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COUPLED FIXED POINT RESULTS INVOLVING ALTERING DISTANCES IN PARTIALLY ORDERED METRIC SPACES WITH APPLICATIONS TO INTEGRAL EQUATIONS

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Abstract. In this article, we study coupled fixed point theorems in partially ordered metric spaces for non linear contraction condition related to a pair of altering distance functions. To illustrate our results, an example and an application to integral equations have also been given.

1 Introduction

The Banach Contraction Principle is the pivotal results of analysis. Generalizations of this principle have been obtained in several directions. Its significance lies in its vast applicability in a number of branches of mathematics.

One of the most interesting fixed point theorems in partially ordered metric spaces was investigated by Ran and Reurings [13] applied their result to linear and nonlinear matrix equations. Then, many authors obtained several interesting results in partially ordered metric spaces, eg., in [7, 11, 12].

In 1984, Khan et. al [9] initiated the use of a control function that alters distance between two points in a metric space. Such mappings are called an altering distances. Altering distance has been used in metric fixed point theory in a number of papers (see [5, 6, 10]). It has also been extended in the context of multivalued and fuzzy mappings. The concept of altering distance function has also been introduced in Menger spaces. Recently, Harjani and Sadarangani [8] used these functions where they proved some fixed point theorems in partially ordered metric spaces with applications to ordinary differential equations.

Bhaskar and Lakshmikantham [1] initiated the study of a coupled fixed point theorem in ordered metric spaces and applied their results to prove the existence and uniqueness of a solutions for a periodic boundary value problem. Many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the framework of partial metric spaces [2, 3, 4, 14, 15].

At first we need the following definitions and results.

Definition 1.1. [9]. An altering distance function is a function $\Psi : [0,\infty) \to [0,\infty)$ satisfying:

(i) Ψ is continuous and nondecreasing.

(ii) $\Psi(t) = 0$ if and only if $t = 0$.

Definition 1.2. [1]. An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 1.3. [1]. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \implies F(x, y_1) \geq F(x, y_2).$$

Theorem 1.4. *[1]. Let $(X, \leq)$ be a partially ordered set and suppose there exits a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.1}$$

$\forall\ x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. If there exists two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Theorem 1.5. *[1]. Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Assume that X has the following properties:*

(i) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, $\forall n$,

(ii) if a nonincreasing sequence $y_n \to y$, then $y_n \geq y$, $\forall n$.

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.2}$$

$\forall\ x \geq u$ and $y \leq v$. If there exists $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

The aim of this paper is to prove some unique coupled fixed point theorems for mappings having the mixed monotone property in partially ordered metric spaces involving altering distance functions. Lastly, we present an application to integral equations.

2 Main Theorem

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping on X having the mixed monotone property such that*

$$\varphi(d(F(x, y), F(u, v))) \leq \varphi(M((x, y), (u, v))) - \phi(M((x, y), (u, v))) \tag{2.1}$$

where

$$M((x, y), (u, v)) = max\{d(x, u), d(y, v), d(F(x, y), x), d(F(u, v), u)\}$$

$\forall\ x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where φ and ϕ are altering distance functions. Suppose that there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then by (2.1), we have

$$\begin{aligned}\varphi(d(x_{n+1}, x_n)) &= \varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\leq \varphi(M((x_n, y_n), (x_{n-1}, y_{n-1}))) - \phi(M((x_n, y_n), (x_{n-1}, y_{n-1}))),\end{aligned}$$

and

$$\begin{aligned}\varphi(d(y_{n+1}, y_n)) &= \varphi(d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \\ &\leq \varphi(M((y_n, x_n), (y_{n-1}, x_{n-1}))) - \phi(M((y_n, x_n), (y_{n-1}, x_{n-1}))),\end{aligned}$$

where,

$$\begin{aligned} M((x_n, y_n), (x_{n-1}, y_{n-1})) &= max\{d(x_n, x_{n-1}), d(y_n, y_{n-1}), \\ &\qquad d(F(x_n, y_n), x_n), d(F(x_{n-1}, y_{n-1}), x_{n-1})\} \\ &= max\{d(x_n, x_{n-1}), d(y_n, y_{n-1}), d(x_{n+1}, x_n), d(x_n, x_{n-1})\} \\ &= max\{d(x_n, x_{n-1}), d(y_n, y_{n-1}), d(x_{n+1}, x_n)\}. \end{aligned}$$

Now, let us consider two cases.
Case I: If

$$M((x_n, y_n), (x_{n-1}, y_{n-1})) = max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}.$$

We have

$$\varphi(d(x_{n+1}, x_n)) \leq \varphi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}) - \phi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}), \tag{2.2}$$

and

$$\varphi(d(y_{n+1}, y_n)) \leq \varphi(max\{d(y_n, y_{n-1}), d(x_n, x_{n-1})\}) - \phi(max\{d(y_n, y_{n-1}), d(x_n, x_{n-1})\}). \tag{2.3}$$

Case II: If

$$M((x_n, y_n), (x_{n-1}, y_{n-1})) = d(x_{n+1}, x_n).$$

We claim that

$$M((x_n, y_n), (x_{n-1}, y_{n-1})) = d(x_{n+1}, x_n) = 0.$$

In fact if $d(x_{n+1}, x_n) \neq 0$, then

$$\varphi(d(x_{n+1}, x_n)) \leq \varphi(d(x_{n+1}, x_n)) - \phi(d(x_{n+1}, x_n)) < \varphi(d(x_{n+1}, x_n)) \text{ as } \phi \geq 0.$$

