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Energy of a rotating Bose-Einstein condensate in a harmonic trap

Ayman Kachmar

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Abstract The state of a rotating Bose-Einstein condensate in a harmonic trap is modeled by a wave function that minimizes the Gross-Pitaevskii functional. The resulting minimization problem has two new features compared to other similar functionals arising in condensed matter physics, such as the Ginzburg-Landau functional. Namely, the wave function is defined in all the plane and is normalized relative to the L^2-norm. This paper deals with the situation when the coupling constant tends to 0 (Thomas-Fermi regime) and the rotation speed is large compared with the first critical speed. It is given the leading order estimate of the ground state energy together with the location of the vortices of the minimizing wave function in the bulk of the condensate. When the rotation speed is inversely proportional to the coupling constant, the condensate is confined in an elliptical region whose conjugate diameter shrinks and whose transverse diameter expands as the rotation speed increases.

1 Introduction

The analysis of energy functionals modeling rotating Bose-Einstein condensation is currently an important field of mathematical physics. A lot of mathematical papers addressed several questions related to this physical phenomenon. In [14, 7], it is proved that the Gross-Pitaevskii frame work is a valid approximation of the N-body model of rotating Bose-Einstein condensation. The monograph [1] contains original results as well as many open questions regarding various models in the subject (see also the papers [2, 3, 4] and the references therein). A series of important contributions ([10, 16] and references therein) contain a deep analysis that describes the various critical speeds of rotating Bose-Einstein condensates in *anharmonic* traps.

When the atoms of the condensate are confined in a *harmonic* trap, the Gross-Pitaevskii functional to study is:

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left(|(\nabla - i\Omega \mathbf{A}_0)u|^2 + \frac{1}{2\varepsilon^2}\Big([a(x)-|u|^2]^2 - [a_-(x)]^2\Big) - \frac{\Omega^2}{4}|x|^2|u|^2 \right) dx\,. \tag{1.1}$$

The functional in (1.1) is defined for functions satisfying the *mass constraint*,

$$\int_{\mathbb{R}^2} |u|^2\,dx = 1\,. \tag{1.2}$$

The parameter $\varepsilon > 0$ is the coupling constant; ε is the ratio of two characteristic lengths. The parameter Ω measures the rotational speed, $\mathbf{A}_0(x) = x^\perp/2 = (-x_2/2, x_1/2)$, $a(x) = a_0 - |x|^2_\Lambda$, $a_0 = \sqrt{2\Lambda/\pi}$, $|x|_\Lambda = \sqrt{x_1^2 + \Lambda^2 x_2^2}$.

The parameter $\Lambda \in (0,1]$ is fixed as well as the term a_0 in the function a. The choice of the term a_0 forces the function a to satisfy the normalization condition $\displaystyle\int_{\mathbb{R}^2} (a(x))_-\,dx = 1$.

The form of the functional given in (1.1) is adequate to apply the techniques developed for the Ginzburg-Landau functional. In non-dimensional units, the functional that appears in the physical literature is actually the sum of three terms: the kinetic energy, the potential energy and the non-linear interaction term (see e.g. [15]),

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}\Big([a(x) - |u|^2]^2 - [a_-(x)]^2\Big) - \Omega\, x^\perp \cdot (iu, \nabla u) \right) dx\,. \tag{1.3}$$

In the regime $\varepsilon \ll 1$ and $\varepsilon\Omega \ll 1$, the condensate is confined in the region

$$\mathcal{D} = \{x \in \mathbb{R}^2 \ : \ a(x) > 0\}\,. \tag{1.4}$$

The ground state energy is:

$$\mathrm{E}_{\mathrm{gs}}(\varepsilon, \Omega) = \inf\,\{\, F_\varepsilon(u) \ : \ u \in H^1(\mathbb{R}^2)\,,\ |x|^2 u \in L^2(\mathbb{R}^2) \ \& \int_{\mathbb{R}^2} |u|^2\,dx = 1\,\}\,. \tag{1.5}$$

The minimization problem in (1.5) is studied in [11] when $\varepsilon \to 0_+$ and $\Omega \approx |\ln\varepsilon|$. Among other things, it is found a critical speed $\Omega_c = \omega_c|\ln\varepsilon|$ such that minimizers start to have zeros when $\Omega > \Omega_c$. In this paper, the focus will be on the regime when $\varepsilon \to 0_+$ and $\Omega \gg \Omega_c$. Part of the results of this paper are qualitatively very similar to those of [10, 9, 8] where *flat* and *anharmonic* traps are treated. However, a regime in the *harmonic* trap discussed in this paper seems to display a new behavior of the concentration of the condensate's wave function. This is explicitly discussed in Remark 1.3 below.

It is established in [11, Prop. 3.1] that there is a minimizer of the problem (1.5) when $\Omega < 2\Lambda/\varepsilon$. The functional in (1.5) is *not* bounded from below when $\Omega > 2\Lambda/\varepsilon$.

Setting $\Omega = 0$ into the *magnetic term* in F_ε, it is obtained the reduced functional:

$$E_{\varepsilon,\Omega}(u) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}\Big([a(x) - |u|^2]^2 - [a_-(x)]^2\Big) - \Omega^2 \frac{|x|^2}{4}|u|^2 \right) dx\,. \tag{1.6}$$

The ground state energy of this functional is:

$$e_{\varepsilon,\Omega} = \inf\,\{E_\varepsilon(u) \ : \ u \in H^1(\mathbb{R}^2)\,,\ |x|^2 u \in L^2(\mathbb{R}^2) \ \& \int_{\mathbb{R}^2} |u|^2\,dx = 1\,\}\,. \tag{1.7}$$

The reduced functional in (1.6) is studied in [11, Thm. 2.2] when $\Omega = 0$, where it is established that (1.7) has a positive minimizer $\widetilde{\eta}_\varepsilon$. In Section 2, it will be constructed a positive minimizer $\widetilde{\eta}_{\varepsilon,\Omega}$ of the functional in (1.6). Following an idea of [13] and writing $u = \widetilde{\eta}_{\varepsilon,\Omega} v$, there holds the following decomposition:

$$F_\varepsilon(u) = E_\varepsilon(\widetilde{\eta}_\varepsilon) + \mathcal{G}_\varepsilon(v)\,, \tag{1.8}$$

with

$$\mathcal{G}_\varepsilon(v) = \int_{\mathbb{R}^2} \left(\widetilde{\eta}_{\varepsilon,\Omega}^2 |(\nabla - i\Omega\mathbf{A}_0)v|^2 + \frac{\widetilde{\eta}_{\varepsilon,\Omega}^4}{2\varepsilon^2}(1 - |v|^2)^2 \right) dx\,. \tag{1.9}$$

Also, if u is selected as a minimizer of (1.5), then v will be a minimizer of $\mathcal{G}_\varepsilon$ under the *weighted* mass constraint,

$$\int_{\mathbb{R}^2} \widetilde{\eta}_{\varepsilon,\Omega}^2 |v|^2\,dx = 1\,. \tag{1.10}$$

More precisely, the minimization problem (1.5) is equivalent to

$$C_0(\varepsilon, \Omega) = \inf\,\{\mathcal{G}_\varepsilon(v) \ : \ v \in H^1(\mathbb{R}^2)\,,\ \widetilde{\eta}_{\varepsilon,\Omega}|x|v \in L^2(\mathbb{R}^2) \ \& \int_{\mathbb{R}^2} \widetilde{\eta}_{\varepsilon,\Omega}^2 |v|^2\,dx = 1\,\}\,. \tag{1.11}$$

The main theorem of this paper is:

Theorem 1.1. Let $M \in (0, 2\Lambda)$ and $b : (0,1) \to (0,\infty)$ satisfies $\lim\limits_{\varepsilon\to 0_+} b(\varepsilon) = \infty$. Suppose that the rotational speed satisfies:

$$b(\varepsilon)|\ln\varepsilon| \le \Omega \le \frac{M}{\varepsilon}\,, \quad \big(\varepsilon \in (0,1)\big)\,.$$

There exist a constant $\varepsilon_0 > 0$ and a function $\mathrm{err} : (0, \varepsilon_0] \to \mathbb{R}$ such that,

$$\lim_{\varepsilon\to 0_+} \mathrm{err}(\varepsilon) = 0\,,$$

and

$$\mathrm{E_{gs}} = e_{\varepsilon,\Omega} + \Omega\left[\ln \frac{1}{\varepsilon\sqrt{\Omega}}\right]\Big(1+\mathrm{err}(\varepsilon)\Big)\,, \quad (\varepsilon\in(0,\varepsilon_0))\,. \tag{1.12}$$

Here $\mathrm{E_{gs}}$ is introduced in (1.5) and $e_{\varepsilon,\Omega}$ in (1.7).

