\Gamma_j^* g_j\|^2 = \|S_{\Upsilon\Lambda}(\sum_{j\in F}\Gamma_j^* g_j)\|^2 \\ &\leq A_2 C_2 \|\sum_{j\in F}\Gamma_j^* g_j\|^2 \leq A_2 B_2 C_2 \sum_{j\in F}\|g_j\|^2.\end{aligned}$$

Similarly, we can get the following result

$$\|\sum_{j\in F} {\Phi_j^\Lambda}^* g_j\|^2 \geq A_1 B_1 C_1 \sum_{j\in F}\|g_j\|^2.$$

Therefore $(\Phi_j^\Lambda)_{j\in\mathcal{I}}$ is a g-Riesz sequence in H.
Conversely, let $(\Phi_j^\Lambda)_{j\in\mathcal{I}}$ be a g-Riesz sequence with g-Riesz bounds $0<D_1\leq D_2$ in H. Suppose that $f\in span_{j\in\mathcal{I}}(\Gamma_j^*H_j)$, then there is a finite set $F\subset I$ and $\{g_j\in H_j : j\in F\}$ such that $f=\sum_{j\in F}\Gamma_j^* g_j$. We have

$$\begin{aligned}\sum_{i\in\mathcal{I}}\|\Lambda_i f\|^2 &= \sum_{i\in\mathcal{I}}\|\Lambda_i(\sum_{j\in F}\Gamma_j^* g_j)\|^2=\sum_{i\in\mathcal{I}}\|\sum_{j\in F}\Lambda_i(\Gamma_j^* g_j)\|^2\\ &\leq \frac{1}{C_1}\|\sum_{i\in\mathcal{I}}\sum_{j\in F}\Upsilon_i^*\Lambda_i\Gamma_j^* g_j\|^2=\frac{1}{C_1}\|\sum_{j\in F}\sum_{i\in\mathcal{I}}\Upsilon_i^*\Lambda_i\Gamma_j^* g_j\|^2\\ &= \frac{1}{C_1}\|\sum_{j\in F}\Phi_j^{\Lambda^*}g_j\|^2\leq\frac{D_2}{C_1}\sum_{j\in F}\|g_j\|^2\\ &\leq \frac{D_2}{B_1C_1}\|\sum_{j\in F}\Gamma_j^* g_j\|^2=\frac{D_2}{B_1C_1}\|f\|^2.\end{aligned}$$

Similarly, we can get the following result

$$\sum_{i\in\mathcal{I}}\|\Lambda_i f\|^2\geq\frac{D_1}{B_2C_2}\|f\|^2.$$

Since $\overline{span_{j\in\mathcal{I}}(\Gamma_j^*H_j)}=H$, then Λ is a g-frame in H. □

In the following theorem, we give a characterization of g-Riesz bases in terms of their g-R-dual of type IV.

Theorem 2.6. *Let $\Lambda=\{\Lambda_i\in L(H,H_i) : i\in\mathcal{I}\}$ be a g-Bessel sequence for H and $\Phi^\Lambda=\{\Phi_j^\Lambda\in L(H,H_j) : j\in\mathcal{I}\}$ be the g-R-dual sequence of type IV of Λ with respect to g-Riesz bases $\Gamma=\{\Gamma_i\in L(H,H_i) : i\in\mathcal{I}\}$ and $\Upsilon=\{\Upsilon_i\in L(H,H_i) : i\in\mathcal{I}\}$. Then $\Lambda=\{\Lambda_i\in L(H,H_i) : i\in\mathcal{I}\}$ is a g-Riesz basis if and only if Φ^Λ is a g-Riesz basis.*

Proof. We know that Λ is a g-Bessel sequence if and only if Φ^Λ is a g-Bessel sequence. For every $f\in H$, we have

$$S_{\Lambda\Upsilon}f=\sum_{j\in\mathcal{I}}\widetilde{\Gamma}_j^*\Gamma_j(S_{\Lambda\Upsilon}f)=\sum_{j\in\mathcal{I}}\widetilde{\Gamma}_j^*\Phi_j^\Lambda f=S_{\widetilde{\Gamma}\Phi^\Lambda}f.$$