This implies

$$d(x_{n+1}, x_n) < d(x_{n+1}, x_n),$$

which is a contradiction. Since $M((x_n, y_n), (x_{n-1}, y_{n-1})) = 0$. Then it is obvious that (2.2) and (2.3) hold.
Now, by (2.2) and (2.3), we have

$$\varphi(d(x_{n+1}, x_n)) \leq \varphi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}) - \phi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}). \tag{2.4}$$

As $\phi \geq 0$.

$$\varphi(d(x_{n+1}, x_n)) \leq \varphi(max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}),$$

and using the fact that φ is nondecreasing, we have

$$d(x_{n+1}, x_n) \leq max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}. \tag{2.5}$$

Similarly,

$$\begin{aligned} \varphi(d(y_{n+1}, y_n)) &\leq \varphi(max\{d(y_n, y_{n-1}), d(x_n, x_{n-1})\}) - \phi(max\{d(y_n, y_{n-1}), d(x_n, x_{n-1})\}) \\ &\leq \varphi(max\{d(y_n, y_{n-1}), d(x_n, x_{n-1})\}), \end{aligned} \tag{2.6}$$

and consequently

$$d(y_{n+1}, y_n) \leq max\{d(y_n, y_{n-1}), d(x_n, x_{n-1})\}, \tag{2.7}$$

by (2.5) and (2.7), we have

$$max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\} \le max\{d(x_n,x_{n-1}),d(y_n,y_{n-1})\},$$

and thus, the sequence $max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}$ is nonnegative decreasing. This implies that there exists $r \ge 0$ such that

$$\lim_{n\to\infty} max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\} = r. \tag{2.8}$$

It is easily seen that if $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing, $\varphi(max(a,b)) = max(\varphi(a),\varphi(b))$ for $a,b \in [0,\infty)$. Taking into account this and (2.4) and (2.6), we get

$$\begin{aligned} max\{\varphi(d(x_{n+1},x_n)),\varphi(d(y_{n+1},y_n))\} &= \varphi(max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) \\ &\le \varphi(max\{d(x_n,x_{n-1}),d(y_n,y_{n-1})\}) \\ &\quad - \phi(max\{d(x_n,x_{n-1}),d(y_n,y_{n-1})\}). \end{aligned} \tag{2.9}$$

Letting $n \to \infty$ in (2.9) and taking into account (2.8), we get

$$\varphi(r) \le \varphi(r) - \phi(r) \le \varphi(r),$$

and this implies $\phi(r) = 0$. Since ϕ is an altering distance function, $r = 0$ and this implies

$$\lim_{n\to\infty} max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\} = 0.$$

Thus

$$\lim_{n\to\infty} d(x_{n+1},x_n) = \lim_{n\to\infty} d(y_{n+1},y_n) = 0. \tag{2.10}$$

Next, we claim that $\{x_n\}, \{y_n\}$ are Cauchy sequences.
We will show that for every $\varepsilon > 0$, there exists $k \in N$ such that if $n,m \ge k$,

$$max\{d(x_{m(k)},x_{n(k)}),d(y_{m(k)},y_{n(k)})\} < \varepsilon.$$

Suppose the above statement is false.
Then, there exists an $\varepsilon > 0$ for which we can find sequence $\{x_{m(k)}\}, \{x_{n(k)}\}$ with $n(k) > m(k) > k$ such that

$$max\{d(x_{m(k)},x_{n(k)}),d(y_{m(k)},y_{n(k)})\} \ge \varepsilon. \tag{2.11}$$

Further, we can choose $n(k)$ corresponding to $m(k)$ in such a way that it is smallest integer with $n(k) > m(k)$ and satisfing (2.11). Then

$$max\{d(x_{m(k)},x_{n(k)-1}),d(y_{m(k)},y_{n(k)-1})\} < \varepsilon. \tag{2.12}$$

From triangle inequality

$$d(x_{n(k)},x_{m(k)}) \le d(x_{n(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{m(k)}). \tag{2.13}$$

Similarly

$$d(y_{n(k)},y_{m(k)}) \le d(y_{n(k)},y_{n(k)-1}) + d(y_{n(k)-1},y_{m(k)}). \tag{2.14}$$

From (2.13) and (2.14), we have

$$\begin{aligned} max\{d(x_{n(k)},x_{m(k)}),d(y_{n(k)},y_{m(k)})\} &\le max\{d(x_{n(k)},x_{n(k)-1}),d(y_{n(k)},y_{n(k)-1})\} \\ &\quad + max\{d(x_{n(k)-1},x_{m(k)}),d(y_{n(k)-1},y_{m(k)})\}. \end{aligned} \tag{2.15}$$

From (2.11), (2.12) and (2.15), we get

$$\varepsilon \le max\{d(x_{n(k)},x_{m(k)}),d(y_{n(k)},y_{m(k)})\} \le max\{d(x_{n(k)},x_{n(k)-1}),d(y_{n(k)},y_{n(k)-1})\} + \varepsilon. \tag{2.16}$$

Letting $k \to \infty$ in (2.16) and taking into account (2.10) we have

$$\lim_{k\to\infty} max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} = \varepsilon. \tag{2.17}$$

Again, the triangle inequality, we have

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}), \tag{2.18}$$

and

$$d(y_{n(k)-1}, y_{m(k)-1}) \leq d(y_{n(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1}). \tag{2.19}$$