Remark 1.2. In light of the decomposition in (1.8), the proof of Theorem 1.1 is done by establishing that:

$$C_0(\varepsilon,\Omega) = \Omega\left[\ln\frac{1}{\varepsilon\sqrt{\Omega}}\right]\Big(1+\mathrm{err}(\varepsilon)\Big)\,.$$

Remark 1.3. (**Bulk of the condensate**)
In Section 2, it will be shown that the function $\widetilde{\eta}_{\varepsilon,\Omega}$ is concentrated in the region

$$\mathcal{D}_{\varepsilon\Omega} = \{x\in\mathbb{R}^2\ :\ \alpha_{\varepsilon\Omega} - |x|^2_{\widetilde{\Lambda}_{\varepsilon\Omega}} > 0\}\,,$$

where

$$\alpha_{\varepsilon\Omega} = a_0\left(\frac{1-\frac{\varepsilon^2\Omega^2}{4\Lambda^2}}{1-\frac{\varepsilon^2\Omega^2}{4}}\right)^{1/4} \quad\text{and}\quad \widetilde{\Lambda}_{\varepsilon\Omega} = \Lambda\left(\frac{1-\frac{\varepsilon^2\Omega^2}{4\Lambda^2}}{1-\frac{\varepsilon^2\Omega^2}{4}}\right)^{1/2}\,.$$

It is worthy to discuss the form of the region $\mathcal{D}_{\varepsilon\Omega}$ in the various existing regimes. In the isotropic case $\Lambda = 1$, the region $\mathcal{D}_{\varepsilon\Omega}$ is independent of $\varepsilon\Omega$,

$$\mathcal{D}_{\varepsilon\Omega} = \mathcal{D} = \{x\in\mathbb{R}^2\ :\ a(x) > 0\}\,.$$

In the non-isotropic case, $0<\Lambda<1$, one observes an interesting behavior. If $\varepsilon\Omega \ll 1$, then the region $\mathcal{D}_{\varepsilon\Omega}$ occupies $\mathcal{D}$.

This region shrinks along the x_1-axis and expands along the x_2-axis as $\varepsilon\Omega$ increases. If $\Omega = M/\varepsilon$ and $M\in(0,2\Lambda)$, then as $M\to 2\Lambda$, the region $\mathcal{D}_{\varepsilon\Omega}$ approaches the following region

$$\mathcal{D}_{2\Lambda} = \{0\}\times\mathbb{R}\,.$$

It seems that this kind of bahavior of the 'bulk' of the condensate is new comapred to the existing behavior for anharmonic and flat traps.

Remark 1.4. (**Concentration of the condensate's wave function**)
Let $\delta > 0$ and $\mathcal{N}_\delta = \{x\in\mathcal{D}_{\varepsilon\Omega}\ :\ \alpha_{\varepsilon\Omega} - |x|^2_{\widetilde{\Lambda}_{\varepsilon\Omega}} > \delta\}$. A simple consequence of the energy asymptotics in Remark 1.2 and the discussion in Remark 1.3 is that any minimizer $u = \widetilde{\eta}_{\varepsilon,\Omega}\,v$ of the functional in (1.1) satisfies,

$$|v| = \left|\frac{u}{\widetilde{\eta}_{\varepsilon,\Omega}}\right| \to 1 \quad \text{in } L^2(\mathcal{N}_\delta)\,.$$

Since the functions u and $\widetilde{\eta}_{\varepsilon,\Omega}$ are normalized in L^2, then the function u satisfies

$$\int_{\mathcal{N}_\delta} |u|^2\,dx = 1 + \mathcal{O}(\delta) \quad\text{and}\quad \int_{\mathbb{R}^2\setminus\mathcal{N}_\delta} |u|^2\,dx = \mathcal{O}(\delta)\,,$$

for sufficiently small values of δ. Note that the behavior of $\widetilde{\eta}_{\varepsilon\Omega}$ described in Theorem 2.2 is used.

Remark 1.5. Along the proof of Theorem 1.1, one gets information about the qualitative behavior of the minimizers. More precisely, it is possible to get information about the arrangement of vortices. This is discussed in Section 6.

Remark 1.6. The letter C denotes a positive constant independent of ε and Ω, and whose value is not the same when seen in different formulas. The quantity $\mathcal{O}(B)$ is any expression that remains in the interval $(-C|B|, C|B|)$. Writing $A \ll B$ means that $A = \delta B$ and $\delta\to 0$. The meaning of $A\approx B$ is that A is bounded between $c_1 B$ and $c_2 B$ with c_1 and c_2 being positive constants.

2 Preliminaries

Some basic properties of the positive minimizer $\widetilde{\eta}_{\varepsilon,\Omega}$ of (1.7) as well as of minimizers of the modified problem (1.11) will be used along the proof of Theorem 1.1. These properties are recalled here.

2.1 The unconstrained problem

The first step is to study the minimization of (1.6) without the mass constraint. The results here are given in [11] but for a slightly more particular case on the potential $\widetilde{a}(x)$ defined below. The proofs here are identically the same as in [11] and are not repeated.

Consider the potential

$$\widetilde{a}(x) = \widetilde{a}_0 - |x|^2_{\widetilde{\Lambda}} = \widetilde{a}_0 - x_1^2 - \widetilde{\Lambda}^2 x_2^2\,, \qquad \left(x = (x_1, x_2) \in \mathbb{R}^2\right),$$

where $\widetilde{a}_0$ and $\widetilde{\Lambda}$ are positive parameters. The parameters $\widetilde{a}_0$ and $\widetilde{\Lambda}$ may depend on ε and Ω but they should remain bounded between two positive constants c_1 and c_2 that are independent of ε and Ω. The results in this section are valid under this last assumption.

Consider the functional

$$\widetilde{E}_\varepsilon(u) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left([\widetilde{a}(x) - |u|^2]^2 - [\widetilde{a}_-(x)]^2 \right) \right) dx\,. \tag{2.1}$$

The functional in (2.1) will be minimized over configurations in the space

$$\mathcal{H} = \{u \in H^1(\mathbb{R}^2) \ : \ |x|^2 u \in L^2(\mathbb{R}^2)\}\,.$$

The proof of Theorem 2.1 below is given in [11, Proposition 2.1].

Theorem 2.1. There exist two positive constants $\varepsilon_0 > 0$ and $C > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then there is a real-valued minimizer $\eta_\varepsilon = \eta_{\varepsilon,\widetilde{a}} \in \mathcal{H}$ of (2.1) satisfying:

(i) $E_\varepsilon(\eta_\varepsilon) \le C|\ln\varepsilon|$ and $\eta_\varepsilon > 0$ in $\mathbb{R}^2$;

(ii) η_ε is the unique solution of

$$-\Delta\eta_\varepsilon = \frac{1}{\varepsilon^2}(\widetilde{a} - \eta_\varepsilon^2)\eta_\varepsilon \quad \text{and} \quad \eta_\varepsilon > 0 \quad \text{in } \mathbb{R}^2\,.$$

(iii) $\eta_\varepsilon(x) \le C\varepsilon^{1/3} \exp\left(\widetilde{a}(x)/(4\varepsilon^{2/3})\right)$ if $|x|_{\widetilde{\Lambda}} \ge \sqrt{\widetilde{a}_0}$;

(iv) $(1 - C\varepsilon^{1/3})\sqrt{\widetilde{a}(x)} \le \eta_\varepsilon(x) \le \sqrt{\widetilde{a}(x)}$ if $|x|_{\widetilde{\Lambda}} \le \sqrt{\widetilde{a}_0} - \varepsilon^{1/3}$.

(v) $\eta_\varepsilon(x) \le C\varepsilon^{1/3}$ if $\sqrt{\widetilde{a}_0} - \varepsilon^{1/3} \le |x|_{\widetilde{\Lambda}} \le \sqrt{\widetilde{a}_0}$.

2.2 The constrained problem

This section is devoted to the construction of a positive minimizer of the constrained problem in (1.7).

A standard compactness argument shows the existence of a minimizer $u_{\varepsilon,\Omega}$ of (1.7). The details are given in [11]. Since $\big|\nabla |u_{\varepsilon,\Omega}|\big| \le |\nabla u_{\varepsilon,\Omega}|$, then $|u_{\varepsilon,\Omega}|$ is a minimizer of (1.7) too. This discussion leads to the existence of a positive minimizer $\widetilde{\eta}_{\varepsilon,\Omega} = |u_{\varepsilon,\Omega}|$ of (1.7). The Euler-Lagrange equation satisfied by $\widetilde{\eta}_{\varepsilon,\Omega}$ is,

$$-\Delta\widetilde{\eta}_{\varepsilon,\Omega} = \frac{1}{\varepsilon^2}\left(k_\varepsilon\varepsilon^2 + V_{\varepsilon\Omega} - \widetilde{\eta}^2_{\varepsilon,\Omega}\right)\widetilde{\eta}_{\varepsilon,\Omega}\,,$$

where $k_\varepsilon \in \mathbb{R}$ is the Lagrange multiplier and $V_{\varepsilon\Omega}(x) = a_0 - |x|^2_\Lambda + \frac{\varepsilon^2\Omega^2}{4}|x|^2$.

Multiplying both sides of the Euler-Lagrange equation by $\widetilde{\eta}_{\varepsilon,\Omega}$, integrating by parts and using $\int_{\mathbb{R}^2}\widetilde{\eta}_{\varepsilon,\Omega}^2\,dx = 1$ yield that $a_0+k_\varepsilon\varepsilon^2 > \mu_\varepsilon > 0$, where μ_ε is the first eigenvalue of the Schrödinger operator

$$-\Delta+\frac{1}{\varepsilon^2}\left(|x|^2_\Lambda-\frac{\varepsilon^2\Omega^2}{4}|x|^2\right)\quad\text{in } L^2(\mathbb{R}^2)\,.$$

Note that, by the assumption on Ω and Λ, the potential of the operator is positive and goes to ∞ when $|x|\to\infty$.

