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WEAK RELATIVELY COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce weak relatively complemented almost distributive lattices. Certain examples are provided. We characterize the class of weak relatively complemented almost distributive lattices in terms of dense elements and $*$-almost distributive lattices.

1 Introduction

It is well known that Boole axiomatized the two valued propositional calculus into a Boolean algebra in 1854. Boolean algebras are used to construct and simplify electrical circuits, switching circuits, which are used in the design of computer chips. However, there are some situations in which two valued propositional calculus is not adequate. In this context, there can be several generalizations of Boolean algebras (complemented distributive lattices) have come into picture. The concept of an almost distributive lattice was introduced by Swamy and Rao as a common abstraction of lattice theoretic and ring theoretic generalizations of a Boolean algebra like p-rings, regular rings, bi-regular rings, associated rings, p_1-rings, triple systems, Baer rings and m-domain rings.

An almost distributive lattice is an algebraic structure $(L, \wedge, \vee, 0)$ of type $(2, 2, 0)$ which satisfies almost all axioms of a distributive lattice with zero except the commutativity of $\wedge$, $\vee$ and the right distributivity of $\vee$ over $\wedge$. Infact each one is equivalent to each other. It is not even a lattice and hence difficulty to deal with. The associativity of $\vee$ is not yet known. Several authors studied this structure in both algebraic and topological aspects. In this paper, our main motto is to emphasize the importance of the class of weak relatively complemented almost distributive latices.

2 Preliminaries

Let us recall that the notion of almost distributive lattices and certain necessary results which are required in the sequel.

Definition 2.1. [9] By an Almost Distributive Lattice (Shortened: ADL), we mean an algebra $(L, \wedge, \vee, 0)$ of type $(2, 2, 0)$, if it satisfies the following conditions;

(i) $0 \wedge a = 0$

(ii) $a \vee 0 = a$

(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(vi) $(a \vee b) \wedge b = b$

for all $a, b, c \in L$.

Example 2.2. [9] Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define

$$x \wedge y = \begin{cases} x_0 & \text{if } x = x_0 \\ y & \text{if } x \neq x_0 \end{cases} \qquad x \vee y = \begin{cases} y & \text{if } x = x_0 \\ x & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \wedge, \vee, x_0)$ is an ADL with x_0 as its zero element. This ADL is called a discrete ADL, which is not a lattice.

Throughout this paper by L we mean an almost distributive lattice $(L, \wedge, \vee, 0)$ with zero.

Lemma 2.3. *[9] For any $a, b, c \in L$, we have*

(i) $a \wedge 0 = 0$ *and* $0 \vee a = a$

(ii) $a \wedge a = a \vee a = a$

(iii) $a \vee b = a \iff a \wedge b = b$

(iv) $\wedge$ *is associative in* L

(v) $a \wedge b \wedge c = b \wedge a \wedge c$

(vi) $a \wedge b = 0 \iff b \wedge a = 0$

(vii) $a \wedge b = b \wedge a$, *when* $a \wedge b = 0$.

For any $a, b \in L$, define $a \leq b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$. It is easy to observe that $\leq$ is a partial ordering on L. An element m is said to be maximal, if $m \wedge x = x$ for all $x \in L$. L is discrete if and only if every non-zero element is maximal.

Definition 2.4. [9] L is said to be relatively complemented, if for any $a, b \in L$, $a \leq b$, the interval $[a, b] = \{x \in L \mid a \leq x \leq b\}$ is a complemented distributive lattice (that is a Boolean algebra).

Theorem 2.5. *[9] L is relatively complemented if and only if, for any $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$.*

A non-empty subset I of L is said to be an ideal of L, if for any $a, b \in I$ and $x \in L$, $a \vee b,\ a \wedge x \in I$. For any non-empty subset S of L, $(S) = \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L, n \text{ is a positive integer}\}$ is the smallest ideal containing S. In particular, for any $a \in L$, $(a) = \{a \wedge x \mid x \in L\}$ is a principal ideal generated by a. The set $\mathcal{I}(L)$ of ideals of L forms a complete bounded distributive lattice, where $I \cap J$ is the infimum and $I \vee J = \{i \vee j \mid i \in I, j \in J\}$ is the suprimum of I and J in $\mathcal{I}(L)$. The set $\mathcal{PI}(L)$ of principal ideals of L is a sublattice of $\mathcal{I}(L)$, where $(a) \wedge (b) = (a \wedge b)$ and $(a) \vee (b) = (a \vee b)$ for all $a, b \in L$.