From (2.18) and (2.19), we have

$$\begin{aligned} max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} &\leq max\{d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})\} \\ &\quad + max\{d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})\}. \end{aligned} \tag{2.20}$$

From (2.12), we have

$$max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \leq max\{d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})\} + \varepsilon. \tag{2.21}$$

Using the triangle inequality, we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}), \tag{2.22}$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}). \tag{2.23}$$

From (2.22), (2.23) and (2.11), we get

$$\begin{aligned} \varepsilon &\leq max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})\} \\ &\quad + max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \\ &\quad + max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\}. \end{aligned} \tag{2.24}$$

From (2.24) and (2.21), we have

$$\begin{aligned} \varepsilon - max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})\} &- max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\} \\ &\leq max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \\ &< max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\} + \varepsilon. \end{aligned} \tag{2.25}$$

Letting $k \to \infty$ in (2.25) and using (2.10), we get

$$\lim_{k\to\infty} max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} = \varepsilon. \tag{2.26}$$

Since $x_{n(k)-1} \geq x_{m(k)-1}$ and $y_{n(k)-1} \leq y_{m(k)-1}$, using the contractive condition we can obtain

$$\begin{aligned} \varphi(d(x_{n(k)}, x_{m(k)})) &= \varphi(d(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1}))) \\ &\leq \varphi(M((x_{n(k)-1}, y_{n(k)-1}), (x_{m(k)-1}, y_{m(k)-1}))) \\ &\quad - \phi(M((x_{n(k)-1}, y_{n(k)-1}), (x_{m(k)-1}, y_{m(k)-1}))), \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} M((x_{n(k)-1}, y_{n(k)-1}), (x_{m(k)-1}, y_{m(k)-1})) &= max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}), \\ &\qquad d(F(x_{n(k)-1}, y_{n(k)-1}), x_{n(k)-1}), \\ &\qquad d(F(x_{m(k)-1}, y_{m(k)-1}), x_{m(k)-1})\} \\ &= max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}), \\ &\qquad d(x_{n(k)}, x_{n(k)-1}), d(x_{m(k)}, x_{m(k)-1})\}. \end{aligned}$$

Similarly,

$$\begin{aligned}\varphi(d(y_{n(k)},y_{m(k)})) &= \varphi(d(F(y_{n(k)-1},x_{n(k)-1}),d(F(y_{m(k)-1},x_{m(k)-1})))\\ &\leq \varphi(M((y_{n(k)-1},x_{n(k)-1}),(y_{m(k)-1},x_{m(k)-1})))\\ &\quad -\phi(M((y_{n(k)-1},x_{n(k)-1}),(y_{m(k)-1},x_{m(k)-1}))),\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}M((y_{n(k)-1},x_{n(k)-1}),(y_{m(k)-1},x_{m(k)-1}) &= max\{d(y_{n(k)-1},y_{m(k)-1}),d(x_{n(k)-1},x_{m(k)-1}),\\ &\qquad d(F(y_{n(k)-1},x_{n(k)-1}),y_{n(k)-1}),\\ &\qquad d(F(y_{m(k)-1},x_{m(k)-1}),y_{m(k)-1})\}\\ &= max\{d(x_{n(k)-1},x_{m(k)-1}),d(y_{n(k)-1},y_{m(k)-1}),\\ &\qquad d(y_{n(k)},y_{n(k)-1}),d(y_{m(k)},y_{m(k)-1})\}.\end{aligned}$$

From (2.27) and (2.28), we have

$$max\{\varphi(d(x_{n(k)},x_{m(k)}),d(y_{n(k)},y_{m(k)}))\}\leq\varphi(z_n)-\phi(z_n),$$

where

$$\begin{aligned}z_n = max\{&d(x_{n(k)-1},x_{m(k)-1}),d(y_{n(k)-1},y_{m(k)-1}),\\ &d(x_{n(k)},x_{n(k)-1}),d(y_{n(k)},y_{n(k)-1}),\\ &d(x_{m(k)},x_{m(k)-1}),d(y_{m(k)},y_{m(k)-1})\}.\end{aligned}$$

Finally letting $k\to\infty$ in last two inequalities and using (2.26), (2.17) and (2.10) and the continuity of φ and ϕ, we have

$$\varphi(\varepsilon)\leq\varphi(max(\varepsilon,0,0))-\phi(max(\varepsilon,0,0))<\varphi(\varepsilon)$$

and consequently, $\phi(\varepsilon)=0$. Since ϕ is an altering distance function, $\varepsilon=0$ and which is a contradiction.

This proves our claim.

Since X is a complete metric space, $\exists\, x,y\in X$ such that

$$\lim_{n\to\infty}x_n=x \ \text{ and } \ \lim_{n\to\infty}y_n=y.$$

Now we show that (x,y) is a coupled fixed point of F.

As, we have

$$\begin{aligned}x&=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}F(x_n,y_n)=F(\lim_{n\to\infty}x_n,\lim_{n\to\infty}y_n)=F(x,y),\\ y&=\lim_{n\to\infty}y_{n+1}=\lim_{n\to\infty}F(y_n,x_n)=F(\lim_{n\to\infty}y_n,\lim_{n\to\infty}x_n)=F(y,x).\end{aligned}$$

Therefore, (x,y) is a coupled fixed point of F. $\square$

Theorem 2.2. *Suppose all the assumptions of Theorem (2.1) are satisfied . Moreover, assume that X has the following properties*

(a) if a non-decreasing sequence$\{x_n\}$ in X converges to some point $x\in X$, then $x_n\leq x$, $\forall n$,

(b) if a non-increasing sequence $\{y_n\}$ in X converges to some point $y\in X$, then $y_n\geq y$, $\forall n$.