Define

$$\widetilde{\varepsilon}=\left(1-\frac{\varepsilon^2\Omega^2}{4}\right)^{-1/2}\frac{a_0}{a_0+k_\varepsilon\varepsilon^2}\,\varepsilon\,,\quad \nu_\varepsilon(x)=\sqrt{\frac{a_0}{a_0+k_\varepsilon\varepsilon^2}}\,\widetilde{\eta}_{\varepsilon,\Omega}\left(\sqrt{\frac{a_0+k_\varepsilon\varepsilon^2}{a_0}}\,x\right).$$

The function ν_ε satisfies,

$$-\Delta\nu_\varepsilon=\frac{1}{\widetilde{\varepsilon}^2}(\widetilde{a}-\nu_\varepsilon^2)\nu_\varepsilon\,,\quad \nu_\varepsilon>0\quad\text{in }\mathbb{R}^2\,,$$

where

$$\widetilde{a}(x)=\widetilde{a}_{\varepsilon\Omega}=\widetilde{a}_0-|x|^2_{\widetilde{\Lambda}}\,,\quad \widetilde{a}_0=\frac{a_0}{1-\frac{\varepsilon^2\Omega^2}{4}}\,,\quad \widetilde{\Lambda}^2=\frac{\Lambda^2-\frac{\varepsilon^2\Omega^2}{4}}{1-\frac{\varepsilon^2\Omega^2}{4}}\,.$$

The conclusion (2) in Theorem 2.1 asserts that,

$$\nu_\varepsilon(x)=\eta_{\widetilde{\varepsilon},\widetilde{a}}(x)\quad(x\in\mathbb{R}^2)\,,$$

where $\eta_{\widetilde{\varepsilon},\widetilde{a}}$ is the solution of the unconstrained problem. As a consequence, there holds,

$$\widetilde{\eta}_{\varepsilon,\Omega}(x)=\sqrt{\frac{a_0+k_\varepsilon\varepsilon^2}{a_0}}\,\eta_{\widetilde{\varepsilon},\widetilde{a}}\left(\sqrt{\frac{a_0}{a_0+k_\varepsilon\varepsilon^2}}\,x\right).$$

Thanks to the conclusions (3)-(5) in Theorem 2.1 and the mass constraint $\int_{\mathbb{R}^2}\widetilde{\eta}_{\varepsilon,\Omega}^2\,dx=1$, there holds,

$$\begin{aligned}\left(\frac{a_0}{a_0+k_\varepsilon\varepsilon^2}\right)^2&=\left(\int_{\widetilde{a}(x)>0}\widetilde{a}(x)\,dx\right)(1+\mathcal{O}(\varepsilon^{1/3}))\\&=\Lambda\left(\Lambda^2-\frac{\varepsilon^2\Omega^2}{4}\right)^{-1/2}\left(1-\frac{\varepsilon^2\Omega^2}{4}\right)^{-3/2}(1+\mathcal{O}(\varepsilon^{1/3}))\,.\end{aligned}$$

In the sequel, let,

$$\alpha_{\varepsilon\Omega}=a_0\left(\frac{1-\frac{\varepsilon^2\Omega^2}{4\Lambda^2}}{1-\frac{\varepsilon^2\Omega^2}{4}}\right)^{1/4},\quad \widetilde{\Lambda}_{\varepsilon\Omega}=\Lambda\left(\frac{1-\frac{\varepsilon^2\Omega^2}{4\Lambda^2}}{1-\frac{\varepsilon^2\Omega^2}{4}}\right)^{1/2}$$

$$p_{\varepsilon\Omega}(x)=\left(\alpha_{\varepsilon\Omega}-|x|^2_{\widetilde{\Lambda}_{\varepsilon\Omega}}\right)=\sqrt{\frac{a_0+k_\varepsilon\varepsilon^2}{a_0}}\,\widetilde{a}\left(\sqrt{\frac{a_0}{a_0+k_\varepsilon\varepsilon^2}}\,x\right)(1+\mathcal{O}(\varepsilon^{1/3}))\,.\tag{2.2}$$

Now, an immediate application of Theorem 2.1 leads to:

Theorem 2.2. Let $M\in(0,2\Lambda)$. There exist positive constants ε_0, C and δ_0 such that, if $\varepsilon\in(0,\varepsilon_0)$ and $\Omega\in[0,M)$, then there is a real-valued minimizer $\widetilde{\eta}_{\varepsilon,\Omega}$ of the constrained problem (2.1) satisfying:

- (i) $E_\varepsilon(\widetilde{\eta}_{\varepsilon,\Omega})\le C\Omega^2$ and $\widetilde{\eta}_{\varepsilon,\Omega}>0$ in $\mathbb{R}^2$;
- (ii) $\widetilde{\eta}_{\varepsilon,\Omega}(x)\le C\varepsilon^{1/3}\exp\left(\delta_0 p_{\varepsilon\Omega}(x)/(\varepsilon^{2/3})\right)$ if $p_{\varepsilon\Omega}(x)\le-\delta_0\varepsilon^{1/3}$;
- (iii) $(1-C\varepsilon^{1/3})\sqrt{p_{\varepsilon\Omega}(x)}\le\widetilde{\eta}_{\varepsilon,\Omega}(x)\le\sqrt{p_{\varepsilon\Omega}(x)}$ if $p_{\varepsilon\Omega}(x)\ge\delta_0\varepsilon^{1/3}$;
- (iv) $\eta_\varepsilon(x)\le C\varepsilon^{1/3}$ if $-\delta_0\varepsilon^{1/3}\le p_{\varepsilon\Omega}(x)\le\delta_0\varepsilon^{1/3}$.

2.3 A uniform bound of the ground states

Theorem 2.3. Let $M \in (0, 2\Lambda)$. There exist positive constants C, δ, λ and ε_0 such that, if $\varepsilon \in (0, \varepsilon_0)$ and $0 < \Omega \leq M/\varepsilon$, then every minimizer v_ε of (1.11) satisfies:

$$|\widetilde{\eta}_\varepsilon v_\varepsilon(x)| \leq C\left(\sqrt{\frac{1}{2\Lambda - M}} + 1\right) \quad \text{in } \mathbb{R}^2 .$$

Proof. Under the assumption on the rotational speed, Proposition 3.2 in [11] implies that the problem (1.5) has a minimizer u_ε. In light of the decomposition in (1.8), it follows that $v_\varepsilon = u_\varepsilon/\widetilde{\eta}_\varepsilon$ is a minimizer of the problem (1.11). Theorem 2.3 will be proved by establishing properties of u_ε. The function u_ε satisfies

$$-(\nabla - i\Omega\mathbf{A}_0)u_\varepsilon = \frac{1}{\varepsilon^2}\big(a(x) + \frac{1}{4}\varepsilon^2\Omega^2|x|^2 + \varepsilon^2\ell_\varepsilon - |u_\varepsilon|^2\big)u_\varepsilon \quad \text{in } \mathbb{R}^2 , \tag{2.3}$$

where $\ell_\varepsilon \in \mathbb{R}$ is the lagrange multiplier. Furthermore, it holds (see the derivation of [11, (3.7)&(3.11)]):

$$F_\varepsilon(u_\varepsilon) \leq C\Omega^2 , \quad |\ell_\varepsilon| \leq C\varepsilon^{-1}\Omega , \quad \int_{\mathbb{R}^2\setminus\mathcal{D}} |u_\varepsilon|^4\, dx \leq C\varepsilon^2\Omega^2 . \tag{2.4}$$

Let $U_\varepsilon = |u_\varepsilon|^2$ and $b(x) = a(x) + \frac{1}{4}\varepsilon^2\Omega^2|x|^2 + \varepsilon^2\ell_\varepsilon$. In light of the identity,

$$\mathrm{Re}\Big[\overline{u_\varepsilon}\,(\nabla - i\Omega\mathbf{A}_0)^2 u_\varepsilon\Big] = \frac{1}{2}\Delta U_\varepsilon - |(\nabla - i\Omega\mathbf{A}_0)u_\varepsilon|^2 ,$$

the function U_ε satisfies,

$$\frac{1}{2}\Delta U_\varepsilon \geq -\frac{1}{\varepsilon^2}(b(x) - U_\varepsilon)U_\varepsilon \quad \text{in } \mathbb{R}^2 . \tag{2.5}$$

Let $\lambda > \sqrt{a_0}$, $\mathbb{E} = \{x \in \mathbb{R}^2 \,:\, |x| \geq 2\lambda\}$ and $\Theta = \{x \in \mathbb{R}^2 \,:\, |x| > \lambda\}$. The condition on λ ensures that $\Theta \subset \mathbb{R}^2 \setminus \mathcal{D}$. In the set Θ, there holds,

$$b(x) \leq a_0 - \lambda^2(\Lambda^2 - M^2) + \varepsilon^2\ell_\varepsilon \leq -\lambda^2\left(\Lambda^2 - \frac{M^2}{4}\right) + C .$$

As a consequence, it is possible to select the constant $\lambda \geq \sqrt{\frac{2C}{\Lambda^2 - \frac{M^2}{2}}}$ such that the function U_ε is subharmonic in the open set Θ.