For any non empty subset A of L, the set $A^* = \{x \in L \mid a \wedge x = 0 \text{ for all } a \in A\}$ is an annihilator ideal of L. In particular, for any $a \in L$, $\{a\}^* = (a)^*$, where (a) is a principal ideal generated by a. An element d is said to be dense, if $(d)^* = \{0\}$. Denote D be the set of dense elements in L. It is closed under $\wedge$, provided D is non-empty. Moreover, if $d \in D$, then $d \vee x$ & $x \vee d \in D$, for all $x \in L$.

Lemma 2.6. *[7] For any $I, J \in \mathcal{I}(L)$, we have*

(i) $I \subseteq J$ *implies* $J^* \subseteq I^*$

(ii) $I^{***} = I^*$

(iii) $(I \vee J)^* = I^* \cap J^*$

(iv) $(I \cap J)^{**} = I^{**} \cap J^{**}$

Definition 2.7. [4] L is said to be a $*$-*ADL*, if for any $x \in L$, there exists $y \in L$ such that $(x)^{**} = (y)^*$.

Theorem 2.8. *[4] L is a $*$-ADL if and only if, for any $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense.*

Definition 2.9. [4] L is said to be disjunctive, if for any $x, y \in L$, $(x)^* = (y)^*$ implies $x = y$.

3 Weak relatively complemented almost distributive lattices

In this section, we define a weak relatively complemented almost distributive lattice. We obtain necessary and sufficient conditions for an almost distributive lattice to become weak relatively complemented.

Definition 3.1. An ADL L is said to be weak relatively complemented, if for any $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.

Example 3.2. Let $L = \{0, d_1, d_2, m_1, m_2\}$ be an ADL, where $\wedge$ and $\vee$ defined as follows;

$\wedge$	0	d_1	d_2	m_1	m_2
0	0	0	0	0	0
d_1	0	d_1	d_2	d_1	d_2
d_2	0	d_1	d_2	d_1	d_2
m_1	0	d_1	d_2	m_1	m_2
m_2	0	d_1	d_2	m_1	m_2

$\vee$	0	d_1	d_2	m_1	m_2
0	0	d_1	d_2	m_1	m_2
d_1	d_1	d_1	d_1	m_1	m_1
d_2	d_2	d_2	d_2	m_2	m_2
m_1	m_1	m_1	m_1	m_1	m_1
m_2	m_2	m_2	m_2	m_2	m_2

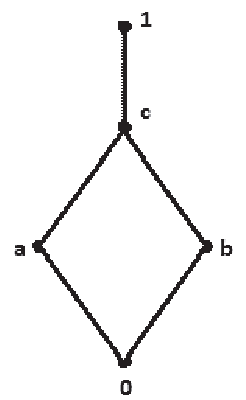
Then $(L, \wedge, \vee, 0)$ is a weak relatively complemented ADL but not a lattice (because $d_2 \wedge d_1 = d_1 \neq d_2 = d_1 \wedge d_2$).

Example 3.3. Let L be a discrete ADL and I is an infinite set. The set $\mathcal{S} = \{f \in L^I \mid |f| = \{i \in I \mid f(i) \neq 0\}$ is finite $\}$ is an ADL with pointwise operations. For any $f, g \in S$, $i \in I$, define

$$x(i) = \begin{cases} g(i) & \text{if } f(i) = 0 \text{ and } g(i) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{S}$ (since $|x| \subseteq |g|$) and $f \wedge x = 0$ and $(f \vee x)^* = (f \vee g)^*$. Thus $\mathcal{S}$ is a weak relatively complemented ADL. Moreover $\mathcal{S}$ has no dense elements.

Example 3.4. Let $L = \{0, a, b, c, 1\}$ be an ADL whose Hasse-diagram is



Then L is a weak relatively complemented ADL.

Remark 3.5. Every relatively complemented ADL is weak relatively complemented. The converse need not be true. For, see Example 3.2., L is weak relatively complemented but not relatively complemented.

In this context, we have the following.