Then the conclusion of Theorem (2.1) also hold.

Proof. Following the proof of Theorem (2.1) we only have to check that (x,y) is a coupled fixed point of F.

In fact, since $\{x_n\}$ is non-decreasing and $x_n \to x$ and $\{y_n\}$ is non-increasing and $y_n \to y$, by our assumption, $x_n \leq x$ and $y_n \geq y \; \forall \; n$.

Applying the contractive condition we have

$$\begin{aligned} \varphi(d(F(x,y),F(x_n,y_n))) &\leq \varphi(M((x,y),(x_n,y_n))) - \phi(M((x,y),(x_n,y_n))) \\ &\leq \varphi(M((x,y),(x_n,y_n))), \end{aligned} \tag{2.29}$$

and as φ is nondecreasing, we obtain

$$d(F(x,y),F(x_n,y_n)) \leq M((x,y),(x_n,y_n)),$$

where

$$M((x,y),(x_n,y_n)) = max\{d(x,x_n),d(y,y_n),d(F(x,y),x),d(F(x_n,y_n),x_n)\}. \tag{2.30}$$

Letting $n \to \infty$ in (2.29) (and hence (2.30)), we obtain

$$d(x,F(x,y)) = 0,$$

and consequently $F(x,y) = x$.

Using a similar argument it can be proved that $y = F(y,x)$ and this finishes the proof. □

Now, we give a sufficient condition for the uniqueness of the coupled fixed point in Theorem (2.1) and (2.2). This condition is

$$\text{for } (x,y),(u,v) \in X \times X \text{ there exists } (z,t) \in X \times X \text{ which is comparable to } (x,y) \text{ and } (u,v). \tag{2.31}$$

Note that in $X \times X$ we consider the partial order relation given by

$$(x,y) \leq (u,v) \iff x \leq u \text{ and } y \geq v.$$

Theorem 2.3. *Adding condition* (2.31) *to the hypotheses of Theorem* (2.1) *(resp. Theorem* (2.2)*) we obtain uniqueness of the coupled fixed point of* F.

Proof. Suppose (x,y) and (x',y') are coupled fixed points of F, that is, $F(x,y) = x, F(y,x) = y, F(x',y') = x'$ and $F(y',x') = y'$. We shall prove that $x = x', y = y'$.

Let (x,y) and (x',y') are not comparable. By assumption there exist $(z,t) \in X \times X$ comparable with both of them. Suppose that $(x,y) \geq (z,t)$.

We define sequences $\{z_n\},\{t_n\}$ as follows

$$z_0 = z, \quad t_0 = t, \quad z_{n+1} = F(z_n,t_n) \;\; and \;\; t_{n+1} = F(t_n,z_n) \;\; \forall n.$$

Since (z,t) is comparable with (x,y). We claim that $(x,y) \geq (z_n,t_n)$ for each $n \in N$.

We will use mathematical induction.

For $n = 0$, as $(x,y) \geq (z,t)$, this means $z_0 = z \leq x$ and $y \geq t = t_0$ and consequently, $(x,y) \geq (z_0,t_0)$.

Suppose that $(x,y) \geq (z_n,t_n)$; then using the mixed monotone property of F, we get

$$\begin{aligned} z_{n+1} &= F(z_n,t_n) \leq F(x,t_n) \leq F(x,y) = x, \\ t_{n+1} &= F(t_n,z_n) \geq F(y,z_n) \geq F(y,x) = y, \end{aligned}$$

and this proves our claim.

Now, since $z_n \leq x$ and $t_n \geq y$, using (2.1), we have

$$\varphi(d(x,z_{n+1})) = \varphi(d(F(x,y),F(z_n,t_n))) \leq \varphi(M((x,y),(z_n,t_n))) - \phi(M((x,y),(z_n,t_n))), \tag{2.32}$$

where

$$\begin{aligned} M((x,y),(z_n,t_n)) &= max\{d(x,z_n),d(y,t_n),d(F(x,y),x),d(F(z_n,t_n),z_n)\} \\ &= max\{d(x,z_n),d(y,t_n)\}. \end{aligned}$$

Therefore

$$\begin{aligned}\varphi(d(x, z_{n+1})) &\leq \varphi(max\{d(x, z_n), d(y, t_n)\}) - \phi(max\{d(x, z_n), d(y, t_n)\}) \\ &\leq \varphi(max\{d(x, z_n), d(y, t_n)\}),\end{aligned} \tag{2.33}$$

and analogously

$$\varphi(d(y, t_{n+1})) \leq \varphi(max\{d(y, t_n), d(x, z_n)\}). \tag{2.34}$$

From (2.33) and (2.34) and using the fact that φ is nondecreasing, we obtain

$$\begin{aligned}\varphi(max\{d(x, z_{n+1}), d(y, t_{n+1})\}) &= max\{\varphi(d(x, z_{n+1}), \varphi(d(y, t_{n+1}))\} \\ &\leq \varphi(max\{d(x, z_n), d(y, t_n)\}) - \phi(max\{d(x, z_n), d(y, t_n)\}) \\ &\leq \varphi(max\{d(x, z_n), d(y, t_n)\}).\end{aligned} \tag{2.35}$$

This imples that

$$max\{d(x, z_{n+1}), d(y, t_{n+1})\} \leq max\{d(x, z_n), d(y, t_n)\},$$

and consequently the sequence $max\{d(x, z_{n+1}), d(y, t_{n+1})\}$ is decreasing and nonnegative and so,

$$\lim_{n\to\infty} max\{d(x, z_{n+1}), d(y, t_{n+1})\} = r, \tag{2.36}$$

for certain $r \geq 0$. Using (2.36) and letting $n \to \infty$ in (2.35), we have

$$\varphi(r) \leq \varphi(r) - \phi(r) \leq \varphi(r),$$

and consequently $\phi(r) = 0$ and thus $r = 0$.