Consider an arbitrary point $x_0 \in \mathbb{E}$. The definition of the set Θ yields that $B(x_0, \lambda) \subset \Theta$ and $\Theta \subset \mathbb{R}^2 \setminus \mathcal{D}$. Since the function U_ε is subharmonic and its L^2-norm is estimated in (2.4), then there exists a constant $C_* > 0$ such that,

$$0 \leq U_\varepsilon(x_0) \leq \frac{1}{|B(x_0, \lambda)|}\int_{B(x_0,\lambda)} U_\varepsilon^2(x)\, dx \leq \mathcal{O}\left(\frac{1}{\lambda}\varepsilon\Omega\right) \leq \frac{C_*}{\lambda} .$$

The next step is to prove that U_ε is bounded in the set

$$B_r = \{x \in \mathbb{R}^2 \setminus \mathcal{D} \,:\, |x| \leq r\}$$

where $r = 3\lambda$. Select a positive constant C such that $b(x) \leq C\lambda + \frac{C_*}{\lambda}$ in B_r. Notice that $\partial B_r \subset \mathbb{E}$ and consequently, $U_\varepsilon \leq C_* \leq C\lambda + \frac{C_*}{\lambda}$ in ∂B_r. Thus, if the maximum of U_ε in B_r is greater than $C\lambda + \frac{C_*}{\lambda}$, then the point of maximum is an interior point in B_r. It is impossible that such a point of maximum exists. In fact, if there exists a point of maximum x_0 satisfying $C\lambda + \frac{C_*}{\lambda} - U_\varepsilon(x_0) < 0$, then $\Delta U_\varepsilon(x_0) \leq 0$. This leads to a contradiction in light of the following inequality,

$$\frac{1}{2}\Delta U_\varepsilon + \frac{1}{\varepsilon^2}\left(C\lambda + \frac{C_*}{\lambda} - U_\varepsilon\right)U_\varepsilon \geq 0 ,$$

which results from (2.5) and the choice of the constant C. □

Remark 2.4. There is a simple consequence of Theorem 2.3 and (3) in Theorem 2.2. Let K be a compact set and $\delta > 0$. If $K \subset \{x \in \mathbb{R}^2 \,:\, p_{\varepsilon\Omega}(x) > \delta\}$ for sufficiently small values of ε, then there exist constants $\varepsilon_{K,\delta}$ and $C_{K,\delta}$ such that, for all $\varepsilon \in (0, \varepsilon_{K,\delta})$, $|v_\varepsilon(x)| \leq C_{K,\delta}$ in K.

Here, the function $p_{\varepsilon\Omega}(x)$ is introduced in (2.2).

3 Reduced Ginzburg-Landau energy

Let $K = (-1/2, 1/2) \times (-1/2, 1/2)$ be a square of unit side length, λ, $h_{\rm ex}$ and ε be positive parameters. Consider the functional defined for all $u \in H^1(K;\mathbb{C})$,

$$E_\lambda^{\rm 2D}(u) = \int_K \left(|(\nabla - ih_{\rm ex}\mathbf{A}_0)u|^2 + \frac{\lambda}{2\varepsilon^2}(1-|u|^2)^2 \right) dx\,. \tag{3.1}$$

Here $\mathbf{A}_0$ is the vector potential whose curl is equal to 1,

$$\mathbf{A}_0(x_1,x_2) = \frac{1}{2}(-x_2,x_1)\,,\quad (x_1,x_2)\in\mathbb{R}^2\,. \tag{3.2}$$

Notice that the functional $E_\lambda^{\rm 2D}$ is a simplified version of the full Ginzburg-Landau functional considered in [18], as the magnetic potential in (3.1) is given and *not* an unknown of the problem.

Minimization of the functional $E_\lambda^{\rm 2D}$ arises naturally over 'magnetic periodic' functions. Let us introduce the following space,

$$E_{h_{\rm ex}} = \{u\in H^1_{\rm loc}(\mathbb{R}^2;\mathbb{C})\ :\ u(x_1+1,x_2) = e^{ih_{\rm ex}x_2/2}u(x_1,x_2)\,,\\ u(x_1,x_2+1) = e^{-ih_{\rm ex}x_1/2}u(x_1,x_2)\}\,, \tag{3.3}$$

together with the ground state energy,

$$m_{\rm p}(h_{\rm ex},\varepsilon) = \inf\{E_\lambda^{\rm 2D}(u)\ :\ u\in E_{h_{\rm ex}}\}\,. \tag{3.4}$$

Minimization of $E_\lambda^{\rm 2D}$ over configurations without prescribed boundary conditions will be needed as well. The ground state energy of this problem is,

$$m_0(h_{\rm ex},\varepsilon) = \inf\{E_\lambda^{\rm 2D}(u)\ :\ u\in H^1(K)\}\,. \tag{3.5}$$

The ground state energies $m_0(h_{\rm ex},\varepsilon)$ and $m_{\rm p}(h_{\rm ex},\varepsilon)$ are estimated in [12] by borrowing tools from [17] and [18]. This is recalled in the next theorem.

Theorem 3.1. Assume that $\lambda_2 > \lambda_1 > 0$ are given constants, $\lambda\in(\lambda_1,\lambda_2)$ and $h_{\rm ex}$ is a function of ε such that

$$|\ln\varepsilon| \ll h_{\rm ex} \ll \frac{1}{\varepsilon^2}\,,\quad \text{as } \varepsilon\to 0\,.$$

As $\varepsilon\to 0$, the ground state energies $m_0(h_{\rm ex},\varepsilon)$ and $m_{\rm p}(h_{\rm ex},\varepsilon)$ satisfy,

$$m_0(h_{\rm ex},\varepsilon) = h_{\rm ex}\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\big(1+o(1)\big)\quad\text{and}\quad m_{\rm p}(h_{\rm ex},\varepsilon) = h_{\rm ex}\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\big(1+o(1)\big)\,.$$

Here, the expression $o(1)$ tends to 0 as $\varepsilon\to 0$ uniformly with respect to λ.

In the forthcoming section, it will be needed a trial state satisfying the mass constraint (L^2-norm equal to 1) and having an energy close to $m_{\rm p}(h_{\rm ex},\varepsilon)$. The next Lemma provides one with a useful trial state whose L^2-norm is *close* to 1.

Lemma 3.2. Suppose that $\lambda > 0$, $h_{\rm ex}$ and ε are as in Theorem 3.1. There exists a function f_ε in $H^1(K)$ such that

$$|f_\varepsilon| \le 1\quad \text{in } K\,,$$

$$\{x\in K\ :\ |f_\varepsilon(x)|<1\}\subset\bigcup_{i=1}^{\mathsf{n}} B(a_i,\varepsilon)\quad\text{and}\quad \mathsf{n} = \mathcal{O}(h_{\rm ex})\,,$$

$$1-\mathcal{O}(\varepsilon^2 h_{\rm ex}) \le \int_K |f_\varepsilon(x)|^2\,dx \le 1\,,$$

and

$$E_\lambda^{\rm 2D}(f_\varepsilon) \le h_{\rm ex}\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\big(1+o(1)\big)\,,$$

as $\varepsilon\to 0_+$. Furthermore, f_ε is independent of λ, and $\mathcal{O}$ is uniform with respect to λ.

Proof. For the convenience of the reader, the construction of f_ε is outlined. Details can be found in [6]. Let N be the largest positive integer satisfying $N \leq \sqrt{h_{\rm ex}/2\pi} < N+1$. Divide the square K into N^2 disjoint squares $(K_j)_{0\leq j\leq N^2-1}$ each of side length equal to $1/N$ and center a_j. Let h be the unique solution of the problem,

$$\begin{cases} -\Delta h + h_{\rm ex} = 2\pi\delta_{a_0} & \text{in} \quad K_0 \\ \dfrac{\partial h}{\partial \nu} = 0 & \text{on} \quad \partial K_0 \\ \displaystyle\int_{K_0} h\,dx = 0. \end{cases}$$

Here ν is the unit outward normal vector of K_0. The function h satisfies periodic conditions on the boundary of K_0, and

$$\int_{K_0\setminus B(a_0,\varepsilon)} |\nabla h|^2\,dx \leq 2\pi \ln\frac{1}{\varepsilon N} + \mathcal{O}(1) = 2\pi\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} + \mathcal{O}(1)\,, \quad \text{as } \varepsilon \to 0_+\,.$$

The function h is extended by periodicity in the square K. Let ϕ be a function (defined modulo 2π) satisfying in $K\setminus\{a_j \,:\, 0\leq j\leq N^2-1\}$,

$$\nabla\phi = -\nabla^\perp h + h_{\rm ex}\mathbf{A}_0\,, \quad \big(\nabla^\perp = (-\partial_{x_2},\partial_{x_1})\big).$$

Here $\mathbf{A}_0$ is the magnetic potential in (3.2). If $x\in K_0$, let $\rho(x) = \min(1, |x-a_0|/\varepsilon)$. The function ρ is extended by periodicity in the square K. Put $f_\varepsilon(x) = \rho(x)e^{i\phi(x)}$ for all $x\in K$. The function f_ε can be extended as a function in the space $E_{h_{\rm ex}}$ in (3.3), see [5, Lemma 5.11] for details.

