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Some properties of the complement of the annihilator graph of a commutative reduced ring

S. Visweswaran and Patat Sarman

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Abstract. Let R be a commutative ring with identity which is not an integral domain. Let $Z(R)$ denote the set of all zero-divisors of R and let $Z(R)^* = Z(R)\backslash\{0\}$. Recall from [5] that the annihilator graph of R, denoted by $AG(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices x, y are joined by an edge in this graph if and only if $ann_R(xy) \neq ann_R x \cup ann_R y$. Let R be a reduced ring. First, in this article, we determine when the complement of the annihilator graph of R is connected and determine its diameter and radius when it is connected. Second, in this article, we determine the girth of the complement of the annihilator graph of R. Finally, we determine rings R such that the clique number of the complement of the annihilator graph of R is finite and also determine a formula for its clique number.

1 Introduction

The rings considered in this note are commutative with identity and which are not integral domains. The idea of associating the elements of a ring with a graph and investigating the interplay between the ring structure and the graph properties of the graph associated with it was introduced by I. Beck in [7]. In [7], Beck was mainly interested in colorings. His research work inspired a lot of work on zero-divisor graphs of rings. For a commutative ring R with identity, we denote the set of all zero-divisors of R by $Z(R)$ and we denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. Recall from [2] that for a commutative ring R with identity, the *zero-divisor graph* of R, denoted by $\Gamma(R)$ is an undirected simple graph whose vertex set is $Z(R)^*$ and distinct vertices x, y are joined by an edge in this graph if and only if $xy = 0$. A lot of research articles authored by several eminent researchers appeared in reputed international journals in Mathematics, especially in Algebra on the zero-divisor graphs of rings and the graphs associated with other algebraic structures. For an excellent and inspiring survey on the work done in the area of zero-divisor graphs of commutative rings, the reader is referred to [3]. The concept of the *annihilator graph* of a commutative ring was introduced by Ayman Badawi in [5] and several interesting interplay between the ring-theoretic properties of a ring R and graph-theoretic properties of its annihilator graph has been well investigated in [5]. Let R be a ring. For an element $a \in R$, we denote the *annihilator* of a in R (that is, $\{r \in R | ar = 0\}$) by $ann_R a$. Recall from [5] that the annihilator graph of R, denoted by $AG(R)$, is an undirected simple graph whose vertex set is $Z(R)^*$ and distinct vertices x, y are joined by an edge in this graph if and only if $ann_R(xy) \neq ann_R x \cup ann_R y$. Let $G = (V, E)$ be a simple graph. Recall from [6] that the *complement* of G, denoted by G^c is a graph whose vertex set is V and distinct elements $x, y \in V$ are joined by an edge in G^c if and only if there is no edge joining x and y in G. The aim of this article is to investigate the effect on the ring structure of R by the graph-theoretic properties of $(AG(R))^c$ (where $(AG(R))^c$ denotes the complement of $AG(R)$). Let R be a ring. It was noted in [5] that $\Gamma(R)$ is a subgraph of $AG(R)$. Hence, it follows that $(AG(R))^c$ is a subgraph of $(\Gamma(R))^c$. In [15], some results on $(\Gamma(R))^c$ were proved and they illustrate the interplay between the ring-theoretic properties of R and the graph-theoretic properties of $(\Gamma(R))^c$. In this note, we investigate on some properties of $(AG(R))^c$, a spanning subgraph of $(\Gamma(R))^c$. It is useful to note that if $x, y \in Z(R)^*$, then

$x - y$ is an edge of $(AG(R))^c$ if and only if $ann_R(xy) = ann_R x \cup ann_R y$ if and only if either $ann_R(xy) = ann_R x$ or $ann_R(xy) = ann_R y$. For a ring R, we denote the *nilradical* of R by $nil(R)$. Recall that a ring R is said to be *reduced* if $nil(R) = (0)$. In this article, we focus our study on $(AG(R))^c$, where R is a reduced ring.

First we recall the following definitions from commutative ring theory which are used in this article. Let R be a ring and I be a proper ideal of R. Recall from [12] that a prime ideal P of R is said to be a *maximal N-prime* of I if P is maximal with respect to the property of being contained in $Z_R(R/I) = \{r \in R : rx \in I \text{ for some } x \in R\backslash I\}$. Thus a prime ideal P of R is a maximal N-prime of (0) if P is maximal with respect to the property of being contained in $Z(R)$. Let $x \in Z(R)$. Then the multiplicatively closed set $S = R\backslash Z(R)$ is such that $Rx \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [13, Theorem 1] that there exists a maximal N-prime P of (0) in R such that $x \in P$. Therefore, we get that $Z(R) = \cup_{\alpha\in\Lambda} P_\alpha$, where $\{P_\alpha\}_{\alpha\in\Lambda}$ is the set of all maximal N-primes of (0) in R.

Let I be a proper ideal of a ring R. Recall from [11] that a prime ideal P of R is said to be a *Bourbaki prime* of I if $P = (I :_R x)$ for some $x \in R$. In such a case, we simply say that P is a B-prime of I.

Before we give a brief account of the results proved in this article, it is useful to recall the following definitions from graph theory. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V, a \neq b$. If there exists a path in G between a and b, then $d(a, b)$ is defined as the length of a shortest path in G between a and b. If there exists no path in G between a and b, then we define $d(a, b) = \infty$. We define $d(a, a) = 0$. Let $G = (V, E)$ be a connected graph. Recall from [6, Definition 4.2.1] that the *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) : a, b \in V\}$. Let $a \in V$. The *eccentricity* of a, denoted by $e(a)$, is defined as $e(a) = sup\{d(a, b) : b \in V\}$. The *radius* of G, denoted by $r(G)$, is defined as $r(G) = min\{e(a) : a \in V\}$.

Let $G = (V, E)$ be a graph. Suppose that G contains a cycle. Recall from [6, p.159] that the *girth* of G, denoted by $girth(G)$ is the length of a shortest cycle in G. If G does not contain any cycle, then we set $girth(G) = \infty$. Recall from [6, Definition 1.2.2] that a *clique* of G is a complete subgraph of G. For a simple graph G, the *clique number* of G, denoted by $\omega(G)$ is defined as the largest positive integer $n \geq 1$ such that G contains a clique on n vertices. If G contains a clique on n vertices for all $n \geq 1$, then we set $\omega(G) = \infty$.

Let $G = (V, E)$ be a graph. Recall from [6, p.129] that a *vertex coloring* of G is a mapping $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper*, if adjacent vertices of G receive distinct colors of S; that is, if u and v are adjacent in G, then $f(u) \neq f(v)$. The *chromatic number* of G, denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G. It is clear that for any graph G, $\omega(G) \leq \chi(G)$.

Let R be a commutative ring with identity which is not an integral domain. Several interesting theorems on $AG(R)$ were proved in [5] by A. Badawi. It was shown in [5, Theorem 2.2] that $AG(R)$ is connected and $diam(AG(R)) \leq 2$. Moreover, it was proved in [5, Theorem 2.9 and Corollary 2.11] that if $AG(R)$ contains a cycle, then $girth(AG(R)) \leq 4$. Furthermore, it was shown that for a reduced ring R, $AG(R) = \Gamma(R)$ if and only if R has exactly two minimal prime ideals [5, Theorem 3.6]. The authors of [14] investigated on the coloring of $AG(R)$. In [14], they studied classes of rings R such that $\omega(AG(R)) = \chi(AG(R))$. An explicit formula was given for $\omega(AG(R))$ for certain classes of rings. The annihilator graph of a semigroup was studied in [1]. Motivated by the above mentioned works, in this article, we try to study the interplay between the graph-theoretic properties of $(AG(R))^c$ and the ring-theoretic properties of R, where R is a reduced ring.

In Section 2, we consider reduced rings R such that R has exactly one maximal N-prime of (0). It is shown in Corollary 2.7 that $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) = r((AG(R))^c) = 2$. It is proved in Lemma 2.10 that $(AG(R))^c$ contains an infinite clique and hence, $girth((AG(R))^c) = 3$. It is illustrated in Example 2.9 that the result regarding $diam((AG(R))^c)$ for reduced rings may fail to hold for a nonreduced ring.

In Section 3, we consider reduced rings R such that R has at least two maximal N-primes of (0). It is shown in Proposition 3.1 that $(AG(R))^c$ is connected if and only if $P_1 \cap P_2 \neq (0)$ for any two maximal N-primes P_1, P_2 of (0) in R. Moreover, it is proved in Proposition 3.1 that if $(AG(R))^c$ is connected, then $2 \leq diam((AG(R))^c) \leq 3$. Furthermore, it is proved in Proposition 3.2 that $diam((AG(R))^c) = 3$ if and only if $Tot(R)$ contains a nontrivial idempotent, where $Tot(R)$ denotes the total quotient ring of R. In Example 3.6, an example of a reduced ring T which admits exactly two maximal N-primes of (0) is provided which satisfies $2 = r((AG(T))^c) < diam((AG(T))^c) = 3$. It is proved in Proposition 3.8 that for a von Neumann regular ring R with at least three prime ideals, $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) = r((AG(R))^c) = 3$. In Theorem 3.19, it is shown that for any reduced ring R which admits at least two maximal N-primes of (0), $girth((AG(R))^c) \in \{3, 6, \infty\}$. Moreover, Theorem 3.19 classifies rings R such that $girth((AG(R))^c) = \infty$.