Therefore $S_{\Lambda\Upsilon}=S_{\widetilde{\Gamma}\Phi^\Lambda}$. Since $S_{\Lambda\Upsilon}=T_\Lambda T_\Upsilon^*$ and Υ is a g-Riesz basis, then $S_{\Lambda\Upsilon}$ is invertible if and only if T_Λ is invertible which is equivalent to Λ is a g-Riesz basis. Therefore Λ is a g-Riesz basis if and only if $S_{\Lambda\Upsilon}$ is invertible. Similarly Φ^Λ is a g-Riesz basis if and only if $S_{\Phi^\Lambda\widetilde{\Gamma}}$ is invertible. Since $S_{\widetilde{\Gamma}\Phi^\Lambda}^*=S_{\Phi^\Lambda\widetilde{\Gamma}}$ by the above relation, Λ is a g-Riesz basis if and only if Φ^Λ is a g-Riesz basis. □

We note that, since every g-orthonormal basis is a g-Riesz basis, the above theorem is a generalization of Proposition 3.10 in [7].

In the following theorem, we characterize all dual g-frames of a g-frame in terms of its g-R-dual of type IV which can be considered as a generalization of Wexler-Raz biorthogonality relations to g-frames.

Theorem 2.7. *Let $\Lambda=\{\Lambda_i\in L(H,H_i) : i\in\mathcal{I}\}$, $\Psi=\{\Psi_i\in L(H,H_i) : i\in\mathcal{I}\}$ be g-frames and $\Gamma=\{\Gamma_i\in L(H,H_i) : i\in\mathcal{I}\}$, $\Upsilon=\{\Upsilon_i\in L(H,H_i) : i\in\mathcal{I}\}$ be g-Riesz bases in H. Let Φ^Ψ be the g-R-dual of type IV of Ψ with respect to g-Riesz bases Γ, Υ and Φ^Λ be the g-R-dual of type IV of Λ with respect to g-Riesz bases $\widetilde{\Gamma}$ and $\widetilde{\Upsilon}$. Then the following statements are equivalent:*

(i) Ψ *and* Λ *are dual g-frames.*

(ii) $S_{\Lambda\Psi}=S_{\Psi\Lambda}=I$.

(iii) $\langle\Phi_j^{\Psi^*}g_j,\Phi_k^{\Lambda^*}g_k\rangle=\delta_{jk}\langle g_j,g_k\rangle\quad\forall g_j\in H_j, g_k\in H_k\quad(j,k\in\mathcal{I})$.

Proof. The equivalence of (1) and (2) is obvious.
Since Υ is a g-Riesz basis, Corollary 3.3 in [9], easily implies that $S_{\Psi\Upsilon}S_{\widetilde{\Upsilon}\Lambda} = S_{\Psi\Lambda}$. For every $g_j \in H_j, g_k \in H_k, j,k \in \mathcal{I}$ we have

$$\langle \Phi_j^{\Psi^*} g_j, \Phi_k^{\Lambda^*} g_k\rangle = \langle S_{\Upsilon\Psi}\Gamma_j^* g_j, S_{\widetilde{\Upsilon}\Lambda}\widetilde{\Gamma}_k^* g_k\rangle = \langle \Gamma_j^* g_j, S_{\Psi\Lambda}\widetilde{\Gamma}_k^* g_k\rangle.$$

Therefore $\langle \Phi_j^{\Psi^*} g_j, \Phi_k^{\Lambda^*} g_k\rangle = \delta_{jk}\langle g_j, g_k\rangle$ if and only if $\langle \Gamma_j^* g_j, S_{\Psi\Lambda}(\widetilde{\Gamma}_k^* g_k)\rangle = \langle \Gamma_j^* g_j, \widetilde{\Gamma}_k^* g_k\rangle$ which is equivalent to $S_{\Psi\Lambda}(\widetilde{\Gamma}_k^* g_k) = \widetilde{\Gamma}_k^* g_k$, for every $k \in \mathcal{I}$. Since $\overline{span_{i\in\mathcal{I}}\widetilde{\Gamma}_i^*(H_i)} = H$ and $S_{\Psi\Lambda}$ is continuous, this is equivalent to $S_{\Psi\Lambda} = I$. Therefore (2) is equivalent to (3). □

In the following lemma we prove that if Λ is a g-Bessel sequence, then there exists a basic relation between its synthesis operator and $span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)$, see [7, Lemma 3.6].