The energy of f_ε is easily computed, since f_ε is 'magnetic periodic' and $N = \sqrt{h_{\rm ex}/2\pi}\big(1+o(1)\big)$. Clearly, in the square K_0, $|f_\varepsilon(x)|<1$ if and only if $|x-a_0|<\varepsilon$. Thus, it is easy to check that f_ε satisfies the requirements in Lemma 3.2. □

4 Upper Bound

4.1 The test configuration

Recall the definition of the ground state energy $C_0(\varepsilon,\Omega)$ in (1.11). The assumption on the rotational speed Ω is $|\ln\varepsilon| \ll \Omega \leq M/\varepsilon$ with $M\in(0,2\Lambda)$. Let

$$L > \sqrt{a_0\left(1-\frac{M^2}{4}\right)^{-1/4}} \quad \text{and} \quad 0<\delta<\min\left(\sqrt{a_0\left(1-\frac{M^2}{4\Lambda^2}\right)},\,\frac{L}{2}\right).$$

Recall the definition of $\alpha_{\varepsilon\Omega}$ in (2.2). The constants δ and L are selected so that

$$\delta < \sqrt{\alpha_{\varepsilon\Omega}} < L \quad \text{and} \quad \sqrt{\alpha_{\varepsilon\Omega}} + \delta < L\,.$$

Define,

$$\mathcal{U}_L = \{x\in\mathcal{D} \,:\, |x|_{\widetilde{\Lambda}_{\varepsilon\Omega}} < L\}\,.$$

Thanks to the assumption on Ω, if ε is sufficiently small, then there holds the inclusion,

$$\mathcal{D}_{\varepsilon\Omega} = \{x\in\mathbb{R}^2 \,:\, p_{\varepsilon\Omega}(x)>0\} \subset \mathcal{U}_L\,,$$

where $\widetilde{\Lambda}_{\varepsilon\Omega}$ and $p_{\varepsilon\Omega}$ are introduced in (2.2) and $\displaystyle\int_{p_{\varepsilon\Omega}(x)>0} p_{\varepsilon\Omega}(x)\,dx = 1$.

Define

$$\ell = \left(\frac{\Omega}{|\ln\varepsilon|}\right)^{1/4}\frac{1}{\sqrt{\Omega}}\,, \quad h_{\rm ex} = \frac{1}{\ell^2}\,. \tag{4.1}$$

Recall the ground state energy $m_{\rm p}(h_{\rm ex},\varepsilon)$ and the space $E_{h_{\rm ex}}$ introduced in (3.4) and (3.3) respectively. Let $f_\varepsilon \in E_{h_{\rm ex}}$ be the test function defined in Lemma 3.2. In particular, f_ε satisfies $E^{\rm 2D}_\lambda(f_\varepsilon) \leq h_{\rm ex}\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\big(1+o(1)\big)$ for any λ varying between two positive constants λ_1 and λ_2.

Define,

$$v(x)=\chi(x)\,f_\varepsilon\Big(\ell\sqrt{\Omega}\,x\Big)\quad(x\in\mathbb{R}^2)\,,$$

where χ is a cut-off function satisfying,

$$0\leq\chi\leq 1\text{ in }\mathbb{R}^2\,,\quad \chi(x)=0\text{ when }|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\geq 2L\,,\quad \chi(x)=1\text{ when }|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq L\,,$$

and

$$|\nabla\chi|\leq\frac{C}{L}\quad\text{in }\mathbb{R}^2\,.$$

Let $(\mathcal{K}_j)$ be the lattice of $\mathbb{R}^2$ generated by the cube,

$$\mathcal{K}=\Big(-\frac{1}{2\ell\sqrt{\Omega}},\frac{1}{2\ell\sqrt{\Omega}}\Big)\times\Big(-\frac{1}{2\ell\sqrt{\Omega}},\frac{1}{2\ell\sqrt{\Omega}}\Big)\,.$$

Let $\mathcal{J}=\{\mathcal{K}_j\ :\ \mathcal{K}_j\cap\mathcal{U}_{2L}\neq\emptyset\}$ and $N=\operatorname{Card}\mathcal{J}$. As $\varepsilon\to0_+$, the number N satisfies,

$$N=|\mathcal{U}_{2L}|\times\big(\ell\sqrt{\Omega}\big)^2\,\big(1+o(1)\big)\,.$$

In light of Lemma 3.2 and the exponential decay of $\widetilde{\eta}_{\varepsilon,\Omega}$ in Lemma 2.1, the function v satisfies,

$$1-\mathcal{O}(\varepsilon^2\Omega)\leq\int_{\mathbb{R}^2}\widetilde{\eta}_{\varepsilon,\Omega}^2|v|^2\,dx\leq 1\,.\tag{4.2}$$

Define the test function,

$$\widetilde{v}(x)=\frac{v(x)}{\sqrt{\displaystyle\int_{\mathbb{R}^2}\widetilde{\eta}_\varepsilon^2|v|^2\,dx}}\,.\tag{4.3}$$

Clearly, the function $\widetilde{v}$ satisfies the weighted mass constraint,

$$\int_{\mathbb{R}^2}\widetilde{\eta}_\varepsilon^2|\widetilde{v}|^2\,dx=1\,,\tag{4.4}$$

and consequently, there holds the upper bound $C_0(\varepsilon,\Omega)\leq\mathcal{G}_\varepsilon(\widetilde{v})$. The rest of the section will be devoted to estimating the energy $\mathcal{G}_\varepsilon(\widetilde{v})$. It will be established that:

$$\limsup_{\varepsilon\to0_+}\left(\frac{\mathcal{G}_\varepsilon(\widetilde{v})}{2\Omega\left[\ln\frac{1}{\varepsilon\sqrt{\Omega}}\right]}-1\right)\leq 0\,.\tag{4.5}$$

The next estimate (4.6) is a consequence of (**??**),

$$C_0(\varepsilon,\Omega)\leq\Omega\left[\ln\frac{1}{\varepsilon\sqrt{\Omega}}\right]\big(1+\mathrm{err}(\varepsilon)\big)\,.\tag{4.6}$$

4.2 Energy of the test configuration: Proof of (4.5)

It will be shown that the term

$$C_\varepsilon=\mathcal{G}_\varepsilon(\widetilde{v})=\int_{\mathbb{R}^2}\left(\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)\widetilde{v}|^2+\frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1-|\widetilde{v}|^2)^2\right)\,dx$$

is of leading order equal to $L_\varepsilon=\Omega\left[\ln\dfrac{1}{\varepsilon\sqrt{\Omega}}\right]$. It is useful to write C_ε as the sum of four terms,

$$C_\varepsilon=C_{\varepsilon,1}+C_{\varepsilon,2}+C_{\varepsilon,3}+C_{\varepsilon,4}\,,\tag{4.7}$$

where

$$C_{\varepsilon,1} = \int_{|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}} \leq \sqrt{\alpha_{\varepsilon\Omega}} - \delta} \left(\widetilde{\eta}_\varepsilon^2 |(\nabla - i\Omega \mathbf{A}_0)\widetilde{v}|^2 + \frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1 - |\widetilde{v}|^2)^2 \right) dx\,, \tag{4.8}$$

$$C_{\varepsilon,2} = \int_{\sqrt{\alpha_{\varepsilon\Omega}} - \delta \leq |x|_{\widetilde{\Lambda}_{\varepsilon\Omega}} \leq \sqrt{\alpha_{\varepsilon\Omega}} + \delta} \left(\widetilde{\eta}_\varepsilon^2 |(\nabla - i\Omega \mathbf{A}_0)\widetilde{v}|^2 + \frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1 - |\widetilde{v}|^2)^2 \right) dx\,, \tag{4.9}$$

$$C_{\varepsilon,3} = \int_{\sqrt{\alpha_{\varepsilon\Omega}} + \delta \leq |x|_{\widetilde{\Lambda}_{\varepsilon\Omega}} \leq 2L} \left(\widetilde{\eta}_\varepsilon^2 |(\nabla - i\Omega \mathbf{A}_0)\widetilde{v}|^2 + \frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1 - |\widetilde{v}|^2)^2 \right) dx\,, \tag{4.10}$$

$$C_{\varepsilon,4} = \int_{|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}} \geq 2L} \left(\widetilde{\eta}_\varepsilon^2 |(\nabla - i\Omega \mathbf{A}_0)\widetilde{v}|^2 + \frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1 - |\widetilde{v}|^2)^2 \right) dx\,, \tag{4.11}$$

and $\alpha_{\varepsilon\Omega}$ is as in (2.2).