In this article, we also consider the problem of characterizing reduced rings R such that $\omega((AG(R))^c) < \infty$. It is shown in Lemma 2.10 that such a ring must admit at least two maximal N-primes of (0). Let R be a reduced ring which admits at least two maximal N-primes of (0). It is proved in Theorem 3.12 that $\omega((AG(R))^c) < \infty$ if and only if there exist finite fields $F_1, F_2, \ldots, F_n (n \geq 2)$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings. For such a ring R, it is shown in Proposition 3.21 that $\omega((AG(R))^c) = \chi((AG(R))^c)$.

A ring R is said to be *quasilocal* if R has only one maximal ideal. A Noetherian quasilocal ring is referred to as a local ring. Whenever a set A is a subset of a set B and $A \neq B$, then we denote it using the notation $A \subset B$. The Krull dimension of a ring is simply denoted by $dimR$. We denote the cardinality of a set A using the notation $|A|$.

2 The case where R has exactly one maximal N-prime of (0)

Let R be a ring which has only one maximal N-prime of (0) (that is, equivalently $Z(R)$ is an ideal of R). With this assumption, we study some graph-theoretic properties of $(AG(R))^c$. As mentioned in the introduction, unless otherwise specified, the rings considered in this article are not integral domains.

Lemma 2.1. *Let R be a ring such that R admits P as its unique maximal N-prime of (0). If $|Z(R)^*| \geq 2$ and P is a B-prime of (0) in R, then $(AG(R))^c$ is not connected.*

Proof. By hypothesis, there exists $x \in R$ such that $P = ((0) :_R x)$. Observe that $x \in Z(R)^*$ and for any $y \in P, y \neq x$, there exists no path in $(\Gamma(R))^c$ between x and y. Since $(AG(R))^c$ is a subgraph of $(\Gamma(R))^c$, it follows that there exists no path in $(AG(R))^c$ between x and y. This proves that $(AG(R))^c$ is not connected. □

Let R be a ring such that R admits P as its unique maximal N-prime of (0). We investigate some properties of $(AG(R))^c$ under the assumption that P is not a B-prime of (0) in R. (Note that P is not a B-prime of (0) in R if R is reduced.)

Remark 2.2. Let R be a ring which admits P as its unique maximal N-prime of (0). Assume that P is not a B-prime of (0) in R. Then it is known that $(\Gamma(R))^c$ is connected and moreover, $diam((\Gamma(R))^c) = 2$ [16, Proposition 1.2]. We are not able to determine whether or not $(AG(R))^c$ is connected. However, we have some partial answers which we now proceed to present.

Lemma 2.3. *Let R be a ring which admits P as its unique maximal N-prime of (0). If $P = nil(R)$, then $(AG(R))^c$ admits no edges.*

Proof. Let $x, y \in P\backslash\{0\}$ be such that $x \neq y$. Since $x, y \in nil(R)$, we can choose $n, m \in \mathbb{N}$ least with the property that $x^n y = 0$ and $xy^m = 0$. Observe that $x^{n-1} \in ann_R(xy)$ but $x^{n-1} \notin ann_R y$ and $y^{m-1} \in ann_R(xy)$ but $y^{m-1} \notin ann_R x$. Hence, $ann_R(xy) \neq ann_R x$ and $ann_R(xy) \neq ann_R y$. Thus as noted in the introduction, it follows that there is no edge of $(AG(R))^c$ joining x and y. This shows that $(AG(R))^c$ has no edges. □

Example 2.4. Let (V, M) be a rank 1 valuation domain which is not discrete. Let $m \in M, m \neq 0$. Let $R = V/mV$. Then $(AG(R))^c$ has no edges.

Proof. It was verified in [16, Example 3.1] that $P = M/mV$ is the only maximal N-prime of the zero ideal in R and moreover, P is not a B-prime of (0) in R. Since P is the only prime ideal of R, it follows that $P = nil(R)$. Hence, it follows from Lemma 2.3 that $(AG(R))^c$ has no edges. □

Let R be a reduced ring with P as its unique maximal N-prime of (0). We prove in Corollary 2.7 that $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) = 2$. We use Proposition 2.5 in the proof of Lemma 2.6, Corollary 2.7 and some other results of this article.

Proposition 2.5. *Let T be a reduced ring which may admit any number of maximal N-primes of (0). Let $a, b \in Z(T)^*, a \neq b$. Then the following statements hold.*

(i) If $ab \neq 0$, then there is a path of length at most two between a and b in $(AG(T))^c$.

(ii) If $ab = 0$ and if $a + b \in Z(T)$, then there is a path of length at most two between a and b in $(AG(T))^c$.

(iii) If the intersection of any two maximal N-primes of (0) in T is nonzero, then there is a path of length at most three between a and b in $(AG(T))^c$.

Proof. (i) We can assume that there is no edge of $(AG(T))^c$ between a and b. Hence, $ann_T(ab) \notin \{ann_T a, ann_T b\}$ and so, $ab \notin \{a, b\}$. Since T is reduced, it follows that $ann_T(a^2b) = ann_T(ab) = ann_T(ab^2)$. Therefore, we obtain that $a - ab - b$ is a path of length two between a and b in $(AG(T))^c$.
(ii) From $ab = 0$ and $a^2 \neq 0$, it follows that $a + b \neq 0$. It is clear that $a + b \notin \{a, b\}$. Note that $(a + b)a = a^2$ and $(a + b)b = b^2$. Moreover, as T is reduced, we obtain that for any $x \in T\backslash\{0\}$, $ann_T x = ann_T x^2$. Therefore, $a-(a+b)-b$ is a path of length two between a and b in $(AG(T))^c$.
(iii) In view of (i) and (ii), we can assume that $ab = 0$ and $a+b \notin Z(T)$. Let P_1, P_2 be maximal N-primes of (0) in T such that $a \in P_1$ and $b \in P_2$. Since $a + b \notin Z(T)$, we obtain that $a \notin P_2$ and $b \notin P_1$. By hypothesis, $P_1 \cap P_2 \neq (0)$. Let $x \in P_1 \cap P_2, x \neq 0$. It follows from $a+b \notin Z(T)$ that either $ax \neq 0$ or $bx \neq 0$. Without loss of generality, we can assume that $ax \neq 0$. Observe that $ann_T(a^2x) = ann_T(ax) = ann_T(a^2x^2)$, and $ax(ax + b) = a^2x^2$, $(ax + b)b = b^2$, and $ann_T b = ann_T b^2$. Hence, it follows that $a - ax - (ax + b) - b$ is a path of length three between a and b in $(AG(T))^c$. □

Lemma 2.6. *Let T be a reduced ring such that $P_1 \cap P_2 \neq (0)$ for any two maximal N-primes P_1, P_2 of (0) in T. Let $a, b \in Z(T)^*$ be such that $ab = 0$ and $a + b \notin Z(T)$. Then $d(a, b) = 3$ in $(AG(T))^c$.*

Proof. We know from the proof of Proposition 2.5(iii) that there exists a path of length at most three between a and b in $(AG(T))^c$. Since $ab = 0$, it is clear that a and b are not adjacent in $(AG(T))^c$. We assert that there exists no path of length two between a and b in $(AG(T))^c$. Suppose that $a - c - b$ is a path of length two between a and b in $(AG(T))^c$. Then $ac \neq 0$ and $bc \neq 0$. As $a - c$ is an edge of $(AG(T))^c$, either $ann_T(ac) = ann_T a$ or $ann_T(ac) = ann_T c$. Note that $b(ac) = 0$ but $bc \neq 0$. Hence, $ann_T(ac) = ann_T a$. Similarly, it follows from $b - c$ is an edge of $(AG(T))^c$, $a(bc) = 0$ but $ac \neq 0$ that $ann_T(bc) = ann_T b$. Since $c \in Z(T)$, there exists $d \in T\backslash\{0\}$ that $cd = 0$. Hence, $(ac)d = (bc)d = 0$. This implies that $ad = bd = 0$. This is impossible since $a + b \notin Z(T)$. Thus there exists no path of length two between a and b in $(AG(T))^c$. This shows that $d(a, b) \geq 3$ in $(AG(T))^c$ and so, $d(a, b) = 3$ in $(AG(T))^c$. □

Corollary 2.7. *Let R be a reduced ring with P as its unique maximal N-prime of (0). Then $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) = r((AG(R))^c) = 2$.*

Proof. Note that $Z(R) = P$. Thus for any $a, b \in Z(R)$, $a + b \in Z(R)$. Let $a, b \in P\backslash\{0\}$ with $a \neq b$. It follows from (i) and (ii) of Proposition 2.5 that there exists a path of length at most two between a and b in $(AG(R))^c$. This proves that $(AG(R))^c$ is connected and $diam((AG(R))^c) \leq 2$. We next verify that $e(x) = 2$ for any $x \in Z(R)^*$. Note that there exists $y \in R\backslash\{0\}$ such that $xy = 0$. As R is reduced, it follows that $x \neq y$. Thus x and y are not adjacent in $(AG(R))^c$. Hence, $d(x, y) = 2$ in $(AG(R))^c$. Since $diam((AG(R))^c) \leq 2$, we obtain that $e(x) = 2$. Therefore, $diam((AG(R))^c) = r((AG(R))^c) = 2$. □

In Example 2.8, we mention an example from [10, Example p.16] which illustrates Corollary 2.7.