Lemma 2.8. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-Bessel sequence with synthesis operator T_Λ and $\Phi^\Lambda = \{\Phi_i^\Lambda \in L(H, H_i) : i \in \mathcal{I}\}$ be the g-R-dual sequence of type IV of Λ with respect to g-Riesz bases $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$, $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$. Let $(h_i)_{i\in\mathcal{I}} \in (\sum_{i\in\mathcal{I}} \oplus H_i)_{\ell^2}$ and $h \in H$. Then*

(i) $h \in ker(T_{\Phi^\Lambda}^*) = span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp$ if and only if $(\Upsilon_i h)_{i\in\mathcal{I}} \in kerT_\Lambda$ (equivalently $S_{\Lambda\Upsilon}h = 0$).

(ii) $(h_i)_{i\in\mathcal{I}} \in kerT_\Lambda$ if and only if $\sum_{i\in\mathcal{I}} \widetilde{\Upsilon}_i^* h_i \in ker(T_{\Phi^\Lambda}^*) = span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp$.

(iii) Φ^Λ is g-complete if and only if T_Λ is injective.

Proof. (1) $h \in ker(T_{\Phi^\Lambda}^*) = span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp$ if and only if for every $j \in \mathcal{I}$, $g_j \in H_j$, $\langle h, \Phi_j^{\Lambda^*} g_j\rangle = 0$. For every $j \in \mathcal{I}$ we have

$$\langle h, \Phi_j^{\Lambda^*} g_j\rangle = \langle h, S_{\Upsilon\Lambda}\Gamma_j^* g_j\rangle = \langle S_{\Lambda\Upsilon}h, \Gamma_j^* g_j\rangle = \langle \sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i h, \Gamma_j^* g_j\rangle.$$

Since $\overline{span\Gamma_j^*(H_j)_{j\in\mathcal{I}}} = H$, then for every $j \in \mathcal{I}$, $\langle \sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i h, \Gamma_j^* g_j\rangle = 0$ if and only if $\sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i h = 0$. Therefore, $h \in span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp$ if and only if $(\Upsilon_i h)_{i\in\mathcal{I}} \in kerT_\Lambda$.
(2) Let $h = \sum_{i\in\mathcal{I}} \widetilde{\Upsilon}_i^* h_i$. Since $(\Upsilon_i)_{i\in\mathcal{I}}$ is a g-Riesz basis, then $(h_i)_{i\in\mathcal{I}} = (\Upsilon_i h)_{i\in\mathcal{I}}$. Thus $(h_i)_{i\in\mathcal{I}} \in kerT_\Lambda$ if and only if $\sum_{i\in\mathcal{I}} \Lambda_i{}^*\Upsilon_i h = 0$, now by using (1) $(h_i)_{i\in\mathcal{I}} \in kerT_\Lambda$ if and only if $h = \sum_{i\in\mathcal{I}} \widetilde{\Upsilon}_i^* h_i \in span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp = ker(T_{\Phi^\Lambda}^*)$.
(3) By (2), T_Λ is injective if and only if $T_{\Phi^\Lambda}^*$ is injective and we know that Φ^Λ is g-complete if and only if $T_{\Phi^\Lambda}^*$ is injective. Therefore, Φ^Λ is g-complete if and only if T_Λ is injective. □

In the following theorem, we give another characterization of dual g-frames.

Theorem 2.9. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-frame with g-frame operator S_Λ and $\Phi^\Lambda = \{\Phi_i^\Lambda \in L(H, H_i) : i \in \mathcal{I}\}$ be the g-R-dual sequence of type I of Λ with respect to g-orthonormal bases $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$, $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$. Then the following statements are equivalent:*

(i) Θ is a dual g-frame of Λ.