Finally, as

$$\lim_{n\to\infty} max\{d(x, z_{n+1}), d(y, t_{n+1})\} = 0. \tag{2.37}$$

This implies

$$\lim_{n\to\infty} d(x, z_{n+1}) = \lim_{n\to\infty} d(y, t_{n+1}) = 0. \tag{2.38}$$

Similarly

$$\lim_{n\to\infty} d(x', z_{n+1}) = \lim_{n\to\infty} d(y', t_{n+1}) = 0. \tag{2.39}$$

From (2.38) and (2.39), we have $x = x', y = y'$. The proof is complete. □

Theorem 2.4. *In addition to the hypotheses of Theorem (2.1)(resp. Theorem (2.2)), suppose that x_0 and y_0 in X are comparable, then $x = y$.*

Proof. Suppose that $x_0 \leq y_0$. We claim that

$$x_n \leq y_n, \forall n \in N. \tag{2.40}$$

From the mixed monotone property of F, we have

$$x_1 = F(x_0, y_0) \leq F(y_0, y_0) \leq F(y_0, x_0) = y_1.$$

Assume that $x_n \leq y_n$, for some n. Now,

$$x_{n+1} = F(x_n, y_n) \leq F(y_n, y_n) \leq F(y_n, x_n) = y_{n+1}.$$

Hence, this proves our claim.

Now, using (2.40) and the contractive condition, we get

$$\begin{aligned}\varphi(d(x_{n+1},y_{n+1})) &= \varphi(d(y_{n+1},x_{n+1})) = \varphi(d(F(y_n,x_n),F(x_n,y_n)))\\ &\le \varphi(M((y_n,x_n),(x_n,y_n))) - \phi(M((y_n,x_n),(x_n,y_n))) \\ &\le \varphi(M((y_n,x_n),(x_n,y_n))),\end{aligned} \tag{2.41}$$

and as φ is nondecreasing,

$$d(x_{n+1},y_{n+1}) \le M((y_n,x_n),(x_n,y_n)),$$

where

$$\begin{aligned}M((y_n,x_n),F(x_n,y_n)) &= max\{d(y_n,x_n),d(x_n,y_n),d(F(y_n,x_n),y_n),d(F(x_n,y_n),x_n)\}\\ &= max\{d(y_n,x_n),d(y_{n+1},y_n),d(x_{n+1},x_n)\}.\end{aligned} \tag{2.42}$$

Thus, $\lim_{n\to\infty} d(x_n,y_n) = r$ for certain $r \ge 0$.

Taking $n \to \infty$ in (2.41)(and hence (2.42)), and using continuity of φ and ϕ, we have

$$\varphi(r) \le \varphi(r) - \phi(r) \le \varphi(r),$$

and this gives us $r = 0$.

As $x_n \to x$ and $y_n \to y$ and $\lim_{n\to\infty} d(x_n,y_n) = 0$. We have $0 = \lim_{n\to\infty} d(x_n,y_n) = d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = d(x,y)$ and thus $x = y$.

This finishes the proof. □

Example 2.5. Let $X = \mathbb{R}$ with usual metric and order. Define $F : X\times X \to X$ as $F(x,y) = \frac{1}{4}(x^2-3y^2)$ for all $x,y \in X$.

Let $\varphi,\phi : [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = t$ and $\phi(t) = \frac{1}{3}(t)$. Clearly, φ,ϕ are altering distance functions.

Now, let $x \le u$ and $y \ge v$. So, we obtain

$$\begin{aligned}\varphi(d(F(x,y),F(u,v))) &= d(F(x,y),F(u,v))\\ &= |\frac{1}{4}(x^2-3y^2) - \frac{1}{4}(u^2-3v^2)|\\ &= \frac{1}{4}|(x^2-u^2) - 3(y^2-v^2)|\\ &\le \frac{1}{4}[d(x,u) + 3d(y,v)]\\ &\le \frac{2}{3}max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\}\\ &= max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\}\\ &\quad - \frac{1}{3}max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\}\\ &= \varphi(max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\})\\ &\quad - \phi(max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\}).\end{aligned}$$

Hence, all of the conditions of Theorem (2.1) are satisfied. Moreover, $(0,0)$ is the coupled fixed point of F.