Theorem 3.6. *Every weak relatively complemented disjunctive ADL is relatively complemented.*

Proof. Suppose that L is a disjunctive weak relatively complemented ADL. Let $a, b \in L$. Then there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$ (since L is weak relatively complemented). $a \vee x = a \vee b$, since L is disjunctive. $\square$

Theorem 3.7. *If every non-zero element is dense in L, then L is weak relatively complemented.*

Proof. Suppose that every non-zero element is dense in L. Let $a, b \in L$, if $a \neq 0$, then choose $x = 0$. So that $a \wedge x = 0$ and $(a \vee x)^* = (a)^* = (a \vee b)^*$ (since $a \vee b$ is dense). Similarly, even if $b \neq 0$. Therefore L is a weak relatively complemented ADL. □

Remark 3.8. The converse of the above theorem need not be true. For, see Example 3.4., L is weak relatively complemented ADL but a is a non-dense element in L.

Definition 3.9. An ideal I of L is said to be a dense complemented, if there exists an ideal J of L such that $I \wedge J = \{0\}$ and $I \vee J$ is an ideal generated by a dense element.

Remark 3.10. In an ADL with maximal elements, every complemented ideal is dense complemented. But the converse need not be true. For, see Example 3.4., (a) is a dense complemented but not a complemented ideal in L.

Lemma 3.11. *For any $d \in D, x \in L$, $(x) = (d)$ implies $x \in D$.*

Proof. From the hypothesis, we have $d = x \wedge d$. So that $x = x \vee d$ (by Lemma 2.3(iii)) is dense (since $x \vee d$ is dense). □

Lemma 3.12. *Every dense complemented ideal is a principal ideal.*

Proof. Let I be a dense complemented ideal in L. Then there exists an ideal J such that $I \cap J = \{0\}$ and $I \vee J = (d)$, where d is dense element in L. So that $d = a \vee b$, for some $a \in I$ and $b \in J$. Clearly $(a) \subseteq I$. Let $x \in I \subseteq (d)$. Then $x = d \wedge x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = a \wedge x$ (since $b \wedge x \in I \cap J = \{0\}$). Therefore $x \in (a)$ and hence $I = (a)$. Thus every dense complemented ideal is a principal ideal. □

Theorem 3.13. *If L has dense elements, then L is weak relatively complemented if and only if every principal ideal is dense complemented.*

Proof. Suppose that L is weak relatively complemented. Let $a \in L$ and d is a dense element in L. Then by our assumption, there exists $b \in L$ such that $a \wedge b = 0$ and $(a \vee b)^* = (a \vee d)^* = (a)^* \cap (d)^* = (0)$. Therefore $a \vee b$ is dense and hence every principal ideal is dense complemented. On the other hand, suppose that every principal ideal is dense complemented. Let $a, b \in L$. Then there exist $c, d \in L$ such that $(a) \cap (c) = \{0\} = (b) \cap (d)$ and $a \vee c$ & $b \vee d$ are dense elements. Take $x = c \wedge b$. Then $a \wedge x = a \wedge c \wedge b = 0$ (since $a \wedge c = 0$) and $(a \vee x) \wedge (a \vee b) = a \vee (x \wedge b) = a \vee (c \wedge b \wedge b) = a \vee x$. So that $(a \vee b)^* \subseteq (a \vee x)^*$. Now, for $t \in L$,

$$
\begin{aligned}
t \in (a \vee x)^* \quad &\Rightarrow \quad t \wedge (a \vee x) = 0 \\
&\Rightarrow \quad t \wedge a = 0 \text{ and } t \wedge c \wedge b = 0 \\
&\Rightarrow \quad t \wedge b \wedge (a \vee c) = 0 \\
&\Rightarrow \quad t \wedge b = 0 \qquad\qquad \text{(since } a \vee c \text{ is dense)} \\
&\Rightarrow \quad t \wedge (a \vee b) = 0 \\
&\Rightarrow \quad t \in (a \vee b)^*.
\end{aligned}
$$

Therefore $(a \vee x)^* \subseteq (a \vee b)^*$ and hence $(a \vee x)^* = (a \vee b)^*$. Thus L is weak relatively complemented. □

Theorem 3.14. *L is weak relatively complemented if and only if $\mathcal{PI}(L)$ is weak relatively complemented.*

In [9], Swamy and Rao proved that an ADL is relatively complemented if and only if every principal ideal is direct summand. In this context, we observe that every principal ideal need not be direct summand in a weak relatively complemented ADL. For, see Example 3.4., $(a) = \{0, a, b\}$ is not a direct summand in L.

4 $B_D(L)$

In this section, we concentrate on the class of almost distributive lattices with dense elements. For an ADL L, we introduce the set $B_D(L) = \{a \in L \mid$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ dense$\}$, where D is the set of dense elements. It is observe that $B_D(L)$ is always a weak relatively complemented subADL of L. We have obtain a necessary and sufficient condition for an ADL to become weak relatively complemented in terms of $B_D(L)$.