Example 2.8. Let K be a field and $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over K. Let $D = \cup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$. Let I be the ideal of D generated by $\{X_iX_j : i, j \in \mathbb{N}, i \neq j\}$. Let $R = D/I$. Then the following statements hold.

(i) R has exactly one maximal N-prime of its zero ideal.
(ii) R is reduced, $(AG(R))^c$ is connected, and $diam((AG(R))^c) = r((AG(R))^c) = 2$.

Proof. (i) For each $i \in \mathbb{N}$, let us denote $X_i + I$ by x_i. Let M denote the ideal of R generated by $\{x_i : i \in \mathbb{N}\}$. It was noted in [10, Example, p.16] that R is quasilocal with M as its unique maximal ideal. Moreover, it was verified in [16, Example 3.4(i)] that R has M as its unique maximal N-prime of its zero ideal.

(ii) Let $i \in \mathbb{N}$. Let P_i denote the ideal of R generated by $\{x_j : j \in \mathbb{N}, j \neq i\}$. It was mentioned in [10, Example, p.16] that $\{P_i : i \in \mathbb{N}\}$ is the set of all minimal prime ideals of R and moreover, $\cap_{i=1}^{\infty} P_i = (0)$. Hence, R is reduced. It now follows from Corollary 2.7 that $(AG(R))^c$ is connected and $diam((AG(R))^c) = r((AG(R))^c) = 2$. □

In Example 2.9, we present an example of a quasilocal ring (R, P) such that P is the unique maximal N-prime of (0) in R, $(AG(R))^c$ is connected, and moreover, $diam((AG(R))^c) = r((AG(R))^c) = 3$. Example 2.9 presented below is from [13, Exercises 6 and 7, pp. 62-63]. Moreover, Example 2.9 illustrates that Corollary 2.7 may fail to hold for a ring which is not reduced.

Example 2.9. Let $S = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $M = SX + SY$. Let $T = S_M$. Let $W = \oplus(T/Tp)$ be the direct sum of the T-modules T/Tp, where p varies over all the nonassociate prime elements of T. Let $R = T \oplus W$ be the ring obtained on using Nagata's principle of idealization. Then the following statements hold.
(i) $P = MT \oplus W$ is the unque maximal ideal of R and moreover, P is the only maximal N-prime of the zero ideal in R.
(ii) $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) = r((AG(R))^c) = 3$.

Proof. Since T is local with MT as its unique maximal ideal, it follows that $P = MT \oplus W$ is the unique maximal ideal of R. Moreover, note that T is a unique factorization domain with an infinite number of nonassociate prime elements. We know from the verification of [17, Example 2.8] that P is the unique maximal N-prime of the zero ideal in R.

(ii) For convenience, let us denote the set of all nonassociate prime elements of T by $\mathbf{P}$. Note that $W = \oplus_{p \in \mathbf{P}}(T/Tp)$. Let $x = (a, w) \in P \backslash \{(0, 0)\}$. Suppose that $a \neq 0$. Let $\mathbf{A}$ denote the set of all $p \in \mathbf{P}$ such that p divides a in T. Then it is easy to verify that $ann_R x = (0) \oplus N$, where $N = \oplus_{p \in \mathbf{A}}(T/Tp)$. Suppose that $a = 0$. Then $w \neq 0$ and $x = (0, w)$. Let $\mathbf{B}$ denote the set of all $p \in \mathbf{P}$ such that the component of w corresponding to p is nonzero. It is clear that $\mathbf{B}$ is a finite nonempty subset of $\mathbf{P}$. Let us denote the ideal $\prod_{p \in \mathbf{B}} Tp$ by I. It is not hard to show that $ann_R x = I \oplus W$.

Let $x, y \in P \backslash \{(0, 0)\}$ with $x \neq y$. Let $x = (a, w)$ and $y = (b, w')$ for some $a, b \in MT$ and $w, w' \in W$. We now show that there exists a path of length at most three between x and y in $(AG(R))^c$. We can assume that x and y are not adjacent in $(AG(R))^c$. We consider the following cases.

Case(I). $a \neq 0$ and $b \neq 0$

Let $z = (ab, 0)$. Then it is clear that $z \in P, z \neq (0, 0)$. Observe that $zx = (a^2 b, abw)$ and $yz = (ab^2, abw')$. Let $\mathbf{A_1}$ denote the set of all $p \in \mathbf{P}$ such that p divides a in T and let $\mathbf{A_2}$ denote the set of all $p \in \mathbf{P}$ such that p divides b in T. Note that $ann_R(xz) = ann_R z = ann_R(yz) = (0) \oplus N$, where $N = \oplus_{p \in \mathbf{A_1} \cup \mathbf{A_2}}(T/Tp)$. Therefore, $x - z - y$ is a path of length two between x and y in $(AG(R))^c$.

Case(II). $a = b = 0$

Note that w, w' are nonzero elements of W. Let $\mathbf{B_1}$ denote the set of all $p \in \mathbf{P}$ such that the component of w corresponding to p is nonzero and let $\mathbf{B_2}$ denote the set of all $p \in \mathbf{P}$ such that the component of w' corresponding to p is nonzero. Observe that $\mathbf{B_1}$ and $\mathbf{B_2}$ are finite and nonempty subsets of $\mathbf{P}$. Since $\mathbf{P}$ is infinite, there exists $p \in \mathbf{P}$ such that $p \notin \mathbf{B_1} \cup \mathbf{B_2}$. Let $z = (p, 0)$. Note that $xz = (0, pw)$ and $yz = (0, pw')$. Observe that $ann_R(xz) = ann_R x = I_1 \oplus W$, where $I_1 = \prod_{p \in \mathbf{B_1}} Tp$. and $ann_R(yz) = ann_R y = I_2 \oplus W$, where $I_2 = \prod_{p \in \mathbf{B_2}} Tp$. Hence, we obtain that $x - z - y$ is a path of length two between x and y in $(AG(R))^c$.

Case(III). $a \neq 0$ but $b = 0$

Let $p \in \mathbf{P}$ be such that p does not divide a in T. Let $w'' \in W$ be such that the component of w'' corresponding to p is $1 + Tp$ and for any $q \in \mathbf{P}$, $q \neq p$, the component of w'' corresponding to q is $0 + Tq$. Let $z = (0, w'')$. Observe that $xz = (0, aw'')$. Note that $ann_R(xz) = ann_R z = Tp \oplus W$. Hence, x and z are adjacent in $(AG(R))^c$. We know from **Case(II)** that there exists a path of length at most two between $z = (0, w'')$ and $y = (0, w')$ in $(AG(R))^c$. This shows that there exists a path of length at most three between x and y in $(AG(R))^c$.