(ii) There exists a g-Bessel sequence $\{M_j^* \in L(\{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp, H_j) : j \in \mathcal{I}\}$ such that for every $g_j \in H_j, j \in \mathcal{I}$

$$\Phi_j^{\Theta^*} g_j - \Phi_j^{\Lambda S_\Lambda^{-1^*}} g_j = M_j g_j.$$

Proof. Let $\Theta = (\Theta_i)_{i\in\mathcal{I}}$ be a dual g-frame of $\Lambda = (\Lambda_i)_{i\in\mathcal{I}}$. Then for every $g_j \in H_j, j \in \mathcal{I}$, we have

$$\begin{aligned}\Gamma_j^* g_j &= S_{\Lambda\Theta}(\Gamma_j^* g_j) = \sum_{i\in\mathcal{I}} \Lambda_i^*\Theta_i\Gamma_j^* g_j = \sum_{i\in\mathcal{I}} \Lambda_i^*(\Theta_i - \Lambda_i S_\Lambda^{-1} + \Lambda_i S_\Lambda^{-1})\Gamma_j^* g_j \\ &= \sum_{i\in\mathcal{I}} \Lambda_i^*(\Theta_i - \Lambda_i S_\Lambda^{-1})\Gamma_j^* g_j + \sum_{i\in\mathcal{I}} \Lambda_i^*\Lambda_i S_\Lambda^{-1}\Gamma_j^* g_j \\ &= \sum_{i\in\mathcal{I}} \Lambda_i^*(\Theta_i - \Lambda_i S_\Lambda^{-1})\Gamma_j^* g_j + S_{\Lambda\Lambda S_\Lambda^{-1}}\Gamma_j^* g_j.\end{aligned}$$

Since $S_{\Lambda\Lambda S_\Lambda^{-1}} = I$, then $\sum_{i\in\mathcal{I}} \Lambda_i^*(\Theta_i - \Lambda_i S_\Lambda^{-1})\Gamma_j^* g_j = 0$ and by Lemma 2.8, we have

$$\sum_{i\in\mathcal{I}} \Upsilon_i^*(\Theta_i - \Lambda_i S_\Lambda^{-1})\Gamma_j^* g_j \in \{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp.$$

Now, define $M_j : H_j \to \{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp \subseteq H$ by

$$M_j g_j = \sum_{i\in\mathcal{I}} \Upsilon_i^*\Theta_i\Gamma_j^* g_j - \sum_{i\in\mathcal{I}} \Upsilon_i^*\Lambda_i S_\Lambda^{-1}\Gamma_j^* g_j \quad (g_j \in H_j, j \in \mathcal{I}).$$

Then $M_j g_j = \Phi_j^{\Theta^*} g_j - \Phi_j^{\Lambda S_\Lambda^{-1}{}^*} g_j \quad (g_j \in H_j, j \in \mathcal{I})$. So $M_j^* : \{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp \mapsto H_j$ and $(M_j^*)_{j\in\mathcal{I}}$ is a g-Bessel sequence for $\{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp$ with respect to $\{H_i; i \in \mathcal{I}\}$. Because, let A' be an upper g-frame bound for Θ. Then for every $f \in \{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp$, we have

$$\begin{aligned}\sum_{j\in\mathcal{I}} \|M_j^* f\|^2 &= \sum_{j\in\mathcal{I}} \|\Phi_j^\Theta f - \Phi_j^{\Lambda S^{-1}} f\|^2 = \sum_{j\in\mathcal{I}} \|\Gamma_j S_{\Theta\Upsilon} f - \Gamma_j S_{\Lambda S_\Lambda^{-1}\Upsilon} f\|^2 \\ &= \sum_{j\in\mathcal{I}} \|\Gamma_j S_{\Theta\Upsilon} f - \Gamma_j S_\Lambda^{-1} \sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i f\|^2,\end{aligned}$$

since $f \in \{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp$ by Lemma 2.8, $\sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i f = 0$. Therefore

$$\sum_{j\in\mathcal{I}} \|M_j^* f\|^2 = \sum_{j\in\mathcal{I}} \|\Gamma_j S_{\Theta\Upsilon} f\|^2 = \|S_{\Theta\Upsilon} f\|^2 \le A'\|f\|^2.$$