Definition 4.1. Given an ADL L with dense elements, define

$$B_D(L) = \{a \in L \mid \text{there exists } b \in L \text{ such that } a \wedge b = 0 \text{ and } a \vee b \text{ is dense}\}.$$

Theorem 4.2. *$B_D(L)$ is a weak relatively complemented subADL of L.*

Proof. Let $a_1, a_2 \in B_D(L)$. Then there exist $b_1, b_2 \in L$ such that $a_1 \wedge b_1 = 0 = a_2 \wedge b_2$ and $a_1 \vee b_1$ & $a_2 \vee b_2$ are dense. Now,

$$(a_1 \wedge a_2) \wedge (b_1 \vee b_2) = (a_1 \wedge a_2 \wedge b_1) \vee (a_1 \wedge a_2 \wedge b_2) = 0$$

and, for $x \in L$,

$$\begin{aligned} x \in [(a_1 \wedge a_2) \vee (b_1 \vee b_2)]^* \quad &\Rightarrow \quad x \wedge a_1 \wedge a_2 = 0, x \wedge b_1 = 0 = x \wedge b_2 \\ &\Rightarrow \quad x \wedge a_2 \wedge (a_1 \vee b_1) = 0 \\ &\Rightarrow \quad x \wedge a_2 = 0 \qquad \text{(since } a_1 \vee b_1 \text{ is dense)} \\ &\Rightarrow \quad x \wedge (a_2 \vee b_2) = 0 \\ &\Rightarrow \quad x = 0 \qquad \text{(since } a_2 \vee b_2 \text{ is dense)} \end{aligned}$$

Therefore $(a_1 \wedge a_2) \vee (b_1 \vee b_2)$ is dense and hence $a_1 \wedge a_2 \in B_D(L)$. Next,

$$(a_1 \vee a_2) \wedge b_1 \wedge b_2 = (a_1 \wedge b_1 \wedge b_2) \vee (a_2 \wedge b_1 \wedge b_2) = 0$$

and, for $x \in L$,

$$\begin{aligned} x \in [(a_1 \vee a_2) \vee (b_1 \wedge b_2)]^* \quad &\Rightarrow \quad x \wedge a_1 = x \wedge a_2 = x \wedge b_1 \wedge b_2 = 0 \\ &\Rightarrow \quad (x \wedge b_2) \wedge (a_1 \vee b_1) = 0 \\ &\Rightarrow \quad x \wedge b_2 = 0 \qquad \text{(since } a_1 \vee b_1 \text{ is dense)} \\ &\Rightarrow \quad x \wedge (a_2 \vee b_2) = 0 \\ &\Rightarrow \quad x = 0. \qquad \text{(since } a_2 \vee b_2 \text{ is dense)} \end{aligned}$$

Therefore $(a_1 \vee a_2) \vee (b_1 \wedge b_2)$ is dense and hence $a_1 \vee a_2 \in B_D(L)$. Thus $B_D(L)$ is a subADL of L. Further, if $a, b \in B_D(L)$, then there exist $c, d \in L$ such that $a \wedge c = 0 = b \wedge d$ and $a \vee c$ & $b \vee d$ are dense. Take $x = c \wedge b$. Then $a \wedge x = a \wedge c \wedge b = 0$ and,

$$(a \vee x) \wedge (a \vee b) = a \vee (x \wedge b) = a \vee (c \wedge b \wedge b) = a \vee (c \wedge b) = a \vee x.$$

So that $(a \vee b)^* \subseteq (a \vee x)^*$. Now, for $t \in L$,

$$\begin{aligned} t \in (a \vee x)^* \quad &\Rightarrow \quad t \wedge a = 0 \text{ and } t \wedge x = 0 \\ &\Rightarrow \quad t \wedge a = 0 \text{ and } t \wedge c \wedge b = 0 \\ &\Rightarrow \quad t \wedge b \wedge (a \vee c) = 0 \\ &\Rightarrow \quad t \wedge b = 0 \qquad \text{(since } a \vee c \text{ is dense)} \\ &\Rightarrow \quad t \wedge (a \vee b) = 0 \\ &\Rightarrow \quad t \in (a \vee b)^*. \end{aligned}$$

Therefore $(a \vee x)^* \subseteq (a \vee b)^*$ and hence $(a \vee x)^* = (a \vee b)^*$. Thus $B_D(L)$ is a weak relatively complemented subADL of L. □

In [8], for an ADL L with maximal elements, Swamy and Ramesh introduced the Birkhoff centre $B(L) = \{a \in L \mid a \wedge b = 0$ and $a \vee b$ is maximal, for some $b \in L\}$. They proved that $B(L)$ is always relatively complemented. Moreover L is relatively complemented if and only if $B(L) = L$.