This proves that $(AG(R))^c$ is connected and $diam((AG(R))^c) \leq 3$. We next verify that $e(x) = 3$ in $(AG(R))^c$ for each $x \in P$, $x \neq (0, 0)$. Note that $x = (a, w)$ for some $a \in MT$ and $w \in W$. We consider two cases. Suppose that $a \neq 0$. Let $\mathbf{A}$ denote the set of all $p \in \mathbf{P}$ such that p divides a in T. It is clear that $\mathbf{A}$ is a finite nonempty subset of $\mathbf{P}$. Let $w' \in W$ be defined as follows: the component of w' corresponding to p equals any nonzero element of T/Tp for each $p \in \mathbf{A}$, whereas, for any $q \in \mathbf{P}$, $q \notin \mathbf{A}$, the component of w' corresponding to q equals $0 + Tq$. Let $y = (0, w')$. Observe that $xy = (0, 0)$. and so, x and y are not adjacent in $(AG(R))^c$. We assert that there exists no path of length two between x and y in $(AG(R))^c$. Suppose that there exists $z \in P\backslash\{(0, 0)\}$ such that $x - z - y$ is a path of length two between x and y in $(AG(R))^c$. Let $z = (b, w'')$ for some $b \in MT$ and $w'' \in W$. As z and y are adjacent in $(AG(R))^c$, it follows that $zy \neq (0, 0)$ and so, $bw' \neq 0$. Hence, we obtain that there exists at least one $p \in \mathbf{A}$ such that p does not divide b in T. Moreover, as y and z are adjacent in $(AG(R))^c$, we obtain that either $ann_R(yz) = ann_R y$ or $ann_R(yz) = ann_R z$. Let $\mathbf{B}$ denote the set of all $p \in \mathbf{P}$ such that p divides b in T. It is clear that $ann_R z = (0) \oplus N$, where $N = \oplus_{p\in\mathbf{B}}(T/Tp)$. Let $\mathbf{A_1}$ be the subset consisting of all $p \in \mathbf{A}$ such that the component of bw' corresponding to p is nonzero. Observe that $ann_R y = (\prod_{p\in\mathbf{A}} Tp) \oplus W$ and $ann_R(yz) = (\prod_{p\in\mathbf{A_1}} Tp) \oplus W$. It follows from the above discussion that $ann_R(yz) = ann_R y$ and so, we obtain that $\mathbf{A} = \mathbf{A_1}$. Hence, b is not divisible by any $p \in \mathbf{A}$. Therefore, $A \cap B = \emptyset$. Note that $ann_R x = (0) \oplus (\oplus_{p\in\mathbf{A}}(T/Tp))$, $ann_R z = (0) \oplus N$, where $N = \oplus_{p\in\mathbf{B}}(T/Tp)$ and $ann_R(xz) = (0) \oplus N'$, where $N' = \oplus_{p\in\mathbf{A}\cup\mathbf{B}}(T/Tp)$. Since $\mathbf{A} \cup \mathbf{B} \notin \{\mathbf{A}, \mathbf{B}\}$, we obtain that $ann_R(xz) \neq ann_R x$ and $ann_R(xz) \neq ann_R z$. This is in contradiction to the assumption that x and z are adjacent in $(AG(R))^c$. This proves that there exists no path of length two between x and y in $(AG(R))^c$. As it is already shown that $diam((AG(R))^c) \leq 3$, it follows that $e(x) = 3$ in $(AG(R))^c$.

Suppose that $a = 0$. Then $x = (0, w)$ and $w \neq 0$. Let $\mathbf{C}$ be the finite nonempty subset of $\mathbf{P}$ consisting of all $p \in \mathbf{P}$ such that the component of w corresponding to p is nonzero. Let $b = \prod_{p\in\mathbf{C}} p$. Let $y = (b, 0)$. It follows as in the previous paragraph that $d(x, y) = 3$ in $(AG(R))^c$ and so, $e(x) = 3$ in $(AG(R))^c$. This proves that $e(x) = 3$ for each vertex x of $(AG(R))^c$ and therefore, $diam((AG(R))^c) = r((AG(R))^c) = 3$. □

Lemma 2.10. *Let R be a reduced ring which admits P as its unique maximal N-prime of* (0). *Then $(AG(R))^c$ contains an infinite clique.*

Proof. Note that $Z(R) = P$. Let $a \in P, a \neq 0$. Since R is reduced, it follows that $a^n \neq 0$ and moreover, $ann_R(a^n) = ann_R a$ for all $n \in \mathbb{N}$. Furthermore, we assert that $a^n \neq a^m$ for all distinct $n, m \in \mathbb{N}$. Suppose that $a^n = a^m$ for some distinct $n, m \in \mathbb{N}$. We can assume without loss of generality that $n < m$. Observe that $a^n(1 - a^{m-n}) = 0$. This implies that $1 - a^{m-n} \in Z(R) = P$. Hence, $1 = a^{m-n} + 1 - a^{m-n} \in P$. This is impossible. Thus $a^n \neq a^m$ for all distinct $n, m \in \mathbb{N}$. Note that the subgraph of $(AG(R))^c$ induced on $\{a^n : n \in \mathbb{N}\}$ is an infinite clique. □

Remark 2.11. Let R be a reduced ring which admits P as its unique maximal N-prime of (0). Then $girth((AG(R))^c) = 3$. Indeed, any edge of $(AG(R))^c$ is an edge of a triangle in $(AG(R))^c$.

Proof. Let $a \in P, a \neq 0$. It follows from the proof of Lemma 2.10 that $a - a^2 - a^3 - a$ is a cycle of length 3 in $(AG(R))^c$. Therefore, $girth((AG(R))^c) = 3$. Let $a - b$ be an edge of $(AG(R))^c$. Then $ab \neq 0$ and as $1 - b, 1 - a \notin P$, it follows that $ab \notin \{a, b\}$. We know from the proof of Proposition 2.5(i) that $a - ab$ and $ab - b$ are edges of $(AG(R))^c$. Hence, $a - ab - b - a$ is a cycle of length 3 in $(AG(R))^c$. □

3 The case where R has at least two maximal N-primes of (0)

In this section, we consider reduced rings R such that R has at least two maximal N-primes of (0) and study the properties of $(AG(R))^c$.

Proposition 3.1. *Let R be a reduced ring which admits at least two maximal N-primes of* (0). *Then* $(AG(R))^c$ *is connected if and only if* $P_1 \cap P_2 \neq (0)$ *for any two maximal N-primes* P_1, P_2 *of* (0) *in* R. *Moreover, if* $(AG(R))^c$ *is connected, then* $2 \leq diam((AG(R))^c) \leq 3$ *and furthermore,* $diam((AG(R))^c) = 3$ *if and only if there exist* $a, b \in Z(R)^*$ *such that* $ab = 0$ *and* $a + b \notin Z(R)$.

Proof. Assume that $P_1 \cap P_2 \neq (0)$ for any two maximal N-primes P_1, P_2 of (0) in R. (This condition is satisfied if R has at least three maximal N-primes of (0).) Let $a, b \in Z(R)^*, a \neq b$. We know from Proposition 2.5(iii) that there exists a path of length at most three between a and b in $(AG(R))^c$. This proves that $(AG(R))^c$ is connected and $diam((AG(R))^c) \leq 3$.

Suppose that $P_1 \cap P_2 = (0)$ for some maximal N-primes P_1, P_2 of (0) in R. In such a case, it is clear that $\{P_1, P_2\}$ is the set of all maximal N-primes of (0) in R. Observe that for any $x \in P_1 \backslash \{0\}$, $P_2 = ((0) :_R x)$ and for any $y \in P_2 \backslash \{0\}$, $P_1 = ((0) :_R y)$. Hence, the subgraph of $(AG(R))^c$ induced on $P_i \backslash \{0\}$ is complete for each $i \in \{1, 2\}$. Let $x \in P_1 \backslash \{0\}$ and $y \in P_2 \backslash \{0\}$. As there is no path in $(\Gamma(R))^c$ between x and y, it follows that there is no path in $(AG(R))^c$ between x and y. This shows that $(AG(R))^c$ is not connected and has exactly two components g_1 and g_2, where g_i equals the subgraph of $(AG(R))^c$ induced on $P_i \backslash \{0\}$ for each $i \in \{1, 2\}$. In this case, it is evident that $(\Gamma(R))^c = (AG(R))^c$.

Assume that $(AG(R))^c$ is connected. It is noted in the first paragraph of this proof that $diam((AG(R))^c) \leq 3$. We next verify that $diam((AG(R))^c) \geq 2$. Indeed, we show that $e(a) \geq 2$ for any $a \in Z(R)^*$. Let $a \in Z(R)^*$. Then there exists $x \in Z(R)^*$ such that $ax = 0$. As R is reduced, it is clear that $a \neq x$. From $ax = 0$, it follows that a and x are not adjacent in $(AG(R))^c$. Hence, $d(a, x) \geq 2$ in $(AG(R))^c$. This shows that $e(a) \geq 2$ for any vertex a of $(AG(R))^c$ and so, $diam((AG(R))^c) \geq 2$. This proves that $2 \leq diam((AG(R))^c) \leq 3$. We now prove the furthermore part. Assume that $diam((AG(R))^c) = 3$. Hence, there exist $a, b \in Z(R)^*$ such that $d(a, b) = 3$ in $(AG(R))^c$. Therefore, a and b are not adjacent in $(AG(R))^c$. If $ab \neq 0$, then we know from Proposition 2.1(i) that $a - ab - b$ is a path of length two between a and b in $(AG(R))^c$. This is impossible, since we are assuming that $d(a, b) = 3$ in $(AG(R))^c$. Therefore, $ab = 0$. If $a + b \in Z(R)$, then we know from Proposition 2.5(ii) that $a - (a + b) - b$ is a path of length two between a and b in $(AG(R))^c$. This is again impossible. Hence, $a + b \notin Z(R)$. Conversely, if $a, b \in Z(R)^*$ are such that $ab = 0$ and $a + b \notin Z(R)$, then we know from Lemma 2.6 that $d(a, b) = 3$ in $(AG(R))^c$ and so, $diam((AG(R))^c) = 3$. □

Recall that for any ring R, the *total quotient ring* of R, denoted by $Tot(R)$, is defined as $S^{-1}R$, where $S = R \backslash Z(R)$. That is, the total quotient ring of R is the ring of fractions of R with respect to the multiplicatively closed subset $R \backslash Z(R)$ of R. Let R be a reduced ring which admits at least two maximal N-primes of (0) such that $(AG(R))^c$ is connected. In Proposition 3.2, we characterize when $diam((AG(R))^c) = 3$ in terms of some ring-theoretic property of $Tot(R)$.