Corollary 2.6. *Let $(X,\le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $F : X\times X \to X$ is a continuous mapping on X having the mixed monotone property such that there exists $k \in [0,1)$ satisfying*

$$d(F(x,y),F(u,v)) \le k\ max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\}$$

$\forall\ x,y,u,v \in X$ with $x \ge u$ and $y \le v$. Suppose either F is continous or X has the following properties

(a) if a non-decreasing sequence $\{x_n\}$ *in* X *converges to some point* $x \in X$*, then* $x_n \leq x$*,* $\forall n$*,*

(b) if a non-increasing sequence $\{y_n\}$ *in* X *converges to some point* $y \in X$*, then* $y_n \geq y$*,* $\forall n$*.*

If there exists $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0, y_0)$ *and* $y_0 \geq F(y_0, x_0)$*, then* F *has a coupled fixed point.*

Proof. Applying Theorems (2.1) and (2.2) and taking as φ= identity and $\phi = (1-k)\varphi$, we obtain the corollary. □

Corollary 2.7. *Let* F *satisfy the contractive condition of Theorems* (2.1) *and* (2.2) *except that condition* (2.1) *is replaced by the following condition. There exists a positive Lebesgue- integrable function* μ *on* $\mathbb{R}_+$ *such that* $\int_0^\varepsilon \mu(t)dt > 0$*, for each* $\varepsilon > 0$ *and that then,* F *has a coupled fixed point.*

$$\int_0^{\varphi(d(F(x,y),F(u,v)))} \leq \int_0^{\varphi(M((x,y),(u,v)))} \mu(t)dt - \int_0^{\phi(M((x,y),(u,v)))} \mu(t)dt, \tag{2.43}$$

where

$$M((x,y),(u,v)) = max\{d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u)\}.$$

Proof. Consider the function $\varphi : [0,\infty) \to [0,\infty)$ defined by

$$\Gamma = \int_0^x \mu(t)dt.$$

is an altering distance function.

Then (2.43) becomes

$$\Gamma(\varphi(d(F(x,y),F(u,v)))) \leq \Gamma(\varphi(M((x,y),(u,v)))) - \Gamma(\phi(M((x,y),(u,v)))),$$

where

$$M((x,y),(u,v)) = max\{d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u)\}.$$

Taking $\varphi_1 = \Gamma o \varphi$, $\phi_1 = \Gamma o \phi$ and applying Theorems (2.1) and (2.2), we obtain the result. □

Corollary 2.8. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *is a continuous mapping on* X *having the mixed monotone property such that there exists* $k \in [0,1)$ *satisfying*

$$\int_0^{d(F(x,y),F(u,v))} \rho(t)dt \leq k \int_0^{max\{d(x,u),d(y,v),d(F(x,y),x),d(F(u,v),u)\}} \rho(t)dt$$

$\forall\, x, y, u, v \in X$ *with* $x \geq u$ *and* $y \leq v$*, where* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *is a Lebesgue- integrable mapping and satisfies that* $\int_0^\varepsilon \rho(t)dt > 0$*, for* $\varepsilon > 0$*. Suppose either* F *is continous or* X *has the following properties*

(a) if a non-decreasing sequence $\{x_n\}$ *in* X *converges to some point* $x \in X$*, then* $x_n \leq x$*,* $\forall n$*,*

(b) if a non-increasing sequence $\{y_n\}$ *in* X *converges to some point* $y \in X$*, then* $y_n \geq y$*,* $\forall n$*.*

If there exists $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0, y_0)$ *and* $y_0 \geq F(y_0, x_0)$*, then* F *has a coupled fixed point.*

Proof. It is easily proven that the function $\varphi : [0,\infty) \to [0,\infty)$ defined by

$$\varphi(t) = \int_0^t \rho(s)ds$$

is an altering distance function.

Applying Theorem (2.1) and (2.2) with the altering distance function φ defined above and $\phi = (1-k)\varphi$, we obtain the desired result. □

3 Application to integral equations

In this section we study the existence and uniqueness of solutions of a nonlinear integral equation using the results proved in Secton 2.

Consider the following integral equation:

$$x(t) = \int_0^1 (k_1(t,s) + k_2(t,s))(f(s,x(s)) + g(s,x(s)))ds + a(t), \quad t \in [0,1]. \tag{3.1}$$

We will analyze (3.1) under the following assumptions:

(i) $k_i : [0,1] \times [0,1] \to \mathbb{R}(i = 1,2)$ are continuous and $k_1(t,s) \geq 0$ and $k_2(t,s) \leq 0$.

(ii) $a \in C[0,1]$.

(iii) $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

(iv) There exist constants $\lambda, \mu > 0$ such that for all $x, y \in \mathbb{R}$ and $x \geq y$

$$0 \leq f(t,x) - f(t,y) \leq \lambda \sqrt{ln[(y-x)^2 + 1]}$$

and

$$-\mu \sqrt{ln[(y-x)^2 + 1]} \leq g(t,x) - g(t,y) \leq 0.$$

(v) There exist $\alpha, \beta \in C[0,1]$ such that

$$\begin{aligned} \alpha(t) &\leq \int_0^1 k_1(t,s)(f(s,\alpha(s)) + g(s,\beta(s)))ds + \int_0^1 k_2(t,s)(f(s,\beta(s)) + g(s,\alpha(s)))ds + a(t) \\ &\leq \int_0^1 k_1(t,s)(f(s,\beta(s)) + g(s,\alpha(s)))ds + \int_0^1 k_2(t,s)(f(s,\alpha(s))) + g(s,\beta(s)))ds + a(t) \leq \beta(t). \end{aligned}$$

(vi) $2.max(\lambda, \mu) \parallel k_1 - k_2 \parallel_\infty \leq 1$, where

$$\parallel k_1 - k_2 \parallel_\infty = sup\{(k_1(t,s) - k_2(t,s)) : t, s \in [0,1]\}.$$

Previously, we considered the space $X = C[0,1]$ of continuous functions defined on [0, 1] with the standard metric given by

$$d(x,y) = \sup_{t \in [0,1]} \mid x(t) - y(t) \mid, \text{for } \; x, y \in C[0,1].$$

This space can also be equipped with a partial order given by

$$x, y \in C[0,1], x \leq y \iff x(t) \leq y(t), \quad \text{for any } \; t \in [0,1].$$

Clearly, if in $X \times X$ we consider the order given by

$$(x,y), (u,v) \in X \times X, (x,y) \leq (u,v) \iff x \leq u \text{ and } \; y \geq v,$$

and since for any $x, y \in X$ we have that $max(x,y), min(x,y) \in X$, condition (2.31) is satisfied.