Remark 4.3. If every dense element is maximal in L, then $B(L) = B_D(L)$. But, we can provided an example, where $B(L)$ is a proper subset of $B_D(L)$. See Example 3.2., $B_D(L) = \{0, d_1, d_2, m_1, m_2\}$ and $B(L) = \{0, m_1, m_2\}$.

Lemma 4.4. *For every non-zero element x in L, there exists a dense element d in L such that $x \leq d$.*

Proof. Let $x \in L$ such that $x \neq 0$. Choose a dense element d in L. Then $x \leq x \vee d$ and $x \vee d$ is dense (because $(x \vee d)^* = (x)^* \cap (d)^* = \{0\}$). □

Theorem 4.5. *L is weak relatively complemented if and only if $B_D(L) = L$.*

Proof. Suppose that L is weak relatively complemented. Let $x \in L$ such that x is non-dense. Then, by Lemma 4.4., there exists a dense element $d \in L$ such that $x \leq d$. For these $x, d \in L$, by our assumption, there exists $y \in L$ such that $x \wedge y = 0$ and $(x \vee y)^* = (x \vee d)^* = \{0\}$. Therefore $x \in B_D(L)$ and hence $B_D(L) = L$. Otherside is trivial. □

Remark 4.6. If L has no dense elements, then there is a weak relatively complemented ADL such that $B_D(L) \neq L$. For, see Example 3.3., $\mathcal{S}$ is weak relatively complemented ADL but $B_D(\mathcal{S}) \neq \mathcal{S}$ (Infact, $B_D(\mathcal{S})$ is empty).

Theorem 4.7. *If every non-zero element is dense in L, then $B_D(L) = L$.*

Proof. Suppose that every non-zero element is dense in L. Then $D = L \setminus \{0\}$. Therefore $B_D(L) = L$. □

Remark 4.8. The converse of the above theorem need not be true. For, see Example 3.4., we have $B_D(L) = L$ but a and b are non-dense elements in L.

In [4], Rao and Rao introduced two congruences

$$\theta = \{(x, y) \in L \times L \mid (x)^* = (y)^*\} \quad \text{and}$$

$$\theta_F = \{(x, y) \in L \times L \mid x \wedge a = y \wedge a, \text{ for some } a \in F\},$$

where F is a non-empty set closed under $\wedge$. Now, we have the following.

Theorem 4.9. *For any $x, y \in L$,*
(i) $(x, y) \in \theta$ and $x \in B_D(L)$ implies $y \in B_D(L)$.
(ii) $(x, y) \in \theta$ and $y \in B_D(L)$ implies $x \in B_D(L)$.

Proof. *(i)* Suppose that $(x, y) \in \theta$ and $x \in B_D(L)$. Then $(x)^* = (y)^*$, $x \wedge t = 0$ and $x \vee t$ is dense for some $t \in L$. So, $t \wedge y = 0$ and $(t \vee y)^* = (t)^* \cap (y)^* = (t)^* \cap (x)^* = (t \vee x)^* = \{0\}$ (since $t \vee x$ is dense). Therefore $t \vee y$ is dense and hence $y \in B_D(L)$. Similarly we can prove *(ii)*. □

Theorem 4.10. *For any $x, y \in L$,*
(i) $(x, y) \in \theta$ and $x \in B_D(L)$ implies $\theta = \theta_D$.
(ii) $(x, y) \in \theta$ and $y \in B_D(L)$ implies $\theta = \theta_D$.