Proposition 3.2. *Let R be a reduced ring which admits at least two maximal N-primes of* (0). *Suppose that* $(AG(R))^c$ *is connected. Then* $diam((AG(R)^c) = 3$ *if and only if* $Tot(R)$ *contains a nontrivial idempotent.*

Proof. Assume that $diam((AG(R))^c) = 3$. Hence, we obtain from Proposition 3.1 that there exist $a, b \in Z(R)^*$ such that $ab = 0$ and $a + b \notin Z(R)$. Consider the elements $x = a/1$ and $y = b/1$ of $Tot(R)$. Observe that $xy = 0/1$ and $x + y$ is a unit in $Tot(R)$. Let $I_1 = Tot(R)x$ and $I_2 = Tot(R)y$. Note that $I_1 + I_2 = Tot(R)$ and $I_1 I_2 = (0/1)$. Therefore, we obtain from the Chinese remainder theorem [4, Proposition 1.10(ii) and (iii)] that the mapping $f : Tot(R) \to Tot(R)/I_1 \times Tot(R)/I_2$ defined by $f(t) = (t + I_1, t + I_2)$ is an isomorphism of rings. This proves that $Tot(R)$ is isomorphic to the direct product of two nonzero rings. Hence, $Tot(R)$ contains an idempotent e such that $e \notin \{0/1, 1/1\}$.

Conversely, assume that $Tot(R)$ contains a nontrivial idempotent. Let $e = r/s$ be a nontrivial idempotent of $Tot(R)$. Then $(r/s)((s - r)/s) = 0/1$. This implies that $r, s - r \in R \backslash \{0\}$, $r(s - r) = 0$, and $r + s - r = s \notin Z(R)$. Hence, we obtain from Proposition 3.1 that $diam((AG(R))^c) = 3$. □

In Example 3.3, we mention an example of a reduced ring S such that $Tot(S)$ admits a nontrivial idempotent but S has no nontrivial idempotent.

Example 3.3. Let R be the ring considered in Example 2.8. Let $S = R[[X]]$ be the power series ring in one variable X over R. Then $Tot(S)$ admits a nontrivial idempotent but S has no nontrivial idempotent.

Proof. In the notation of Example 2.8, R is quasilocal with M as its unique maximal ideal, R is reduced, and moreover, M is generated by $\{x_i : i \in \mathbb{N}\}$. Furthermore, $x_i x_j = 0 + I$ for all distinct $i, j \in \mathbb{N}$. Note that $S = R[[X]]$ is quasilocal with $MS + XS$ as its unique maximal ideal. Therefore, S has no nontrivial idempotent. Since R is reduced, it follows that S is reduced. Let $f(X) = x_1 X$ and $g(X) = \sum_{j=2}^{\infty} x_j X^j$.. Observe that $f(X)g(X) = 0 + I$. Hence, $f(X), g(X) \in Z(S)$. We claim that $f(X) + g(X) \notin Z(S)$. For if $f(X) + g(X) \in Z(S)$, then it follows from [9, Proposition 3.5] that there exists $r \in R\backslash\{0 + I\}$ such that $r(f(X) + g(X)) = 0 + I$. This implies that $rx_i = 0 + I$ for all $i \in \mathbb{N}$. Hence, $rM = (0 + I)$. Therefore, $r^2 = 0 + I$. This is impossible, since R is reduced and $r \in R\backslash\{0+I\}$. This proves that $f(X)+g(X) \notin Z(S)$. Now it follows from Propositions 3.1 and 3.2 that $Tot(S)$ contains a nontrivial idempotent. □

Let R be a reduced ring which admits at least two maximal N-primes of (0) such that $(AG(R))^c$ is connected. In Lemma 3.4, we provide a condition on the nature of maximal N-primes of (0) in R which implies that $diam((AG(R))^c) = 3$.

Lemma 3.4. *Let R be a reduced ring which admits at least two maximal N-primes of (0) such that $(AG(R))^c$ is connected. If R admits at least one maximal N-prime of (0) such that P is a B-prime of (0) in R, then $diam((AG(R))^c) = 3$.*

Proof. Let $u \in R\backslash\{0\}$ be such that $P = ((0) :_R u)$. Since R is reduced, $u^2 \neq 0$ and so, $u \notin P$. As is mentioned in the introduction , we know from Zorn's lemma and [13, Theorem 1] that if I is any ideal of R with $I \subseteq Z(R)$, then $I \subseteq Q$ for some maximal N-prime Q of (0) in R. Hence, it follows that $P + Ru \not\subseteq Z(R)$. Therefore, there exist $p \in P$ and $r \in R$ such that $p + ru \notin Z(R)$. On applying Lemma 2.6 with $a = p$ and $b = ru$, we obtain that $d(p, ru) = 3$ in $(AG(R))^c$. It now follows from Proposition 3.1 that $diam((AG(R))^c) = 3$. □

Let R be a reduced ring which admits at least two maximal N-primes of (0) such that $(AG(R))^c$ is connected. In Lemma 3.5, Example 3.6, Propositions 3.7 and 3.8, we discuss regarding $r((AG(R))^c)$.

Lemma 3.5. *Let R be a reduced ring which admits at least two maximal N-primes of (0) such that $(AG(R))^c$ is connected. If there exists $x \in R\backslash\{0\}$ such that $x \in P$ for any maximal N-prime P of (0) in R, then $r((AG(R))^c) = 2$.*

Proof. We know from the proof of Proposition 3.1 that $e(a) \geq 2$ in $(AG(R))^c$ for any $a \in Z(R)^*$. We now verify that $d(x, y) \leq 2$ in $(AG(R))^c$ for any $y \in Z(R)^*$ with $y \neq x$. Let $y \in Z(R)^*$, $y \neq x$ be such that x and y are not adjacent in $(AG(R))^c$. If $xy \neq 0$, then we obtain from Proposition 2.5(i) that $x - xy - y$ is a path of length two between x and y in $(AG(R))^c$. Suppose that $xy = 0$. Note that $y \in P$ for some maximal N-prime P of (0) in R. By hypothesis, $x \in P$. Hence, $x + y \in P \subseteq Z(R)$. Therefore, we obtain from Proposition 2.5(ii) that $x - (x + y) - y$ is a path of length two between x and y in $(AG(R))^c$. This proves that $e(x) = 2$ in $(AG(R))^c$ and so, $r((AG(R))^c) = 2$. □

For a reduced ring R which has only one maximal N-prime of (0), we know from Corollary 2.7 that $diam((AG(R))^c) = r((AG(R))^c)$. In Example 3.6, we illustrate that this result can fail to hold for a reduced ring T which admits at least two maximal N-primes of (0) such that $(AG(T))^c$ is connected.

Example 3.6. Let R be the reduced ring considered in Example 2.8. Let $T = R \times \mathbb{Z}$. Then $(AG(T))^c$ is connected, and moreover, $r((AG(T))^c) = 2 < diam((AG(T))^c) = 3$.

Proof. In the notation of the proof of Example 2.8(i), we know that R has M as its only one maximal N-prime of its zero ideal and moreover, R is reduced. Note that T is reduced and $Z(T) = (M \times \mathbb{Z}) \cup (R \times (0))$. Hence, it follows that T has exactly two maximal N-primes of its zero ideal and they are given by $P_1 = M\times\mathbb{Z}$ and $P_2 = R\times(0)$. Observe that $P_1\cap P_2 = M\times(0) \neq (0)\times(0)$. Therefore, we obtain from Proposition 3.1 that $(AG(T))^c$ is connected. Moreover, on applying Proposition 3.1 with $a = (1, 0)$ and $b = (0, 1)$, we get that $diam((AG(T))^c) = 3$. Furthermore, for any nonzero $m \in M$, $(m, 0) \in P_1 \cap P_2$ and hence, it follows from Lemma 3.5 that $r((AG(T))^c) = 2$. □

Let R be a reduced ring with at least two maximal N-primes of (0) such that $(AG(R))^c$ is connected. In Propositions 3.7 and 3.8, we provide some sufficient conditions on the ring R in order that $diam((AG(R))^c) = r((AG(R))^c) = 3$.