Moreover, in [11] it is proved that $(C[0,1], \leq)$ satisfies assumption (1).

Now, we formulate our result.

Theorem 3.1. *Under assumptions (i)-(vi), eq.* (3.1) *has a unique solution in* $C[0,1]$.

Proof. We consider the operator $F : X \times X \to X$ defined by

$$\begin{aligned} F(x,y)(t) = &\int_0^1 k_1(t,s)(f(s,x(s)) + g(s,y(s)))ds \\ &+ \int_0^1 k_2(t,s)(f(s,y(s)) + g(s,x(s)))ds + a(t), \quad \text{for } \; t \in [0,1]. \end{aligned}$$

By virtue of our assumptions, F is well defined (this means that for $x, y \in X$ then $F(x, y) \in X$).

Firstly, we prove that F has the mixed monotone property.

In fact, for $x_1 \leq x_2$ and $t \in [0, 1]$, we have

$$\begin{aligned} F(x_1, y)(t) - F(x_2, y)(t) &= \int_0^1 k_1(t,s)(f(s,x_1(s)) + g(s,y(s)))ds \\ &+ \int_0^1 k_2(t,s)(f(s,y(s)) + g(s,x_1(s)))ds + a(t) \\ &- \int_0^1 k_1(t,s)(f(s,x_2(s)) + g(s,y(s)))ds \\ &- \int_0^1 k_2(t,s)(f(s,y(s)) + g(s,x_2(s)))ds - a(t) \\ &= \int_0^1 k_1(t,s)(f(s,x_1(s)) - f(s,x_2(s)))ds \\ &+ \int_0^1 k_2(t,s)(g(s,x_1(s)) - g(s,x_2(s)))ds. \end{aligned} \tag{3.2}$$

Taking into account that $x_1 \leq x_2$ and our assumptions,

$$\begin{aligned} f(s,x_1(s)) - f(s,x_2(s)) \leq 0, \\ g(s,x_1(s)) - g(s,x_2(s)) \geq 0, \end{aligned}$$

and from (3.2) we obtain

$$F(x_1, y)(t) - F(x_2, y)(t) \leq 0$$

and this proves that $F(x_1, y) \leq F(x_2, y)$.

Similarly, if $y_1 \geq y_2$ and $t \in [0, 1]$, we have

$$\begin{aligned} F(x, y_1)(t) - F(x, y_2)(t) &= \int_0^1 k_1(t,s)(f(s,x(s)) + g(s,y_1(s)))ds \\ &+ \int_0^1 k_2(t,s)(f(s,y_1(s)) + g(s,x(s)))ds + a(t) \\ &- \int_0^1 k_1(t,s)(f(s,x(s)) + g(s,y_2(s)))ds \\ &- \int_0^1 k_2(t,s)(f(s,y_2(s)) + g(s,x(s)))ds - a(t) \\ &= \int_0^1 k_1(t,s)(g(s,y_1(s)) - g(s,y_2(s)))ds \\ &+ \int_0^1 k_2(t,s)(f(s,y_1(s)) - f(s,y_2(s)))ds, \end{aligned}$$

and by our assumptions, as $y_1 \geq y_2$,

$$\begin{aligned} g(s,y_1(s)) - g(s,y_2(s)) \leq 0, \\ f(s,y_1(s)) - f(s,y_2(s)) \geq 0, \end{aligned}$$

and thus,

$$F(x, y_1)(t) - F(x, y_2)(t) \leq 0,$$

or, equivalently,

$$F(x, y_1) \leq F(x, y_2).$$

Therefore, F has mixed monotone property.

In what follows, we estimate $d(F(x,y),F(u,v))$ for $x \geq u$ and $y \leq v$.

Indeed, as F has the mixed monotone property, $F(x,y) \geq F(u,v)$ and we can obtain

$$\begin{aligned}
d(F(x,y),F(u,v)) &= \sup_{t\in[0,1]} \mid F(x,y)(t) - F(u,v)(t) \mid \\
&= \sup_{t\in[0,1]} (F(x,y)(t) - F(u,v)(t)) \\
&= \sup_{t\in[0,1]} \Big[\int_0^1 k_1(t,s)(f(s,x(s)) + g(s,y(s)))ds \\
&\quad + \int_0^1 k_2(t,s)(f(s,y(s)) + g(s,x(s)))ds + a(t) \\
&\quad - \int_0^1 k_1(t,s)(f(s,u(s)) + g(s,v(s)))ds \\
&\quad - \int_0^1 k_2(t,s)(f(s,v(s)) + g(s,u(s)))ds - a(t)\Big] \\
&= \sup_{t\in[0,1]} \Big[\int_0^1 k_1(t,s)[(f(s,x(s)) - f(s,u(s))) - (g(s,v(s)) - g(s,y(s)))] \\
&- \int_0^1 k_2(t,s)[(f(s,v(s)) - f(s,y(s))) - (g(s,x(s)) - g(s,u(s)))]ds\Big].
\end{aligned} \tag{3.3}$$