Proof. *(i)* Let $x, y \in L$ such that $(x, y) \in \theta$ and $x \in B_D(L)$. Then there exists $s \in L$ such that $x \wedge s = 0$ and $x \vee s$ is dense. Take $d = (x \wedge y) \vee s$. For $t \in L$,

$$\begin{aligned}
t \in (d)^* &\Rightarrow t \wedge x \wedge y = 0 \text{ and } t \wedge s = 0 \\
&\Rightarrow t \wedge y \wedge (x \vee s) = 0 \\
&\Rightarrow t \wedge y = 0 && \text{(since } x \vee s \text{ is dense)} \\
&\Rightarrow t \wedge x = 0 && \text{(since } (x, y) \in \theta) \\
&\Rightarrow t \wedge (x \vee s) = 0 \\
&\Rightarrow t = 0. && \text{(since } x \vee s \text{ is dense)}
\end{aligned}$$

So that d is dense. Now,

$$x \wedge d = x \wedge ((x \wedge y) \vee s) = (x \wedge y) \vee (x \wedge s) = x \wedge y$$

$$y \wedge d = y \wedge ((x \wedge y) \vee s) = (x \wedge y) \vee (y \wedge s) = x \wedge y \text{ (since } (x, y) \in \theta)$$

Therefore $(x, y) \in \theta_D$. On the other hand, let $s, t \in L$ such that $s \wedge d = t \wedge d$, for some $d \in D$. Then

$$\begin{array}{rcll} (s)^{**} & \Rightarrow & (s)^{**} \cap (0)^{*} & \\ & \Rightarrow & (s)^{**} \cap (d)^{**} & \\ & \Rightarrow & (s \wedge d)^{**} & \\ & \Rightarrow & (t \wedge d)^{**} & \text{(since } s \wedge d = t \wedge d) \\ & \Rightarrow & (t)^{**} \cap (d)^{**} & \\ & \Rightarrow & (t)^{**} \cap (0)^{*} & \\ & \Rightarrow & (t)^{**}. & \end{array}$$

Therefore $(s)^{***} = (t)^{***}$ and hence $(s)^{*} = (t)^{*}$. Therefore $(s, t) \in \theta$. Thus $\theta = \theta_D$. Similarly, we can prove *(ii)*. □

Lemma 4.11. *Let L_1, L_2 be two ADLs. Then d_1 & d_2 are dense elements in L_1 & L_2 respectively if and only if (d_1, d_2) is dense in $L_1 \times L_2$.*

Theorem 4.12. *Let L_1 and L_2 be two ADLs. Then $B_D(L_1 \times L_2) = B_D(L_1) \times B_D(L_2)$.*

The relation $\eta = \{(a, b) \in L \times L \mid a \wedge b = b \text{ and } b \wedge a = a\}$ is a congruence relation on L. This η is the smallest congruence on L such that L/η is a lattice. Now, we have the following.

Theorem 4.13. $B_D(L/\eta) \cong B_D(L)$

Proof. Let $a/\eta \in B_D(L/\eta)$. Then there exists $b \in L$ such that $a/\eta \wedge b/\eta = 0/\eta$ and $a/\eta \vee b/\eta$ is a dense element in L/η. So that $a \wedge b = 0$ and, for $x \in L$,

$$\begin{array}{rcll} (a \vee b) \wedge x = 0 & \Rightarrow & ((a \vee b) \wedge x)/\eta = 0/\eta & \\ & \Rightarrow & (a \vee b)/\eta \wedge x/\eta = 0/\eta & \\ & \Rightarrow & x/\eta = 0/\eta & \text{(since } (a \vee b)/\eta \text{ is dense)} \\ & \Rightarrow & (x, 0) \in \eta & \\ & \Rightarrow & x = 0. & \end{array}$$

Therefore $(a \vee b)$ is dense and hence $a \in B_D(L)$. Define $f : B_D(L) \to B_D(L/\eta)$ by $f(a) = a/\eta$, for all $a \in L$. Then f is well-defined and onto.

$$\begin{array}{rcl} Kerf & = & \{x \in B_D(L) \mid f(x) = 0/\eta\} \\ & = & \{x \in B_D(L) \mid x/\eta = 0/\eta\} \\ & = & \{x \in B_D(L) \mid x = 0\} \\ & = & \{0\}. \end{array}$$

Therefore f is one - one and hence f is an isomorphism from $B_D(L)$ onto $B_D(L/\eta)$. □

Theorem 4.14. *An ideal I is dense complemented if and only if $I = (a]$, for some $a \in B_D(L)$.*

Proof. Let I be a dense complemented ideal in L. Then there exists an ideal J in L such that $I \cap J = \{0\}$ and $I \vee J$ is an ideal generated by a dense element in L. Say d. Then $d = a \vee b$ for some $a \in I$ and $b \in J$. So that $a \wedge b = 0$ and then $a \in B_D(L)$. For $x \in L$,

$$\begin{array}{rcl} x \in I & \Rightarrow & x \wedge b \in J \cap I = \{0\} \\ & \Rightarrow & x \wedge b = 0 \end{array}$$

and,

$$\begin{aligned} x \in I &\Rightarrow x \in I \vee J = (d) \\ &\Rightarrow x = d \wedge x = (a \vee b) \wedge x \\ &\Rightarrow x = a \wedge x \\ &\Rightarrow x \in (a). \end{aligned}$$