Proposition 3.7. *Let R be a reduced ring with a finite number $n \geq 3$ of minimal prime ideals. Then $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) = r((AG(R))^c) = 3$.*

Proof. Let $\{P_1, P_2, P_3, \ldots, P_n\}$ denote the set of all minimal prime ideals of R. Note that $\cap_{i=1}^n P_i = (0)$ and $Z(R) = \cup_{i=1}^n P_i$. Observe that $\{P_1, P_2, \ldots, P_n\}$ is the set of all maximal N-primes of (0) in R. Since $n \geq 3$, it follows that the intersection of any two maximal N-primes of (0) is nonzero. Hence, we obtain from Proposition 3.1 that $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) \leq 3$. We next verify that $e(x) \geq 3$ for any $x \in Z(R)^*$. Let $x \in Z(R)^*$. Let $A = \{i \in \{1, \ldots, n\} : x \in P_i\}$. Note that $A \neq \emptyset$ and $A \subset \{1, \ldots, n\}$. Let $B = \{1, \ldots, n\}\backslash A$. Note that $\cap_{i\in B} P_i \not\subseteq \cup_{i\in A} P_i$. Therefore, there exists $y \in (\cap_{i\in B} P_i)\backslash(\cup_{i\in A} P_i)$. It is clear from the choice of y that $xy = 0$ and $x + y \notin Z(R)$. Hence, we obtain from Lemma 2.6 that $d(x, y) = 3$ in $(AG(R))^c$. This proves that $e(x) \geq 3$ for any $x \in Z(R)^*$. Therefore, $diam((AG(R))^c) = r((AG(R))^c) = 3$. □

Let R be the ring considered in Example 2.8. It is noted in the proof of Example 2.8(ii) that R has an infinite number of minimal prime ideals. Let $T = R \times \mathbb{Z}$. Observe that T is reduced and has infinitely many minimal prime ideals. It is verified in Example 3.6 that $r((AG(T))^c) = 2 < diam((AG(T))^c) = 3$. Thus Example 3.6 illustrates that Proposition 3.7 can fail to hold for a reduced ring with an infinite number of minimal prime ideals. Recall from [8, Exercise 16, p.111] that a ring R is said to be *von Neumann regular* if for any given $x \in R$, there exists $y \in R$ such that $x = x^2y$. It is known that a ring R is von Neumann regular if and only if R is reduced and $dimR = 0$ [8, Exercise 16(d), p.111]. We prove in Proposition 3.8 that for any von Neumann regular ring R with at least three prime ideals, $(AG(R))^c$ is connected and moreover, $r((AG(R))^c) = diam((AG(R))^c) = 3$.

Proposition 3.8. *Let R be a von Neumann regular ring with at least three prime ideals. Then $(AG(R))^c$ is connected and moreover, $r((AG(R))^c) = diam((AG(R))^c) = 3$.*

Proof. As R is von Neumann regular, each prime ideal of R is minimal and maximal. Thus each prime ideal of R is a maximal N-prime of (0) in R. By hypothesis, we obtain that R has at least three maximal N-primes of (0) and therefore, it follows from Proposition 3.1 that $(AG(R))^c$ is connected and moreover, $diam((AG(R))^c) \leq 3$. We next verify that $e(x) \geq 3$ for each $x \in Z(R)^*$. Let $x \in Z(R)^*$. Since R is von Neumann regular, there exists $y \in R$ such that $x = x^2y$. This implies that $xy = e$ is an idempotent element of R. Observe that $x = ex = (x + 1 - e)e = ue$ with $u = x + 1 - e$ is a unit in R. It is clear that $e \notin \{0, 1\}$. Let $z = 1 - e$. Note that $xz = 0$ and $x + z$ is a unit in R and hence, $x + z \notin Z(R)$. Therefore, we obtain from Lemma 2.6 that $d(x, z) = 3$ in $(AG(R))^c$. This proves that $e(x) \geq 3$ for any vertex x of $(AG(R))^c$. Hence, we get that $r((AG(R))^c) = diam((AG(R))^c) = 3$. □

Let R be a reduced ring which admits at least two maximal N-primes of (0). We discuss regarding $girth((AG(R))^c)$ in Lemmas 3.9 and 3.10. We need some lemmas to arrive at the conclusion of Theorem 3.12. It is clear that if $(AG(R))^c$ contains an infinite clique, then $girth((AG(R))^c) = 3$. In keeping with our focus on the present need and also on the problem of determining the clique number of $(AG(R))^c$ in the later part of this section, we provide some sufficient conditions in Lemmas 3.9 and 3.10 in order that $(AG(R))^c$ to contain an infinite clique.

Lemma 3.9. *Let R be a reduced ring. Let P_1, P_2 be distinct prime ideals of R such that $P_1 \cup P_2 \subseteq Z(R)$. If $P_1 + P_2 \neq R$, then $(AG(R))^c$ contains an infinite clique and so, $girth((AG(R))^c) = 3$.*

Proof. Observe that either $P_1 \not\subseteq P_2$ or $P_2 \not\subseteq P_1$. Without loss of generality, we can assume that $P_1 \not\subseteq P_2$. Let $a \in P_1 \backslash P_2$. Hence, $a^n \in P_1 \backslash P_2$ for all $n \in \mathbb{N}$ and so, $a^n \neq 0$. We claim that $a^n \neq a^m$ for all distinct $n, m \in \mathbb{N}$. Suppose that $a^n = a^m$ for some distinct $n, m \in \mathbb{N}$. Without loss of generality, we can assume that $n < m$. From $a^n = a^m$, it follows that $a^n(1 - a^{m-n}) = 0$. This implies from the choice of a that $1 - a^{m-n} \in P_2$. Therefore, $1 = a^{m-n} + 1 - a^{m-n} \in P_1 + P_2$. This is in contradiction to the assumption that $P_1 + P_2 \neq R$. Thus $a^n \neq a^m$ for all distinct $n, m \in \mathbb{N}$. Since R is reduced, $ann_R a = ann_R a^n$ for all $n \in \mathbb{N}$. Moreover, as $P_1 \subseteq Z(R)$, it follows that $a^n \in Z(R)^*$ for all $n \in \mathbb{N}$. Hence, we obtain that a^n and a^m are adjacent in $(AG(R))^c$ for all distinct $n, m \in \mathbb{N}$. Therefore, it follows that the subgraph of $(AG(R))^c$ induced on $\{a^n : n \in \mathbb{N}\}$ is an infinite clique. Observe that $a - a^2 - a^3 - a$ is a cycle of length 3 in $(AG(R))^c$ and so, $girth((AG(R))^c) = 3$. □

Lemma 3.10. *Let R be a reduced ring. If $\Gamma(R)$ contains an infinite clique, then so does $(AG(R))^c$.*

Proof. By hypothesis, there exists an infinite subset A of $Z(R)^*$ such that the subgraph of $\Gamma(R)$ induced on A is a clique. Hence, for each $i \in \mathbb{N}$, there exists $a_i \in A$ such that $a_i \neq a_j$ for all distinct $i, j \in \mathbb{N}$. Moreover, $a_i a_j = 0$ for all distinct $i, j \in \mathbb{N}$. Let $i \in \mathbb{N}$. Define $x_i = \sum_{k=1}^{i} a_k$. From $x_i a_{i+1} = 0$, it follows that $x_i \in Z(R)$ for each $i \in \mathbb{N}$. We assert that $x_i \neq 0$ for each $i \in \mathbb{N}$. This is clear if $i = 1$. Let $i \geq 2$. If $x_i = 0$, then we obtain that $a_i = -\sum_{k=1}^{i-1} a_k$. This implies that $a_i^2 = 0$. This is a contradiction since $a_i \neq 0$ and R is reduced. Let $i, j \in \mathbb{N}, i \neq j$. We verify that x_i and x_j are adjacent in $(AG(R))^c$. Without loss of generality, we can assume that $i < j$. First, note that $x_i \neq x_j$. For if $x_i = x_j$, then we get that $a_j = -\sum_{i+1 \leq k < j} a_k$. This implies that $a_j = 0$. This is impossible. Hence, $x_i \neq x_j$. From $x_i x_j = x_i^2$, it follows that $ann_R(x_i x_j) = ann_R x_i$. This shows that x_i and x_j are adjacent in $(AG(R))^c$. Therefore, the subgraph of $(AG(R))^c$ induced on $\{x_i : i \in \mathbb{N}\}$ is an infinite clique. □

With the help of [7, Proposition 3.7], in Theorem 3.12, we determine necessary and sufficient conditions in order that $(AG(R))^c$ does not contain any infinite clique, where R is a reduced ring which admits at least two maximal N-primes of (0). We use Lemma 3.11 in the proof of Theorem 3.12.

Lemma 3.11. *Let R_1 be an integral domain and let R_2 be a nonzero ring. Let $R = R_1 \times R_2$. If R_1 is infinite, then $(AG(R))^c$ contains an infinite clique.*

Proof. Since R_1 is infinite, there exist $a_i \in R_1 \backslash \{0\}$ for each $i \in \mathbb{N}$ such that $a_i \neq a_j$ for all distinct $i, j \in \mathbb{N}$. Observe that the subgraph of $(AG(R))^c$ induced on $\{(a_i, 0) : i \in \mathbb{N}\}$ is an infinite clique. □

Theorem 3.12. *Let R be a reduced ring which admits at least two maximal N-primes of (0). Then the following statements are equivalent:*

(i) $\omega((AG(R))^c) < \infty$.