The term $C_{\varepsilon,1}$:

Let $\mathcal{J}_0 = \{j \in \mathcal{J} \ :\ \mathcal{K}_j \cap \{x \ :\ |x|_{\widetilde{\Lambda}_{\varepsilon\Omega}} \leq \sqrt{\alpha_{\varepsilon\Omega}} - \delta\} \neq \emptyset\}$. Since δ is selected independently of ε, then in light of Theorem 2.2, there holds in every square $\mathcal{K}_j$ with $j \in \mathcal{J}_0$,

$$\widetilde{\eta}_\varepsilon^2(x) \leq p_{\varepsilon\Omega}(x)\,.$$

The mean value theorem applied to the function $p_{\varepsilon\Omega}$ yields,

$$p_{\varepsilon\Omega}(x) \leq p_{\varepsilon\Omega}(x_j) + \frac{C}{\ell\sqrt{\Omega}}\,,$$

where x_j is an arbitrary point in $\mathcal{K}_j$ and $j \in \mathcal{J}_0$. The above two estimates applied successively yield an upper bound of the term $C_{\varepsilon,1}$ as follows:

$$C_{\varepsilon,1} \leq \sum_{j\in\mathcal{J}_0} \left[p_{\varepsilon\Omega}(x_j) + \frac{C}{\ell\sqrt{\Omega}} \right] \int_{\mathcal{K}_j} \left(|(\nabla - i\Omega \mathbf{A}_0)\widetilde{v}|^2 + \frac{\lambda_\varepsilon}{2\varepsilon^2}(1 - |\widetilde{v}|^2)^2 \right) dx\,,$$

where

$$\lambda_\varepsilon = \max_{j\in\mathcal{J}_0} \left(\frac{p_{\varepsilon\Omega}(x_j)}{p_{\varepsilon\Omega}(x_j) + \frac{C}{\ell\sqrt{\Omega}}} \right)\,.$$

In the domain $\mathcal{U}_L$, the function χ is equal to 1 and $v(x) = f_\varepsilon(\ell\sqrt{\Omega}\,x)$. By using successively the estimate in (4.2), the 'magnetic' periodicity of v over the lattice $(\mathcal{K}_j)_j$ and the bound $|v| \leq 1$, one gets the following upper bound,

$$\begin{aligned}
&\int_{\mathcal{K}_j} \left(|(\nabla - i\Omega \mathbf{A}_0)\widetilde{v}|^2 + \frac{\lambda_\varepsilon}{2\varepsilon^2}(1 - |\widetilde{v}|^2)^2 \right) dx \\
&\qquad \leq (1 + C\varepsilon^2\Omega) \int_{\mathcal{K}_j} \left(|(\nabla - i\Omega \mathbf{A}_0)v|^2 + \frac{\lambda_\varepsilon}{2\varepsilon^2}(1 - |v|^2)^2 \right) dx + C\Omega \int_{\mathcal{K}_j} |v|^4\, dx \\
&\qquad \leq (1 + C\varepsilon^2\Omega) \int_{\mathcal{K}} \left(|(\nabla - i\Omega \mathbf{A}_0)v|^2 + \frac{\lambda_\varepsilon}{2\varepsilon^2}(1 - |v|^2)^2 \right) dx + C\Omega|\mathcal{K}_j|\,.
\end{aligned} \tag{4.12}$$

The integral term in (4.12) is computed by the change of variable $y = \ell\sqrt{\Omega}\,x$ that transforms it to

$$\int_K \left(|(\nabla - ih_{\rm ex}\mathbf{A}_0)f_\varepsilon|^2 + \frac{\lambda_\varepsilon}{2\tilde{\varepsilon}^2}(1 - |f_\varepsilon|^2)^2 \right) dx\,, \tag{4.13}$$

where $\tilde{\varepsilon} = \varepsilon\ell\sqrt{\Omega}$ and $h_{\rm ex} = \frac{1}{\ell^2}$. As $\varepsilon \to 0_+$, $\tilde{\varepsilon} \gg \varepsilon$ and $h_{\rm ex}$ satisfies $|\ln\varepsilon| \ll h_{\rm ex} \ll \varepsilon^{-2}$. Also, λ_ε remains inside a fixed interval $[\lambda_1, \lambda_2]$. Consequently, it is possible to use Lemma 3.2 and get that $\big(1+o(1)\big)h_{\rm ex}\ln\dfrac{1}{\varepsilon\sqrt{h_{ex}}}$ is an upper bound of the term in (4.13). As a consequence, it is obtained the following upper bound of $C_{\varepsilon,1}$,

$$C_{\varepsilon,1}\leq(1+C\varepsilon^2\Omega)\sum_{j\in\mathcal{J}_0}\left[p_{\varepsilon\Omega}(x_j)+C\varepsilon^2|\ln\varepsilon|+\frac{C}{\ell\sqrt{\Omega}}\right]\left(\big(1+o(1)\big)h_{\rm ex}\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}+C\Omega|\mathcal{K}_j|\right). \tag{4.14}$$

Recall that, as $\varepsilon\to0_+$, the number of squares $\mathcal{K}_j$ satisfies $N=|\mathcal{U}_{2L}|\times\ell^2\Omega\big(1+o(1)\big)$. Since $|\mathcal{K}_j|=\dfrac{1}{\ell^2\Omega}$ for every j, then $\sum_{j\in\mathcal{J}}|\mathcal{K}_j|=|\mathcal{U}_{2L}|\big(1+o(1)\big)$. Also, all the extra terms appearing in (4.14) are $o(1)$ as $\varepsilon\to0_+$, and this leads one to,

$$\begin{aligned}C_{\varepsilon,1}&\leq\big(1+o(1)\big)\sum_{j\in\mathcal{J}_0}\frac{1}{|\mathcal{K}_j|}p_{\varepsilon\Omega}(x_j)\ell^2\Omega h_{\rm ex}\ln\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\\&=\big(1+o(1)\big)\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\sum_{j\in\mathcal{J}_0}\frac{1}{|\mathcal{K}_j|}a(x_j)\,.\end{aligned}$$

Since each point x_j is arbitrarily selected in the square $\mathcal{K}_j$, then the sum $\sum_j\frac{1}{|\mathcal{K}_j|}p_{\varepsilon\Omega}(x_j)$ becomes a Riemann sum. Select the points (x_j) such that the sum is a lower Riemann sum. That way,

$$\sum_{j\in\mathcal{J}'}\frac{1}{|\mathcal{K}_j|}p_{\varepsilon\Omega}(x_j)\leq\int_{|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}-\delta}p_{\varepsilon\Omega}(x)\,dx\leq\int_{p_{\varepsilon\Omega}(x)>0}p_{\varepsilon\Omega}(x)\,dx=1\,.$$

As a consequence, the term $C_{\varepsilon,1}$ satisfies,

$$C_{\varepsilon,1}\leq\big(1+o(1)\big)\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\qquad\text{as }\varepsilon\to0_+\,. \tag{4.15}$$

The term $C_{\varepsilon,2}$:

To estimate the term $C_{\varepsilon,2}$, it is used the result of Theorem 2.2 that the function $\widetilde{\eta}_\varepsilon$ is bounded independently of ε to get that,

$$C_{\varepsilon,2}\leq C\int_{\sqrt{\alpha_{\varepsilon\Omega}}-\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}+\delta}\left(|(\nabla-i\Omega\mathbf{A}_0)\widetilde{v}|^2+\frac{1}{2\varepsilon^2}(1-|\widetilde{v}|^2)^2\right)dx\,.$$

The definition of $\widetilde{v}$ and the estimate in (4.2) together yield,

$$\begin{aligned}C_{\varepsilon,2}\leq C(1+C\varepsilon^2\Omega)\int_{\sqrt{\alpha_{\varepsilon\Omega}}-\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}+\delta}&\left(|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{1}{2\varepsilon^2}(1-|v|^2)^2\right)dx\\&+C\Omega\int_{\sqrt{\alpha_{\varepsilon\Omega}}-\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}+\delta}|v|^2\,dx\,.\end{aligned}$$

The function χ is equal to 1 in $\{\sqrt{\alpha_{\varepsilon\Omega}}-\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}+\delta\}\subset\mathcal{U}_L$. As a consequence $v(x)=f_\varepsilon(\ell\sqrt{\Omega}\,x)$. As is done for the term $C_{\varepsilon,1}$, one gets that,

$$C_{\varepsilon,2}\leq C\big(1+o(1)\big)\left(\int_{\sqrt{\alpha_{\varepsilon\Omega}}-\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}+\delta}dx\right)\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\leq C\delta\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\,. \tag{4.16}$$

The term $C_{\varepsilon,3}$:

When $\sqrt{\alpha_{\varepsilon\Omega}}+\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq2L$, the function χ is no more constant and the function v is *small*. As a consequence, it is not useful to estimate the 'Ginzburg-Landau' energy of v along the same procedure as done before. However, as Theorem 2.1 states, the function $\widetilde{\eta}_\varepsilon$ decays exponentially, and this will be the key to estimate the term $C_{\varepsilon,3}$. Thanks to (4.2), the function $\widetilde{v}$ satisfies the uniform inequality $|1-|\widetilde{v}|^2| \leq 1+\mathcal{O}(\varepsilon^2\Omega)$. This and the exponential decay of $\widetilde{\eta}_\varepsilon$ in Theorem 2.1 together yield when $\varepsilon \to 0_+$,

$$\frac{1}{2\varepsilon^2}\int_{\sqrt{\alpha_{\varepsilon\Omega}}+\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq 2L}\widetilde{\eta}_\varepsilon^4(1-|\widetilde{v}|^2)^2\,dx \leq C\frac{1}{\varepsilon^2}\exp\left(-\frac{\delta}{\varepsilon^{1/2}}\right)\int_{\sqrt{a_0}+1/2\leq|x|_\Lambda\leq\sqrt{a_0}+1}dx = o(1)\,.$$

Using a similar reasoning, the kinetic energy term is estimated as follows,

$$\begin{aligned}\int_{\sqrt{\alpha_{\varepsilon\Omega}}+\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq 2L}&\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)\widetilde{v}|^2\,dx\\ &\leq C\exp\left(-\frac{\delta}{\varepsilon^{1/2}}\right)\int_{\sqrt{\alpha_{\varepsilon\Omega}}+\delta\leq|x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq 2L}\Big(|(\nabla-i\Omega\mathbf{A}_0)v|^2+|\nabla\chi|^2|v|^2\Big)\,dx\\ &\leq C\exp\left(-\frac{\delta}{\varepsilon^{1/2}}\right)\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}} = o(1)\,,\end{aligned}$$

thereby obtaining that $C_{\varepsilon,3}=o(1)$ as $\varepsilon\to 0_+$.