Therefore $I = (a)$. On the other hand, suppose that $I = (a)$, for some $a \in B_D(L)$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Take $J = (b)$. Then $I \cap J = \{0\}$ and $I \vee J = (a \vee b)$. Therefore I is a dense complemented ideal in L. □

Theorem 4.15. *The following are equivalent for L,*
(i) L is a weak relatively complemented ADL
(ii) Every principal ideal is dense complemented
(iii) $B_D(L) = L$.

Proof. It is sufficient to prove *(ii)* ⇒ *(iii)*. Assume *(ii)* Let $a \in L$. Then, by our assumption, there exists $b \in L$ such that $(a) \cap (b) = (0)$ and $(a) \vee (b) = (a \vee b)$ is a principal ideal generated by a dense element. Therefore $a \wedge b = 0$ and $a \vee b$ is dense. Hence $a \in B_D(L)$. Thus $B_D(L) = L$. □

5 Characterization of $*$-ADLs

In this section, we prove several characterizations of $*$-ADLs in terms of weak relatively complemented ADLs and $B_D(L)$.

$*$-ADL always possesses dense elements where as weak relatively complemented ADL may not have dense elements (See Example 3.3.). In this regard, we observe that the class of $*$-ADLs coincides with the class of weak relatively complemented ADLs provided L has dense elements.

Theorem 5.1. *L is a $*$-ADL if and only if L is weak relatively complemented (provided L should has dense elements).*

Proof. Suppose that L is a $*$-ADL. Let $a, b \in L$. Then there exist $a', b' \in L$ such that $(a)^{**} = (a')^*$ and $(b)^{**} = (b')^*$. Take $x = a' \wedge b$. Then $a \wedge x = a \wedge a' \wedge b = 0$ and,

$$(a \vee x) \wedge (a \vee b) = a \vee (x \wedge b) = a \vee (a' \wedge b \wedge b) = a \vee (a' \wedge b) = a \vee x.$$

So that $(a \vee b)^* \subseteq (a \vee x)^*$. Now, for $t \in L$,

$$\begin{aligned} t \in (a \vee x)^* &\Rightarrow a \wedge t = 0 \text{ and } x \wedge t = 0 \\ &\Rightarrow a \wedge t = 0 \text{ and } a' \wedge b \wedge t = 0 \\ &\Rightarrow t \wedge b \wedge (a \vee a') = 0 \\ &\Rightarrow t \wedge b = 0 \qquad \text{(since } a \vee a' \text{ is dense)} \\ &\Rightarrow t \wedge (a \vee b) = 0 \\ &\Rightarrow t \in (a \vee b)^*. \end{aligned}$$

Therefore $(a \vee x)^* \subseteq (a \vee b)^*$ and hence $(a \vee b)^* = (a \vee x)^*$. Thus L is weak relatively complemented. On the other hand, suppose L is weak relatively complemented. Let $a \in L$. Choose a dense element d in L. Then there exists $b \in L$ such that $a \wedge b = 0$ and $(a \vee b)^* = (a \vee d)^* = \{0\}$. Therefore $a \vee b$ is dense and hence L is a $*$-ADL. □

Remark 5.2. If L has no dense elements, then the above theorem need not be true. For, see Example 3.3., $\mathcal{S}$ is a weak relatively complemented but $\mathcal{S}$ is not a $*$-ADL.

Corollary 5.3. *If L has dense elements, then every relatively complemented ADL is a $*$-ADL.*

Remark 5.4. If L has no dense elements, then the above corollary need not be true. For, see Example 3.2., L is a $*$-ADL but not a relatively complemented (because, for $d_2, m_1 \in L$, there is no element x such that $d_2 \wedge x = 0$ and $d_2 \vee x = d_2 \vee m_1$).

