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The Laplace equation with nonlinear oblique boundary conditions

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Abstract In the present work, we deal with the harmonic problems in a bounded domain of $\mathbb{R}^2$ with the nonlinear boundary integral conditions. After applying the Boundary integral method, a nonlinear boundary integral equation is obtained, the existence and uniqueness of the solution will be a consequence of applying theory of monotone operators.

1 Introduction

For the harmonic problem the simplest boundary condition we can impose specifies u at all points on the boundary Γ and is known as the Dirichlet boundary condition. The Dirichlet problem for the Laplace equation can easily be solved using the boundary integral equation [15]. If the normal derivative of u i,e. $\frac{\partial u}{\partial n}$, where n is the outward normal to the boundary Γ, is specified at all points on the boundary Γ, i,e. the Neumann boundary condition, with $\int_\Gamma \frac{\partial u}{\partial n} ds = 0$, then given the value of u at one point on Γ enables a unique solution to be obtained [15].
In this work, we impose more general boundary conditions, namely the nonlinear integral equation of Urysohn type[8; 11].
Much attention has been paid to the resolution of boundary value problems for partial differential operators with nonlinear boundary conditions by the method of integral equations, in many directions (see for example, K. E. Atkinson [2; 3] and Ruotssalainen and Wendland [12]).

Problems involving nonlinearities form a basis of mathematical models of various steady-state phenomena and processes in mechanics, physics and many other areas of science. Among these is the steady-state heat transfer. Also some electromagnetic problems contain nonlinearities in the boundary conditions, for instance problems, where the electrical conductivity of the boundary is variable [5]. Further applications arise in heat radiation and heat transfer [4; 5].

In the present paper, we look for the solution of the Laplacian equation with nonlinear data of the form:

$$\nabla^2 u(x) = 0 \quad , \quad x \in \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial n}(x) + \frac{\partial u}{\partial \tau}(x) + \int_\Gamma K\left(x, y, u(y)\right) ds_y = f(x) \ \ , \ \ x \in \Gamma. \tag{1.2}$$

We recall that the nonlinear boundary integral operator defined by

$$A(x, u(x)) = \int_\Gamma K\left(x, y, u(y)\right) ds_y \ \ , \ \ x \in \Gamma \tag{1.3}$$

is the nonlinear integral operator of Urysohn type.
In (1), we assume Ω is an open bounded region in $\mathbb{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$, and

$$f : \Gamma \to \mathbb{R} \quad , \quad K : \Gamma \times \Gamma \times \mathbb{R} \to \mathbb{R}$$

are given real value functions.
By the Green representation formula we formulate a nonlinear integral equation on the boundary

Γ of the domain Ω. Under some assumptions on the Kernel of the nonlinear integral equation of Urysohn $K(x,y,u)$ we prove the existence and uniqueness of the solution.

1.1 Definitions and notations

Definition 1.1. ,[1,15] Let $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), |\alpha| \leq m\}$$

Definition 1.2. ,[1,15] Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the Sobolev space :

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); (1+|\xi|^2)^{\frac{s}{2}}|F[u]| \in L^2(\mathbb{R}^n)\}.$$

and the associated norm:

$$\|u\|_{H^s} = (\int_{\mathbb{R}^n}(1+|\xi|^2)^s|F[u]|^2 d\xi)^{\frac{1}{2}}.$$

with $F[.]$ the Fourier transform.

Definition 1.3. ,[1,15] Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\Gamma := \partial\Omega$, we defined

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^n)\}, s \in \mathbb{R}$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+\frac{1}{2}}(\mathbb{R}^n)\}, s > 0 \\ L^2(\Gamma), \quad s = 0 \\ (H^{-s}(\Gamma))' \ \text{(dual space)}, s < 0 \end{cases}$$