The term $C_{\varepsilon,4}$:

Recall the definition of this term in (4.11) and that the function $\widetilde{v}=0$ here. As a consequence, $C_{\varepsilon,4} = \int_{|x|_\Lambda\geq\sqrt{a_0}+1}\frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}\,dx$ and this is equal to $o(1)$ as $\varepsilon\to 0_+$ after using the exponential decay of $\widetilde{\eta}_\varepsilon$ stated in Theorem 2.2.

Conclusion:

Collecting the estimates $C_{\varepsilon,4}=o(1)$, $C_{\varepsilon,3}=o(1)$, (4.16) and (4.14) and inserting them into (4.7) yields an upper bound of C_ε. Inserting this bound into the expression of $\mathcal{G}_\varepsilon(\widetilde{v})$ yields the upper bound

$$C_0(\varepsilon,\Omega) \leq (1+C\delta+o(1))\,\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}+o(1)\,,$$

as $\varepsilon\to 0_+$. This yields (4.6) by taking the successive limits as $\varepsilon\to 0_+$ and then as $\delta\to 0_+$.

5 Lower Bound

Suppose that v is a minimizer of the functional $\mathcal{G}_\varepsilon$ introduced in (1.9), and that the rotational speed Ω satisfies the assumption of Theorem 1.1. The aim of this section is to write a lower bound of $\mathcal{G}_\varepsilon(v)$.

The assumption on the rotational speed is still $|\ln\varepsilon| \ll \Omega \leq M/\varepsilon$ with $0 < M < 2\Lambda$. Consider a positive constant

$$0<\delta<\sqrt{a_0\left(1-\frac{M^2}{4\Lambda^2}\right)^{1/4}}$$

and the following subset of $\mathcal{D}_{\varepsilon\Omega}$,

$$\mathcal{U}_\delta = \{x\in\mathbb{R}^2\ :\ |x|_{\widetilde{\Lambda}_{\varepsilon\Omega}}\leq\sqrt{\alpha_{\varepsilon\Omega}}-\delta\}\,,$$

where $\alpha_{\varepsilon\Omega}$ and $\widetilde{\Lambda}_{\varepsilon\Omega}$ are introduced in (2.2).

Recall the lattice of squares $\mathcal{K}_j$ introduced in Section 4. The parameters ℓ and $h_{\rm ex}$ are still as in (4.1). Put

$$\mathcal{J}' = \{j\ :\ \mathcal{K}_j\subset\mathcal{U}_\delta\}\,. \tag{5.1}$$

There holds the obvious lower bound,

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{\widetilde{\eta}_\varepsilon^4}{\varepsilon^2}(1-|v|^2)^2\right)dx\\ \geq\int_{\mathcal{U}_\delta}\left(\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1-|v|^2)^2\right)dx\\ \geq\sum_{j\in\mathcal{J}'}\int_{\mathcal{K}_j}\left(\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1-|v|^2)^2\right)dx\,.\end{aligned}\tag{5.2}$$

Lower bound of the 'Ginzburg-Landau' energy:

For each $j\in\mathcal{J}'$, it will be obtained a lower bound of the term,

$$\mathcal{G}_\varepsilon(v,\mathcal{K}_j)=\int_{\mathcal{K}_j}\left(\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{\widetilde{\eta}_\varepsilon^2}{2\varepsilon^2}(1-|v|^2)^2\right)dx\,.\tag{5.3}$$

By Theorem 2.2, one can write for an arbitrary point x_j in $\mathcal{K}_j$,

$$\widetilde{\eta}_\varepsilon^2(x)\geq(1-C\varepsilon^{1/3})p_{\varepsilon\Omega}(x)\geq\left(1-C\varepsilon^{1/3}-\frac{C}{\ell\sqrt{\Omega}}\right)p_{\varepsilon\Omega}(x_j)\quad\text{in }\mathcal{K}_j\,,$$

and consequently,

$$\mathcal{G}_\varepsilon(v,\mathcal{K}_j)\geq\left(1-C\varepsilon^{1/3}-\frac{C}{\ell\sqrt{\Omega}}\right)\int_{\mathcal{K}_j}\left(p_{\varepsilon\Omega}(x_j)|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{p_{\varepsilon\Omega}(x_j)^2}{2\varepsilon^2}(1-|v|^2)^2\right)dx\,.\tag{5.4}$$

Let y_j be the center of the square $\mathcal{K}_j$, $K=(-1/2,1/2)^2$, $\tilde{\varepsilon}=\ell\sqrt{\Omega}\,\varepsilon$ and $h_{\rm ex}=1/\ell^2$. Using the re-scaled function $f(x)=v(y_j+\ell\sqrt{\Omega}\,x)$, $(x\in K)$, it is possible to express (5.4) in the following form,

$$\mathcal{G}_\varepsilon(v,\mathcal{K}_j)\geq\left(1-C\varepsilon^{1/3}-\frac{C}{\ell\sqrt{\Omega}}\right)p_{\varepsilon\Omega}(x_j)\int_K\left(|(\nabla-ih_{\rm ex}\mathbf{A}_0)f|^2+\frac{p_{\varepsilon\Omega}(x_j)}{2\tilde{\varepsilon}^2}(1-|f|^2)^2\right)dx\,.\tag{5.5}$$

Notice that the term $p_{\varepsilon\Omega}(x_j)$ remains in a constant interval $[\lambda_1,\lambda_2]$ as $j\in\mathcal{J}'$ and ε vary. Also, as $\varepsilon\to0$, $\tilde{\varepsilon}$ and $h_{\rm ex}$ satisfy $|\ln\tilde{\varepsilon}|\ll h_{\rm ex}\ll\tilde{\varepsilon}^{-2}$. Thus, it is possible to bound the integral on the right side of (5.5) by the ground state energy $m_0(h_{\rm ex},\tilde{\varepsilon})$ in (3.5), which is estimated from below in Theorem 3.1. Therefore, it is inferred from (5.5),

$$\mathcal{G}_\varepsilon(v,\mathcal{K}_j)\geq\big(1+o(1)\big)p_{\varepsilon\Omega}(x_j)h_{\rm ex}\ln\frac{1}{\tilde{\varepsilon}\sqrt{h_{\rm ex}}}=\big(1+o(1)\big)p_{\varepsilon\Omega}(x_j)\frac{1}{\ell^2}\ln\frac{1}{\varepsilon\sqrt{\Omega}}\,.\tag{5.6}$$

Inserting this into (5.3) and then into (5.2) yields,

$$\int_{\mathbb{R}^2}\left(\widetilde{\eta}_\varepsilon^2|(\nabla-i\Omega\mathbf{A}_0)v|^2+\frac{\widetilde{\eta}_\varepsilon^2}{2\varepsilon^2}(1-|v|)^2\right)dx\geq\big(1+o(1)\big)\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\sum_{j\in\mathcal{J}'}\frac{1}{\ell^2\Omega}p_{\varepsilon\Omega}(x_j).\tag{5.7}$$

The sum on the right side of (5.7) is estimated as follows. As $\varepsilon\to0_+$, the term $\sum_{j\in\mathcal{J}'}\frac{1}{\ell^2\Omega}a(x_j)$ is a Riemann sum. Select the points (x_j) such that the sum is an upper Riemann sum. As a consequence, there holds,

$$\begin{aligned}\sum_{j\in\mathcal{J}'}p_{\varepsilon\Omega}(x_j)h_{\rm ex}\ln\frac{1}{\tilde{\varepsilon}\sqrt{h_{\rm ex}}}&=\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\sum_{j\in\mathcal{J}'}\frac{1}{\ell^2\Omega}p_{\varepsilon\Omega}(x_j)\\&=\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\int_{\mathcal{U}_{2\delta}}p_{\varepsilon\Omega}(x)\,dx\,.\end{aligned}$$

Therefore, it results from (5.7),

$$\int_{\mathbb{R}^2}\left(\widetilde{\eta}_\varepsilon^2|(\nabla - i\Omega\mathbf{A}_0)v|^2 + \frac{\widetilde{\eta}_\varepsilon^2}{2\varepsilon^2}(1-|v|)^2\right)dx \geq \Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\left(\int_{\mathcal{U}_{2\delta}} p_{\varepsilon\Omega}(x)\,dx\right). \tag{5.8}$$

Recall that the function $p_{\varepsilon\Omega}$ in (2.2) satisfies $\displaystyle\int_{p_{\varepsilon\Omega}(x)>0} p_{\varepsilon\Omega}(x)\,dx = 1$. Thus,

$$\int_{\mathcal{U}_{2\delta}} p_{\varepsilon\Omega}(x)\,dx = \int_{p_{\varepsilon\Omega}(x)>0} p_{\varepsilon\Omega}(x)\,dx - \int_{p_{\varepsilon\Omega}(x)>2\delta} p_{\varepsilon\Omega}(x)\,dx \geq 1 - C\delta\,.$$

That way, (5.8) becomes,

$$\int_{\mathbb{R}^2}\left(\widetilde{\eta}_\varepsilon^2|(\nabla - i\Omega\mathbf{A}_0)v|^2 + \frac{\widetilde{\eta}_\varepsilon^2}{2\varepsilon^2}(1-|v|)^2\right)dx \geq \Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\left(1 - C\delta\right). \tag{5.9}$$

Conclusion:

It is obtained by collecting the estimate in (5.9),

$$C_0(\varepsilon,\Omega) \geq \Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\left(1 - C\delta\right).$$

As a consequence, it is obtained by taking the limit as $\varepsilon \to 0_+$,

$$\liminf_{\varepsilon\to 0_+}\frac{C_0(\varepsilon,\Omega)}{\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}} \geq 1 - C\delta\,.$$

By Taking $\delta \to 0_+$, it results the lower bound:

$$\liminf_{\varepsilon\to 0_+}\frac{C_0(\varepsilon,\Omega)}{\Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}} \geq 1\,.$$

The conclusion of this section and Section 4 finishes the proof of Theorem 1.1.