(ii) $(AG(R))^c$ *does not contain any infinite clique.*

(iii) *There exist finite fields $F_1, F_2, \ldots, F_n (n \geq 2)$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.*

Proof. $(i) \Rightarrow (ii)$ This is clear.

$(ii) \Rightarrow (iii)$ We know from Lemma 3.10 that $\Gamma(R)$ does not contain any infinite clique. Hence, we obtain from [7, Proposition 3.7] that R can admit only a finite number of minimal prime ideals. Let $\{P_1, \ldots, P_n\}$ denote the set of all minimal prime ideals of R. It is clear that $n \geq 2$ and $\cap_{i=1}^{n} P_i = (0)$. Observe that $\{P_1, P_2, \ldots, P_n\}$ is the set of all maximal N-primes of (0) in R. It follows from Lemma 3.9 that $P_i + P_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Now on applying the Chinese remainder theorem [4, Proposition 1.10(ii) and (iii)], we obtain that the mapping $f : R \to R/P_1 \times R/P_2 \times \cdots \times R/P_n$ defined by $f(r) = (r + P_1, r + P_2, \ldots, r + P_n)$ is an isomorphism of rings. Let us denote R/P_i by D_i for each $i \in \{1, 2, \ldots, n\}$. Note that D_i is an integral domain for each $i \in \{1, 2, \ldots, n\}$ and $R \cong D_1 \times D_2 \times \cdots \times D_n$ as rings. From Lemma 3.11, we obtain that D_i is finite and as any finite integral domain is a field, it follows that D_i is a finite field for each $i \in \{1, 2, \ldots, n\}$. Therefore, with $F_i = D_i$, we obtain that F_i is a finite field for each $i \in \{1, 2, \ldots, n\}$ and $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.

$(iii) \Rightarrow (i)$ From (iii), it is clear that R is finite and so, $\omega((AG(R))^c) < \infty$. □

Let R be a reduced ring which admits at least two maximal N-primes of (0). If $(AG(R))^c$ contains an infinite clique, then it is clear that $girth((AG(R))^c) = 3$. Hence, in determining rings R such that $girth((AG(R))^c) = 3$, it is enough to consider rings R such that $(AG(R))^c$ does not contain any infinite clique. Therefore, by Theorem 3.12, we can assume that there exist $n \geq 2$ and finite fields $F_1, F_2, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots F_n$ as rings. With this assumption, we proceed to determine rings R which are direct product of n ($n \geq 2$) finite fields such that $girth((AG(R))^c) = 3$.

Lemma 3.13. *Let R_1, R_2, R_3 be rings and let $R = R_1 \times R_2 \times R_3$. If at least one among R_1, R_2, R_3 contains at least three elements, then $girth((AG(R))^c) = 3$.*

Proof. Without loss of generality, we can assume that R_1 contains at least three elements. Let $a \in R_1 \backslash \{0, 1\}$. Let $x = (1, 0, 0), y = (a, 0, 0)$, and $z = (1, 1, 0)$. Note that $x - y - z - x$ is a cycle of length 3 in $(AG(R))^c$ and so, $girth((AG(R))^c) = 3$. □

Corollary 3.14. *Let $F_1, F_2, \ldots, F_n$ be finite fields and let $R = F_1 \times F_2 \times \cdots \times F_n$. If $n \geq 4$, then $girth((AG(R))^c) = 3$.*

Proof. If $n \geq 4$, then it is clear that $|F_3 \times F_4 \times \cdots \times F_n| \geq 3$ and $R \cong R_1 \times R_2 \times R_3$ with $R_1 = F_1, R_2 = F_2$, and $R_3 = F_3 \times F_4 \times \cdots \times F_n$. It follows immediately from Lemma 3.13 that $girth((AG(R))^c) = 3$. □

In Lemma 3.16, we characterize rings of the form $R = F_1 \times F_2$, where F_1 and F_2 are finite fields in order that $girth((AG(R))^c) = 3$. It is convenient to denote the collection of rings $\{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3\}$ by **A**.

Lemma 3.15. *Let D_1, D_2 be integral domains and let $R = D_1 \times D_2$. If either D_1 or D_2 contains at least three nonzero elements, then $girth((AG(R))^c) = 3$.*

Proof. Without loss of generality, we can assume that D_1 contains at least three nonzero elements. Let $a, b, c \in D_1 \backslash \{0\}$ be any three distinct elements. Then it is clear that $(a, 0) - (b, 0) - (c, 0) - (a, 0)$ is a cycle of length 3 in $(AG(R))^c$. Therefore, $girth((AG(R))^c) = 3$. □

Lemma 3.16. *Let $R = F_1 \times F_2$, where F_1 and F_2 are finite fields. Then the following statements are equivalent:*
(i) *$girth((AG(R))^c) = 3$.*
(ii) *$(AG(R))^c$ contains a cycle.*
(iii) *R is not isomorphic to any of the rings in* **A**.

Proof. $(i) \Rightarrow (ii)$ This is clear.
$(ii) \Rightarrow (iii)$ Let $T_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Observe that $(AG(T_1))^c$ is a graph on two isolated vertices $\{(1, 0), (0, 1)\}$. Let $T_2 = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $(AG(T_2))^c$ has exactly two components g_1 and g_2, where g_1 is the complete graph on the single vertex $\{(1, 0)\}$ and g_2 is the complete graph on the vertices $\{(0, 1), (0, 2)\}$. Let $T_3 = \mathbb{Z}_3 \times \mathbb{Z}_3$. Note that $(AG(T_3))^c$ has exactly two components h_1 and h_2, where h_1 is the complete graph on two vertices $\{(0, 1), (0, 2)\}$ and h_2 is the complete graph on two vertices $\{(1, 0), (2, 0)\}$. Thus $(AG(T_i))^c$ does not contain any cycle for each $i \in \{1, 2, 3\}$ and so, $girth((AG(T_i))^c) = \infty$. Therefore, R is not isomorphic to any of the rings in **A**.
$(iii) \Rightarrow (i)$ Assume that R is not isomorphic to any of the rings in **A**. Hence, we obtain that either F_1 or F_2 must contain at least three nonzero elements. In such a case , it follows from Lemma 3.15 that $girth((AG(R))^c) = 3$. □

Let F_1, F_2, F_3 be finite fields and $R = R_1 \times F_2 \times F_3$. In Lemma 3.18, we characterize such rings in order that $girth((AG(R))^c) = 3$.

Lemma 3.17. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $(AG(R))^c$ is a cycle of length* 6.

Proof. It is convenient to denote $(1, 0, 0)$ by e_1, $(0, 1, 0)$ by e_2, and $(0, 0, 1)$ by e_3. Observe that the vertex set of $(AG(R))^c = Z(R)^* = \{e_1, e_2, e_3, e_1 + e_2, e_2 + e_3, e_3 + e_1\}$. Since $e_i^2 = e_i$ for each $i \in \{1, 2, 3\}$ and $e_ie_j = (0, 0, 0)$ for all distinct $i, j \in \{1, 2, 3\}$, it follows that $(AG(R))^c$ is the cycle of length 6 given by $e_1 - (e_1 + e_2) - e_2 - (e_2 + e_3) - e_3 - (e_1 + e_3) - e_1$. □

Lemma 3.18. *Let $R = F_1 \times F_2 \times F_3$, where F_1, F_2, F_3 are finite fields. Then $girth((AG(R))^c) = 3$ if and only if R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.*

Proof. Assume that $girth((AG(R))^c) = 3$. Let us denote the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by T. We know from Lemma 3.17 that $(AG(T))^c$ is a cycle of length 6. Hence, $girth((AG(T))^c) = 6$. Therefore, R is not isomorphic to T as rings.

Conversely, assume that R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. Hence, at least one among F_1, F_2, F_3 contains at least three elements. In such a case, we obtain from Lemma 3.13 that $girth((AG(R))^c) = 3$. □

With the help of Theorem 3.12, Corollary 3.14, Lemmas 3.16 and 3.18, we prove in Theorem 3.19 that for a reduced ring R which admits at least two maximal N-primes of (0), $girth((AG(R))^c) \in \{3, 6, \infty\}$. Moreover, in Theorem 3.19, we classify the rings R for which $girth((AG(R))^c) = \infty$.

Theorem 3.19. *Let R be a reduced ring with at least two maximal N-primes of (0). Then the following statements are equivalent:*

(i) $girth((AG(R))^c) \in \{3, 6\}$.

(ii) $(AG(R))^c$ contains a cycle.

(iii) R is not isomorphic to any of the rings in $\mathbf{A}$, where $\mathbf{A} = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.

Proof. $(i) \Rightarrow (ii)$ This is clear.

$(ii) \Rightarrow (iii)$ If $(AG(R))^c$ contains an infinite clique, then it is clear that (iii) holds. Suppose that $(AG(R))^c$ does not contain any infinite clique. Then it follows from Theorem 3.12, Corollary 3.14, and Lemmas 3.16, 3.17, 3.18 that R is not isomorphic to any of the rings in $\mathbf{A}$.