2 The Boundary Integral method

2.1 Representative formula and boundary operator

We need the fundamental solution of operator Laplacian Δ in the plane, defined by:

$$E(x,y) = \frac{1}{2\pi}\log|x-y| \tag{2.1}$$

We first consider some standard boundary integral operators. For $x \in \Omega$, the single layer potential is

$$S_\Omega u(x) := -\int_\Gamma E(x,y)\,u(y)\,ds_y$$

and the double layer potential is

$$D_\Omega u(x) := \int_\Gamma u(y)\frac{\partial}{\partial n_y}E(x,y)\,ds_y.$$

Using the Green's identity for harmonic functions

$$u(x) = \int_\Gamma u(y)\frac{\partial}{\partial n_y}E(x,y)\,ds_y - \int_\Gamma \frac{\partial u(y)}{\partial n_y}E(x,y)\,ds_y$$

for $x \in \Omega$, or in the forme

$$u(x) = D_\Omega u(x) + S_\Omega \frac{\partial u(x)}{\partial n} \quad \text{, for } x \in \Omega. \tag{2.2}$$

Letting x tend to point on the boundary Γ and with the continuously of the simple layer potential S_Ω and the jump relation of the double layer potential D_Ω . We can write the integral equation on the boundary Γ.

$$u(x) - Du(x) = S\frac{\partial u(x)}{\partial n} \quad , \quad x \in \Gamma. \tag{2.3}$$

where

$$S\frac{\partial u(x)}{\partial n} := -2\int_\Gamma E(x,y)\frac{\partial u(y)}{\partial n}ds_y \quad , \quad x \in \Gamma$$

and

$$Du(x) := 2\int_\Gamma u(y)\frac{\partial}{\partial n_y}E(x,y)\,ds_y \quad , \quad x \in \Gamma.$$

Clearly, if $u \in H^1(\Omega)$ is the solution of (1), then the Cauchy data $u|_\Gamma$ and $\frac{\partial u}{\partial n}|_\Gamma$ satisfies the integral equation (6).
Then the boundary conditions

$$\frac{\partial u}{\partial n}(x) = -\frac{\partial u}{\partial \tau}(x) - A(x,u(x)) + f(x), x \in \Gamma$$

yields

$$u(x) - Du(x) = -S\frac{\partial u}{\partial \tau}(x) - SA(x,u(x)) + Sf(x) \quad , \quad x \in \Gamma. \tag{2.4}$$

the equation (7) can be written as

$$(I-D)u(x) + S\frac{\partial u}{\partial \tau}(x) + SA(x,u(x)) = Sf(x) \quad , \quad x \in \Gamma. \tag{2.5}$$

we have

$$S\frac{\partial u(x)}{\partial \tau} := -2\int_\Gamma E(x,y)\frac{\partial u(y)}{\partial \tau}ds_y \quad , \quad x \in \Gamma$$

and

$$0 = -2\int_\Gamma \frac{\partial}{\partial \tau}\{u(y)E(x,y)\}ds_y$$

$$= -2\int_\Gamma \frac{\partial u(y)}{\partial \tau}E(x,y)\,ds_y - 2\int_\Gamma \frac{\partial E(x,y)}{\partial \tau}u(y)ds_y$$

then we have

$$S\frac{\partial u(x)}{\partial \tau} = D'u(x) := 2\int_\Gamma \frac{\partial E(x,y)}{\partial \tau}u(y)ds_y \tag{2.6}$$

hence the equation 8 can be written as

$$(I - D + D')u(x) + SA(x,u(x)) = Sf(x) \quad , \quad x \in \Gamma. \tag{2.7}$$

For studying the solvability of the nonlinear equation(10), we give some assumptions to be made here.
(**H1**) We assume a $diam(\Omega) < 1$.
(**H2**) The Kernel $K(.,.,.)$ of the Urysohn operator is a Caratheodory function[11].
(**H3**) We assume that $\frac{\partial K(x,y,u)}{\partial u}$ is measurable satisfying

$$0 < a \leq \frac{\partial K(x,y,u)}{\partial u} \leq b < +\infty,$$

for some constants a, b.