Remark 5.1. If $U \subset \mathcal{D}_{\varepsilon\Omega}$ and $u \in H^1(U)$, define the local energy:

$$\mathcal{E}_\varepsilon(u;U) = \int_U\left(\widetilde{\eta}_\varepsilon^2|(\nabla - i\mathbf{A}_0)u|^2 + \frac{\widetilde{\eta}_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2\right)dx\,.$$

The analysis of this section allows one to prove the following. If v is a minimizer of (1.11), $U \subset \mathcal{D}_{\varepsilon\Omega}$ is an open set, $\overline{U} \subset \mathcal{D}_{\varepsilon\Omega}$, $|\partial U| = 0$, and U is independent of ε and Ω, then,

$$\mathcal{E}_\varepsilon(v;U) \geq \Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\left(\int_U p_{\varepsilon\Omega}(x)\,dx + o(1)\right) \quad \text{as } \varepsilon \to 0_+\,.$$

Combine this lower bound with the upper bound (4.6) to obtain the 'local' energy asymptotics:

$$\mathcal{E}_\varepsilon(v;U) = \Omega\ln\frac{1}{\varepsilon\sqrt{\Omega}}\left(\int_U p_{\varepsilon\Omega}(x)\,dx + o(1)\right) \quad \text{as } \varepsilon \to 0_+\,.$$

6 Vortices and their density

The assumption on the rotational speed is as in Theorem 1.1. Recall the definition of the domain $\mathcal{D}$ in (1.4). Let $\beta > 0$. Suppose that U is an open set in $\mathbb{R}^2$ satisfying the properties in Remark 5.1 and

$$\operatorname{dist}(U, \partial\mathcal{D}_{\varepsilon\Omega}) \geq \beta\,.$$

According to Theorem 2.2, the function $\widetilde{\eta}_\varepsilon$ satisfies the pointwise bound $\widetilde{\eta}_\varepsilon \geq c_0(U) > 0$ in U. The constant $c_0(U)$ depends only on U.

Let v be a minimizer of (1.11). By borrowing the results of [17, 18], it will be given some details regarding the location and 'density' of the zeros of the minimizer v inside U.

Consider the lattice of squares $(\mathcal{K}_j)$ generated by the square $\mathcal{K} = (-\delta,\delta)\times(-\delta,\delta)$, where $\delta = \frac{1}{2}\big(|\ln\varepsilon|/\Omega\big)^{-1/4}$. Suppose that x_j is the center of the square $\mathcal{K}_j$.

By Theorem 3.1, there exists a positive function $g(\varepsilon)$ such that, as $\varepsilon\to 0_+$, $g(\varepsilon)\ll 1$ and

$$\mathrm{GL}_\varepsilon(v;\mathcal{K}_j) := \int_{\mathcal{K}_j}\left(|(\nabla - i\Omega\mathbf{A}_0)v|^2 + \frac{\widetilde{\eta}_\varepsilon^2(x_j)}{2\varepsilon^2}(1-|v|^2)^2\right)dx \geq \big(1-g(\varepsilon)\big)\Omega\delta^2\ln\frac{1}{\varepsilon\sqrt{\Omega}}\,.$$

One distinguishes between *good* squares and *bad* squares in U; *good* squares are those satisfying that

$$\mathrm{GL}_\varepsilon(v;\mathcal{K}_j) := \int_{\mathcal{K}_j}\left(|(\nabla - i\Omega\mathbf{A}_0)v|^2 + \frac{\widetilde{\eta}_\varepsilon^2(x_j)}{2\varepsilon^2}(1-|v|^2)^2\right)dx \leq \big(1+\sqrt{g(\varepsilon)}\,\big)\Omega\delta^2\ln\frac{1}{\varepsilon\sqrt{\Omega}}\,,$$

while *bad* squares satisfy the reverse condition that $\mathrm{GL}_\varepsilon(v;\mathcal{K}_j) > \Omega\delta^2\big(1+\sqrt{g(\varepsilon)}\,\big)\ln\frac{1}{\varepsilon\sqrt{\Omega}}$. The number of *bad* squares N_{b} is small compared to the number of *good* squares N_{g}, namely $N_{\mathrm{b}}\ll N_{\mathrm{g}}$ as $\varepsilon\to 0_+$. Proposition 5.1 in [18] gives one the following. There exists a constant $C>0$ and a positive function $\hat{g}(\varepsilon)$ such that, if $\mathcal{K}_j$ is a *good* square then there exists a finite family of discs $\big(B(a_{i,j},r_{i,j})\big)_i$ with the following properties,

(i) $\displaystyle\sum_i r_{i,j}\leq C\Omega^{-1/2}$;

(ii) $\displaystyle\{x\in\mathcal{K}_j\ :\ |v(x)|<\tfrac{1}{2}\}\subset\bigcup_i B(a_{i,j},r_{i,j})$;

(iii) If $d_{i,j}$ is the winding number of $v/|v|$ when $B(a_{i,j},r_{i,j})\subset\mathcal{K}_j$ and 0 otherwise, then,

$$\sum_i d_{i,j}\geq\Omega\delta^2\big(1-\hat{g}(\varepsilon)\big)\quad\text{and}\quad\sum_i|d_{i,j}|\leq\Omega\delta^2\big(1+\hat{g}(\varepsilon)\big)\,.$$

(iv) $\hat{g}(\varepsilon)\ll 1$ as $\varepsilon\to 0_+$.

Let $\mathcal{J}_{\mathrm{g}}$ be the collection of all indices j such that $\mathcal{K}_j$ is a good square and $\mathcal{K}_j\subset U$. Define the measure

$$\mu_\varepsilon = \sum_{\substack{i,j\\ j\in\mathcal{J}_{\mathrm{g}}}} d_{i,j}\delta_{a_{i,j}}\,, \tag{6.1}$$

where $\delta_{a_{i,j}}$ is the dirac measure supported at a_i. The measure μ_ε is called the vorticity measure in U: It indicates the existence of vortices (when $\mu_\varepsilon\neq 0$), its support indicates the location of vortices, and its norm indicates their density.

Notice that the aforementioned construction indicates the location and density of vortices for minimizers of (1.5), since $v = u/\widetilde{\eta}_\varepsilon$ and u is a minimizer of (1.5). Thus, v and u have the same zeros (vortices).

It is possible to prove that:

Theorem 6.1. Under the assumption of Theorem 1.1, the vorticity measure in U fulfills the weak convergence:

$$\frac{1}{\Omega}\mu_\varepsilon\rightharpoonup\mathbf{1}_U\,dx\quad\text{as }\varepsilon\to 0_+,$$

where dx is the Lebesuge measure in $\mathbb{R}^2$ and $\mathbf{1}_U$ the characteristic function of U.

Proof. Notice that the upper bound in (3) and the fact that the number of indices j is asymptotically proportional to δ^{-2} together yield that $\Omega^{-1}\sum_{i,j}|d_{i,j}|$ is bounded independently of ε and Ω. Consequently, by passing to a subsequence, one can suppose that $\Omega^{-1}\mu_\varepsilon$ converges weakly to a measure μ. It suffices to prove that $\mu = \mathbf{1}_U\,dx$.

Since the number of good squares satisfies $N_g \times \delta^2 = |U| + o(1)$ as $\varepsilon \to 0_+$, then the two-sided estimate of $\sum_{i,j} d_{i,j}$ in (3) above leads to the following. If S is an open set in U and $|\partial S| = 0$, then

$$\Omega|S|\big(1+o(1)\big) \le \sum_{i,j} d_{i,j} \le \big(1+o(1)\big)\mu_\varepsilon(S)$$

$$\le \big(1+o(1)\big)\sum_{i,j}|d_{i,j}| \le \Omega|S|\big(1+o(1)\big)\,, \quad \text{as } \varepsilon \to 0_+\,.$$

This proves that $\Omega^{-1}\mu_\varepsilon$ converges weakly to the Lebesgue measure restricted to U. □

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Author information

Ayman Kachmar, Lebanese University, Department of Mathematics, Hadat, Lebanon, and Lebanese International University, School of Arts and Sciences, Beirut, LEBANON.
E-mail: `ayman.kashmar@gmail.com`

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