Conversely, suppose that (2) holds. Since for every $g \in H, j \in \mathcal{I}$, $\Gamma_j g \in H_j$, then we have

$$M_j\Gamma_j g = \Phi_j^{\Theta^*}\Gamma_j g - \Phi_j^{\Lambda S_\Lambda^{-1}{}^*}\Gamma_j g$$

Therefore by [7, Lemma 3.3], for every $i \in \mathcal{I}$

$$(\Theta_i - \Lambda_i S_\Lambda^{-1})g = \sum_{j\in\mathcal{I}} \Upsilon_i M_j \Gamma_j g.$$

So for every $g_l \in H_l, l \in \mathcal{I}$ we have

$$\begin{aligned}\sum_{i\in\mathcal{I}} \Lambda_i^*\Theta_i\Gamma_l^* g_l &= \sum_{i\in\mathcal{I}} \Lambda_i^*(\Lambda_i S_\Lambda^{-1} + \Theta_i - \Lambda_i S_\Lambda^{-1})\Gamma_l^* g_l \\ &= \Gamma_l^* g_l + \sum_{i\in\mathcal{I}} \Lambda_i^*(\Theta_i - \Lambda_i S_\Lambda^{-1})\Gamma_l^* g_l \\ &= \Gamma_l^* g_l + \sum_{i\in\mathcal{I}} \Lambda_i^*(\sum_{j\in\mathcal{I}} \Upsilon_i M_j(\Gamma_j\Gamma_l^* g_l)) \\ &= \Gamma_l^* g_l + \sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i \sum_{j\in\mathcal{I}} M_j(\Gamma_j\Gamma_l^* g_l) \\ &= \Gamma_l^* g_l + \sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i M_l g_l,\end{aligned}$$

since $M_l g_l \in \{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)\}^\perp$, then by Lemma 2.8, $\sum_{i\in\mathcal{I}} \Lambda_i^*\Upsilon_i M_l g_l = 0$. Therefore Θ is a dual g-frame of Λ and this implies (1). □

Now, we present a characterization of the canonical dual g-frames.

Corollary 2.10. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a g-frame with the canonical dual $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Phi^\Lambda = \{\Phi_j^\Lambda \in L(H, H_j) : j \in \mathcal{I}\}$ be the g-R-dual sequence of type I of Λ with respect to g-orthonormal bases $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathcal{I}\}$. Let $\Theta = \{\Theta_i \in L(H, H_i) : i \in \mathcal{I}\}$ be a dual g-frame of Λ. Then for every $g_j \in H_j, j \in \mathcal{I}$*

$$\|\Phi_j^{\Theta^*} g_j\| \ge \|\Phi_j^{\widetilde{\Lambda}^*} g_j\|,$$

with equality if and only if $\Theta = \widetilde{\Lambda}$.

Proof. Let T_Λ and $T_{\widetilde{\Lambda}}$ be the synthesis operators of Λ and $\widetilde{\Lambda}$, respectively. Easily we can see that $kerT_\Lambda = kerT_{\widetilde{\Lambda}}$, then by Lemma 2.8, $\overline{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)} = \overline{span_{j\in\mathcal{I}}\Phi_j^{\widetilde{\Lambda}^*}(H_j)}$, so $Ran\Phi_j^{\widetilde{\Lambda}^*} \subseteq \overline{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)}$. On the other hand by the above theorem, for every $g_j \in H_j, j \in \mathcal{I}$ we have $\Phi_j^{\Theta^*} g_j = \Phi_j^{\widetilde{\Lambda}^*} g_j + M_j g_j$, where $RanM_j \subseteq span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp$. But $\overline{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)}^\perp = span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)^\perp$, so $RanM_j \subseteq \overline{span_{j\in\mathcal{I}}\Phi_j^{\Lambda^*}(H_j)}^\perp$. Then for every $g_j \in H_j, j \in \mathcal{I}$ we have

$$\|\Phi_j^{\Theta^*} g_j\|^2 = \|\Phi_j^{\widetilde{\Lambda}^*} g_j\|^2 + \|M_j g_j\|^2 \geq \|\Phi_j^{\widetilde{\Lambda}^*} g_j\|^2.$$

By the above theorem, the equality holds if and only if $\Theta = \widetilde{\Lambda}$. □

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