Theorem 5.5. *Every disjunctive $*$-ADL is relatively complemented.*

Proof. Let L be a disjunctive $*$-ADL. Then L has dense elements and, by Theorem 5.1., it is weak relatively complemented. Given $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. $a \vee x = a \vee b$, since L is disjunctive. Therefore L is relatively complemented. □

In [10], Rao and Rao introduced the pseudo complementation on ADL L with maximal elements. A unary operation $* : L \to L$ is said to be pseudo complementation on L, if for any $a, b \in L$, it satisfies the following conditions;
(i) $a \wedge a^* = 0$
(ii) $a \wedge b = 0$ implies $a^* \wedge b = b$
(iii) $(a \vee b)^* = a^* \wedge b^*$.
They have obtained a one to one correspondence between the set of pseudo complementations on L and the set of maximal elements in L.

Theorem 5.6. *For any pseudo complementation $*$ on L and $x \in L$, $x^* = 0$ if and only if x is dense.*

Proof. Let $*$ be a pseudo complementation on L and $x \in L$. Suppose that $x^* = 0$. For $t \in L$,

$$t \in (x)^* \Rightarrow x \wedge t = 0 \Rightarrow x^* \wedge t = t \Rightarrow t = 0.$$

Therefore x is a dense element in L. On the other hand, suppose that x is a dense element in L. Then $x^* \wedge x = 0$ (since $*$ is a pseudo complementation on L). Therefore $x^* = 0$, since x is dense. □

Lemma 5.7. *For any pseudo complementation $*$ on L and $x \in L$, $x \vee x^*$ is dense.*

Proof. For $t \in L$,

$$\begin{aligned} t \in (x \vee x^*)^* &\Rightarrow x \wedge t = 0 \ \& \ x^* \wedge t = 0 \\ &\Rightarrow x^* \wedge t = t \ \& \ x^* \wedge t = 0 \\ &\Rightarrow t = 0. \end{aligned}$$

Therefore $x \vee x^*$ is a dense element in L. □

Theorem 5.8. *Every pseudo complemented ADL is a $*$-ADL and hence weak relatively complemented.*

Proof. Let $*$ be a pseudo complementation on L. Let $x \in L$. Then $x \wedge x^* = 0$ and $x \vee x^*$ is dense (by Lemma [5.7]). Therefore $x \in B_D$ and $B_D = L$. Hence L is a $*$-ADL. Since every pseudo complemented ADL possess a maximal(dense) elements, $*$-ADL is weak relatively complemented (Theorem 5.1.). Thus L is weak relatively complemented. □

Remark 5.9. The converse of the above theorem need not be true. That is, every $*$-ADL need not be a pseudo complemented ADL. For example, let $(\mathbb{N}, \leq)$ be an ADL with least element 1, where $\mathbb{N}$ is the set of all natural numbers, $\leq$ is the natural ordering on $\mathbb{N}$. Then $(\mathbb{N}, \leq)$ is a $*$-ADL but not a pseudo complemented (because $\mathbb{N}$ has no maximal elements).

Lemma 5.10. *If every non-zero element is dense in L, then L is a $*$-ADL.*

Remark 5.11. The converse of the above lemma need not be true. For, see Example 3.4., L is a $*$-ADL but a and b are non-dense elements.

Theorem 5.12. *L is a $*$-ADL if and only if every principal ideal of L is dense complemented.*

Proof. Suppose that L is a $*$-ADL. Let $a \in L$. Then there exists $a' \in L$ such that $a \wedge a' = 0$ and $a \vee a'$ is dense (by Theorem 2.8.). Therefore $(a] \cap (a'] = \{0\}$ and $(a] \vee (a'] = (a \vee a']$ is an ideal generated by a dense element in L. Other side is trivial. □

In [6], Rao, Rao and Lakshman introduced quasi complemented ADLs. That is, by a quasi complemented ADL we mean an ADL L with maximal elements in which for any $a \in L$, there exists an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is maximal. Now, we have the following.

Theorem 5.13. *Let L be an ADL with maximal elements. Then the following are equivalent;*
(i) L is a $*$-ADL in which every dense element is maximal
(ii) L is a quasi complemented ADL
(iii) $B = B_D$
(iv) L is relatively complemented.

Theorem 5.14. *B_D is always a $*$-SubADL of L.*

Proof. In the Theorem 4.2., we confirm that B_D is a subADL of L. Let $a \in B_D$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Therefore $(a)^{**} = (b)^*$ and hence B_D is a $*$-ADL. □

Theorem 5.15. *L is a $*$-ADL if and only if $B_D = L$.*

Proof. Suppose that L is a $*$-ADL. Let $a \in L$. Then there exists $a' \in L$ such that $(a)^{**} = (a')^*$. So that $a \wedge a' = 0$ and $a \vee a'$ is dense. Therefore $a \in B_D$. The converse is trivial. □

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