Remark 2.1. 1) The operator S may have eigenfunctions [15], then (**H1**) ensure that the integral operator

$$S : H^s(\Gamma) \to H^{s+1}(\Gamma)$$

is an isomorphism for every $s \in \mathbb{R}$ and

$$(S\mu, \mu) \geq c\|\mu\|^2_{H^{-1/2}}$$

for all $\mu \in H^{-1/2}$ with some positive constant $c > 0$, [15]. By $(.,.)$ we denote the $L^2(\Gamma)$ scalar product.
2) The Kernel $K(.,.,.)$ is a Caratheodory function (**H2**) (i.e) $K(.,.,u)$ is measurable for all $u \in \mathbb{R}$ and $K(x,y,.)$ is continuous for almost all $x, y \in \Gamma$.
3) The assumption (**H3**) implies that the Nemytski operator

$$A : L^2(\Gamma) \to L^2(\Gamma)$$

is Lipschitz continuous and strongly monotonous such that

$$\|Au - Av\|_0 \leq b \;\; mes(\Gamma)\|u - v\|_0$$

and

$$(Au - Av, u - v) \geq a \;\; mes(\Gamma)\|u - v\|_0^2. \tag{2.8}$$

for all $u, v \in L^2(\Gamma)$.

Theorem 2.2. *Let assumptions (H1), (H2) and (H3) hold. Then, for every $f \in H^{-1/2}$ the non-linear boundary integral equation (10) has a unique solution in $H^{\frac{1}{2}}(\Gamma)$.*

Proof. The proof follows from the well-known theorem by Browder and Minty on monotone operators [12, 13].
Since the simple layer potential operator on Γ

$$S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is an isomorphism it is sufficient to consider the unique solvability of equation

$$Bu(x) := S^{-1}(I - D + D')u(x) + A(x, u(x)) = f(x) \quad , \quad x \in \Gamma. \tag{2.9}$$

We shall prove that the operator

$$B : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is continuous and strongly monotonous.
i- in the first we show that B is continuous:
It is clear from the continuity of the mapping properties of the simple and double layer operators, that

$$S^{-1}(I - D + D') : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is continuous. And from (H3)

$$A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is continuous. Hence the boundary integral operator

$$B : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is continuous.

ii- In the second we show that B is strongly monotonous operator.
Let $\mu \in H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\mu(x) := S^{-1}(I - D + D')u(x)$$

for all $u(x) \in H^{\frac{1}{2}}(\Gamma)$, is the normal derivative of the harmonic function

$$w(x)=\int_{\Gamma}u(y)\frac{\partial}{\partial n_y}E\left(x,y\right)ds_y-\int_{\Gamma}\mu(y)E\left(x,y\right)ds_y$$

for $x\in\Omega$, this means that w satisfies the problem

$$\begin{cases}\Delta w(x)=0 \quad , \quad x\in\Omega\\ w\left(x\right)=u\left(x\right) \quad , \quad x\in\Gamma.\end{cases}$$

Then Green's theorem yields

$$(S^{-1}(I-D+D')u,u)=\int_{\Gamma}\mu u ds=\int_{\Gamma}\frac{\partial w}{\partial n}u ds=\int_{\Gamma}\frac{\partial w}{\partial n}w ds=\int_{\Omega}(\nabla w)^2dx.$$

Hence, for all $u,v\in H^{\frac{1}{2}}\left(\Gamma\right)$

$$(S^{-1}(I-D+D')(u-v),u-v)=\int_{\Omega}\left(\nabla(w_1-w_2)\right)^2dx=|w_1-w_2|^2_{H^1(\Omega)} \tag{2.10}$$

where (w_1-w_2) denotes the harmonic function corresponding to the Cauchy data $u-v$ and $S^{-1}(I-D)(u-v)$.
In an other hand, we note that there exists $(\nu_1-\nu_2)\in H^{-\frac{1}{2}}\left(\Gamma\right)$, such that