$(iii) \Rightarrow (i)$ If $(AG(R))^c$ contains an infinite clique, then it is clear that $girth((AG(R))^c) = 3$. Suppose that $(AG(R))^c$ does not contain any infinite clique. Then it follows from Theorem 3.12, Corollary 3.14, and Lemmas 3.16, 3.17 , 3.18 that $girth((AG(R))^c) \in \{3, 6\}$. Indeed, it follows from the above mentioned results that $girth((AG(R))^c) = 6$ if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. □

Let $n \geq 2$ and let $F_1, F_2, \ldots, F_n$ be finite fields. Let $R = F_1 \times F_2 \times \cdots \times F_n$. In view of Theorem 3.12, it is natural to determine $\omega((AG(R))^c)$. We determine $\omega((AG(R))^c)$ in Proposition 3.21 and moreover, we prove that $\omega((AG(R))^c) = \chi((AG(R))^c)$. We use Lemma 3.20 in the proof of Proposition 3.21.

Let T be a ring. It is well-known that the relation $\sim$ defined on $Z(T)^*$ by $x \sim y$ if and only if $ann_T x = ann_T y$ is an equivalence relation and hence, it determines a partition of $Z(T)^*$ into equivalence classes. For an element $x \in Z(T)^*$, we denote the equivalence class determined by $\sim$ containing x by $[x]$.

Lemma 3.20. *Let R be a reduced ring. Let $\sim$ be the equivalence relation defined on $Z(R)^*$ as mentioned above. Let $x \in Z(R)^*$. Then the subgraph of $(AG(R))^c$ induced on $[x]$ is a clique.*

Proof. Let $a, b \in [x]$, $a \neq b$. Note that $a \sim b$ and so, $ann_R a = ann_R b$. It is clear that $ann_R a \subseteq ann_R(ab)$. Let $r \in R$ be such that $r(ab) = 0$. Then $ra \in ann_R b = ann_R a$. Hence, $ra^2 = 0$ and so, $ra = 0$. This proves that $ann_R(ab) \subseteq ann_R a$ and therefore, $ann_R(ab) = ann_R a$. This shows that a and b are adjacent in $(AG(R))^c$. Hence, the subgraph of $(AG(R))^c$ induced on $[x]$ is a clique. □

Proposition 3.21. *Let $n \geq 2$ and $F_1, F_2, \ldots, F_n$ be finite fields with $|F_1| \geq |F_2| \geq \cdots \geq |F_n|$. Let $R = F_1 \times F_2 \times \cdots \times F_n$. Then $\omega((AG(R))^c) = \chi((AG(R))^c) = \sum_{i=1}^{n-1} k_i$, where $k_i = \prod_{j=1}^{i} |F_j^*|$.*

Proof. Let $i \in \{1, \ldots, n-1\}$ and let $C_i = \{(\alpha_1, \ldots, \alpha_i, 0, \ldots, 0) : \alpha_1 \in F_1^*, \ldots, \alpha_i \in F_i^*\}$. Let $C = \cup_{i=1}^{n-1} C_i$. We assert that the subgraph of $(AG(R))^c$ induced on C is a clique. Let $x, y \in C$ with $x \neq y$. We verify that x and y are adjacent in $(AG(R))^c$. Observe that for any $i \in \{1, \ldots, n-1\}$ and any $a, b \in C_i$, $ann_R a = ann_R b$. Hence, by Lemma 3.20, we obtain that the subgraph of $(AG(R))^c$ induced on C_i is a clique. Thus if $x, y \in C_i$ for some $i \in \{1, \ldots, n-1\}$, then we obtain that x and y are adjacent in $(AG(R))^c$. Suppose that $x \in C_i$ and $y \in C_j$ for some distinct $i, j \in \{1, \ldots, n-1\}$. Without loss of generality, we can assume that $i < j$. Observe that $xy \in C_i$ and so, $ann_R(xy) = ann_R x$. Therefore, x and y are adjacent in $(AG(R))^c$. This proves that the subgraph of $(AG(R))^c$ induced on C is a clique. As $|C| = \sum_{i=1}^{n-1} k_i$, where $k_i = \prod_{j=1}^{i} |F_j|^*$, we get that $\omega((AG(R))^c) \geq \sum_{i=1}^{n-1} k_i$. We next show that $\chi((AG(R))^c) \leq \sum_{i=1}^{n-1} k_i$. Let $\{c_{11}, \ldots, c_{1k_1}, c_{21}, \ldots, c_{2k_2}, \ldots, c_{n-11}, \ldots, c_{n-1k_{n-1}}\}$ be a set of $\sum_{i=1}^{n-1} k_i$ distinct colors. Let $i \in \{1, \ldots, n-1\}$ and let $A_i = \{r \in R$: exactly i coordinates of r are nonzero$\}$. It is clear that $Z(R)^* = \cup_{i=1}^{n-1} A_i$ and $C_i \subseteq A_i$ for each $i \in \{1, \ldots, n-1\}$. Observe that $|C_i| = k_i$ and it is already noted in this proof that the subgraph of $(AG(R))^c$ induced on C_i is a clique. Hence, the elements of C_i can be colored using the set of colors $\{c_{i1}, \ldots, c_{ik_i}\}$. For any choice of integers $1 \leq m_1 < \cdots < m_i \leq n$ from $\{1, 2, \ldots, n\}$, let us denote the set consisting of all $r \in A_i$ such that m_kth coordinate of r is nonzero for each $k \in \{1, \ldots, i\}$ by $A_{i(m_1, \ldots, m_i)}$. Observe that A_i is the disjoint union of the sets $A_{i(m_1, \ldots, m_i)}$, where $1 \leq m_1 < \cdots < m_i \leq n$ varies over all possible choice of i elements from $\{1, 2, \ldots, n\}$ and it is clear that $C_i = A_{i(1, \ldots, i)}$. By hypothesis, $|F_1| \geq |F_2| \geq \cdots \geq |F_n|$ and so, $|A_{i(m_1, \ldots, m_i)}| \leq k_i = |C_i|$ for any choice of i integers $1 \leq m_1 < \cdots < m_i \leq n$ from $\{1, 2, \ldots, n\}$. For any distinct choices of integers say, $1 \leq m_1 < \cdots < m_i \leq n$ and $1 \leq n_1 < \cdots < n_i \leq n$, it is not hard to verify that a and b are not adjacent in $(AG(R))^c$ for any $a \in A_{i(m_1, \ldots, m_i)}$ and $b \in A_{i(n_1, \ldots, n_i)}$. Moreover, for any $x, y \in A_{i(m_1, \ldots, m_i)}$, $xy \in A_{i(m_1, \ldots, m_i)}$ and $ann_R(xy) = ann_R x$. Hence, the subgraph of $(AG(R))^c$ induced on $A_{i(m_1, \ldots, m_i)}$ is a clique. As $|A_{i(m_1, \ldots, m_i)}| \leq k_i$, the elements of $A_{i(m_1, \ldots, m_i)}$ can be colored using any subset of $\{c_{i1}, \ldots, c_{ik_i}\}$ containing exactly $|A_{i(m_1, \ldots, m_i)}|$ colors. Thus the vertices of $(AG(R))^c = Z(R)^* = \cup_{i=1}^{n-1} A_i$ can be colored using a set of $\sum_{i=1}^{n-1} k_i$ colors. It is clear that the above assignment of colors is indeed a proper vertex coloring of $(AG(R))^c$. This proves that $\chi((AG(R))^c) \leq \sum_{i=1}^{n-1} k_i$. Therefore, $\sum_{i=1}^{n-1} k_i \leq \omega((AG(R))^c) \leq \chi((AG(R))^c) \leq \sum_{i=1}^{n-1} k_i$. This shows that $\omega((AG(R))^c) = \chi((AG(R))^c) = \sum_{i=1}^{n-1} k_i$. □

Corollary 3.22. *Let* $n \geq 2$ *and* $F_i = \mathbb{Z}_2$ *for each* $i \in \{1, 2, \ldots, n\}$. *Let* $R = F_1 \times F_2 \times \cdots \times F_n$. *Then* $\omega((AG(R))^c) = \chi((AG(R))^c) = n - 1$.

Proof. As $|F_i^*| = 1$ for each $i \in \{1, 2, \ldots, n\}$, in the notation of Proposition 3.21, we obtain that $k_i = 1$ for each $i \in \{1, \ldots, n-1\}$. Therefore, it follows from Proposition 3.21 that $\omega((AG(R))^c) = \chi((AG(R))^c) = \sum_{i=1}^{n-1} k_i = n - 1$. □

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Author information

S. Visweswaran and Patat Sarman, Department of Mathematics, Saurashtra University, Rajkot, Gujarat 360 005, India.
E-mail: s_visweswaran2006@yahoo.co.in

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