By our assumptions(notice that $x \geq u$ and $y \leq v$)

$$\begin{aligned}
f(s,x(s)) - f(s,u(s)) &\leq \lambda\sqrt{ln[(x(s)-u(s))^2+1]} \\
g(s,v(s)) - g(s,y(s)) &\geq -\mu\sqrt{ln[(y(s)-v(s))^2+1]} \\
f(s,v(s)) - f(s,y(s)) &\leq \lambda\sqrt{ln[(v(s)-y(s))^2+1]} \\
g(s,x(s)) - g(s,u(s)) &\geq -\mu\sqrt{ln[(x(s)-u(s))^2+1]}.
\end{aligned}$$

Taking into account these last inequalities, $k_2 \leq 0$ and (3.3), we get

$$\begin{aligned}
d(F(x,y),F(u,v)) &\leq \sup_{t\in[0,1]} \Big[\int_0^1 k_1(t,s)[\lambda\sqrt{ln[(x(s)-u(s))^2+1]} + \mu\sqrt{ln[(y(s)-v(s))^2+1]}]ds \\
&\quad + \int_0^1 (-k_2(t,s))[\lambda\sqrt{ln[(v(s)-y(s))^2+1]} + \mu\sqrt{ln[(x(s)-u(s))^2+1]}]ds\Big] \\
&= max(\lambda,\mu) \sup_{t\in[0,1]} [\int_0^1 (k_1(t,s) - k_2(t,s))\sqrt{ln[(x(s)-u(s))^2+1]}ds \\
&\quad + \int_0^1 (k_1(t,s) - k_2(t,s))\sqrt{ln[(y(s)-v(s))^2+1]}ds\Big].
\end{aligned} \tag{3.4}$$

Defining

$\mathrm{I} = \int_0^1 (k_1(t,s) - k_2(t,s))\sqrt{ln[(x(s)-u(s))^2+1]}ds$

$\mathrm{II} = \int_0^1 (k_1(t,s) - k_2(t,s))\sqrt{ln[(y(s)-v(s))^2+1]}ds$

and using the Cauchy - Schwartz inequality in (I) we obtain

$$\begin{aligned} (I) &\leq \Big(\int_0^1 (k_1(t,s) - k_2(t,s))^2 ds \Big)^{\frac{1}{2}} . \Big(\int_0^1 ln[(x(s) - u(s))^2 + 1] ds \Big)^{\frac{1}{2}} \\ &\leq \| k_1 - k_2 \|_\infty .(ln \| x - u \|^2 + 1))^{\frac{1}{2}} = \| k_1 - k_2 \|_\infty .(ln(d(x,u)^2 + 1))^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

Similarly, we can obtain the following estimate for (II):

$$(II) \leq \| k_1 - k_2 \|_\infty .(ln(d(y,v)^2 + 1))^{\frac{1}{2}}. \tag{3.6}$$

from (3.4)- (3.6), we have

$$\begin{aligned} d(F(x,y),F(u,v)) &\leq max(\lambda,\mu) \| k_1 - k_2 \|_\infty [(ln(d(x,u)^2 + 1))^{\frac{1}{2}} + (ln(d(y,v)^2 + 1))^{\frac{1}{2}}] \\ &\leq max(\lambda,\mu) \| k_1 - k_2 \|_\infty [(ln(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 + 1))^{\frac{1}{2}} \\ &\qquad + (ln(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 + 1))^{\frac{1}{2}}] \\ &= 2max(\lambda,\mu) \| k_1 - k_2 \|_\infty [(ln(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 + 1))^{\frac{1}{2}}]. \end{aligned}$$

The last inequality and assumption (vi) give us

$$d(F(x,y),F(u,v)) \leq (ln(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 + 1))^{\frac{1}{2}},$$

and this implies

$$d(F(x,y),F(u,v))^2 \leq (ln(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 + 1)),$$

or, equivalently,

$$\begin{aligned} d(F(x,y),F(u,v))^2 \leq\ & (max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 \\ & - [(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 \\ & - ln(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))^2 + 1))]. \end{aligned} \tag{3.7}$$

Put $\varphi(x) = x^2$ and $\phi(x) = x^2 - ln(x^2 + 1)$. Obviously, φ and ϕ are altering distance functions and from (3.7) we get

$$\begin{aligned} \varphi(d(F(x,y),F(u,v)) \leq\ & \varphi(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u)) \\ & - \phi(max(d(x,u), d(y,v), d(F(x,y),x), d(F(u,v),u))) \end{aligned}$$

This proves that the operator F satisfies the contractive condition appearing in Theorem (2.1).

Finally, let α, β be the functions appearing in assumption (v); then, by (v), we get

$$\alpha \leq F(\alpha,\beta) \leq F(\beta,\alpha) \leq \beta.$$

Theorem (2.3) gives us that F has a unique coupled fixed point $(x,y) \in X \times X$. Since $\alpha \leq \beta$, Theorem (2.4) says us that $x = y$ and this implies $x = F(x,x)$ and x is the unique solution of eq. (3.1).

This finishes the proof. □

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