$$S(\nu_1-\nu_2)=u-v$$

on Γ, [15]. Hence for all $x\in\Omega$, we have

$$S_\Omega(\nu_1-\nu_2)=w_1-w_2.$$

The simple layer potential

$$S_\Omega:H^s\left(\Gamma\right)\to H^{s+3/2}\left(\Omega\right)$$

is continuous,for all $s\in\mathbb{R}$ [15]. Hence for $s=-3/2$ we find

$$\begin{aligned}\|w_1-w_2\|_{L^2(\Omega)}&\le c_1\|\nu_1-\nu_2\|_{H^{-3/2}(\Gamma)}\\&\le c_2\|u-v\|_{H^{-1/2}(\Gamma)}\le c_3\|u-v\|_0,\end{aligned}$$

for some positive constants c_1,c_2 and c_3.
Hence we have

$$\|u-v\|_0\ge\frac{1}{c_3}\|w_1-w_2\|_{L^2(\Omega)}. \tag{2.11}$$

Then with (10) and (11) we get

$$\begin{aligned}(Bu-Bv,u-v)&=(S^{-1}(I-D+D')(u-v),u-v)+(Au-Av,u-v)\\&=|w_1-w_2|^2_{H^1(\Omega)}+(Au-Av,u-v)\end{aligned}$$

and with (9) we get the inequality

$$(Bu-Bv,u-v)\ge|w_1-w_2|^2_{H^1(\Omega)}+a\ \ mes(\Gamma)\|u-v\|_0^2$$

hence with (12) we have

$$\begin{aligned}(Bu-Bv,u-v)&\ge|w_1-w_2|^2_{H^1(\Omega)}+\frac{a\ \ mes(\Gamma)}{c_3^2}\|w_1-w_2\|^2_{L^2(\Omega)}\\&\ge\min\{1,\frac{a\ \ mes(\Gamma)}{c_3^2}\}\left(|w_1-w_2|^2_{H^1(\Omega)}+\|w_1-w_2\|^2_{L^2(\Omega)}\right)\\&\ge\min\{1,\frac{a\ \ mes(\Gamma)}{c_3^2}\}\|w_1-w_2\|^2_{H^1(\Omega)}\\&\ge c_4\|u-v\|^2_{H^{1/2}(\Gamma)}\end{aligned}$$

by the trace theorem [1, 15]. Which completes the proof. □

Example 2.3. Here we give an example to illustrate the theoretical results.
We consider the harmonic problems

$$\Delta u\,(x) = 0 \quad , \quad x \in \Omega$$

$$\frac{\partial u}{\partial n}\,(x) + \frac{\partial u}{\partial \tau}\,(x) + \int_{\Gamma} (2u(y) + \sin u(y))\, ds_y = f(x) \;\; , \;\; x \in \Gamma$$

where the nonlinear boundary integral equation of Urysohn type defined by

$$Au(x) = \int_{\Gamma} (2u(y) + \sin u(y))\, ds_y \;\; , \;\; x \in \Gamma$$

and the domain is

$$\Omega = \{x = (x_1, x_2) | x_1^2 + x_2^2 < r^2 < \frac{1}{4}\}$$

Clearly, the nonlinearity satisfies our assumptions $(H_1), (H_2)$ and (H_3) such that $diam(\Omega) = 2r < 1$.
The Kernel $(2u(y) + \sin u(y))$ of the nonlinear boundary integral equation of Urysohn type is a Caratheodory function. And

$$\frac{\partial K(x, y, u)}{\partial u} = 2 + \cos u(y)$$

is measurable satisfying

$$1 \leq \frac{\partial\,(2u(y) + \sin u(y))}{\partial u} \leq 3 < +\infty.$$

implies that the Nemytski operator

$$A : L^2(\Gamma) \to L^2(\Gamma)$$

is Lipschitz continuous and strongly monotonous such that

$$2\pi r \|u - v\|_0^2 \leq (Au - Av, u - v)$$

$$\|Au - Av\|_0 \leq 6\pi r \|u - v\|_0$$

for all $u, v \in L^2(\Gamma)$.

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