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Intersection Number of Some Comaximal Graphs

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Abstract. Let S be a set and $\mathcal{F} = \{S_1, S_2, \dots, S_p\}$ be a non-empty family of distinct non-empty subsets of S whose union is S. The intersection graph of $\mathcal{F}$ is denoted by $\mathscr{I}(\mathcal{F})$ and defined by $V(\mathscr{I}(\mathcal{F})) = \mathcal{F}$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. Then a graph G is an intersection graph on S if there exists a family $\mathcal{F}$ of subsets for which G and $\mathscr{I}(\mathcal{F})$ are isomorphic graphs. The intersection number $\omega(G)$ of a given graph G is the minimum number of elements in a set S such that G is an intersection graph on S.

Let R be a commutative ring with unity $1 \neq 0$. We associate a simple graph $\Omega(R)$ to R whose vertices are the elements of R, where two distinct vertices x and y of R are adjacent if and only if $Rx + Ry = R$.

We find the intersection number of a complete tripartite graph and $\Omega(R)$ for some classes of R.

1 Introduction

The idea of relating a commutative ring to a graph was introduced by Istvan Beck. In [1] Beck considered $\Gamma(R)$ as a graph with vertices as elements of R where R is a commutative ring with unity and two different vertices a and b are adjacent if and only if $ab = 0$.

In [7] Sharma and Bhatwadekar define another graph on R with vertices as elements of R and two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Later in [6] Maimani and others have studied some of the properties of the same graph and termed the graph as comaximal graph. We denote it by $\Omega(R)$. The authors have studied the structure of comaximal graphs in [5] and found all commutative rings with unity whose comaximal graph is split.

Intersection number of graphs is studied mainly by Chaudam and Parthasarathy in [3]. Intersection number of a triangle free graph with at least three vertices is its number of edges. So, intersection number of a complete bipartite graph other than $K_{1,1}$ is the product of the sizes of its partitions. But when the graph is multipartite, the situation is not very simple as it contains triangles. Thomas in [8] investigated the intersection number of certain classes of multipartite graphs.

In this paper we find the intersection number of a complete tripartite graph. We use this to find the intersection number of $\Omega(R)$ for certain classes of R namely, the direct product of finite fields. Also we find the intersection number of $\Omega(\mathbb{Z}_{p^2})$, when p is any prime and $\Omega(\mathbb{Z}_{4p})$, when p is an odd prime.

2 Preliminaries

In this section we list some concepts and results in graph theory which are useful in the subsequent discussion.

Definition 2.1. Let S be a set and $\mathcal{F} = \{S_1, S_2, \dots, S_p\}$ be a non-empty family of distinct non-empty subsets of S whose union is S. The intersection graph of $\mathcal{F}$ is denoted by $\mathscr{I}(\mathcal{F})$ and defined by $V(\mathscr{I}(\mathcal{F})) = \mathcal{F}$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. Then a graph G is an intersection graph on S if there exists a family $\mathcal{F}$ of subsets for which G and $\mathscr{I}(\mathcal{F})$ are isomorphic graphs.

Definition 2.2. The intersection number $\omega(G)$ of a given graph G is the minimum number of elements in a set S such that G is an intersection graph on S.

Theorem 2.3 ([4]). *Every finite graph is an intersection graph.*

Theorem 2.4 ([4]). *If G is a connected (p, q)-graph and $p \geq 3$ then $\omega(G) \leq q$.*

Theorem 2.5 ([4]). *Let G be a connected (p, q)-graph with $p > 3$. Then $\omega(G) = q$ if and only if G has no triangles.*

Theorem 2.6 ([3]). *If H is an induced subgraph of a graph G; then $\omega(G) \geq \omega(H)$.*

Theorem 2.7 ([3]). $\omega(K_p) = \lceil 1 + \log_2 p \rceil$

Theorem 2.8 ([2]). *Every k-regular bipartite graph $(k > 0)$ has a perfect matching and (hence by induction) is 1-factorable.*

3 Intersection Number of $K_{l,m,n}$

It follows from theorem 2.5 that $\omega(K_{m,n}) = mn$ if $m > 1$ or $n > 1$. In this section we investigate the intersection number of a complete tripartite graph. The following result may be known but we are not able to find a reference. Hence we include a proof of the result.

Theorem 3.1. $\omega(K_{l,m,n}) = mn$ *where* $l \leq m \leq n$ *and* $m \geq 2$.

Proof. Let $U = \{u_1, \ldots, u_l\}$, $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_n\}$ be the partition sets of $V(K_{l,m,n})$. Since $K_{m,n}$ is an induced subgraph of $K_{l,m,n}$, $\omega(K_{l,m,n}) \geq mn$. Now let,

$$B_i = \text{Set of edges through } v_i \text{ in the induced subgraph } K_{m,n} \ (1 \leq i \leq m)$$

and

$$C_j = \text{Set of edges through } w_j \text{ in the induced subgraph } K_{m,n} \ (1 \leq j \leq n).$$

First we assign B_i to v_i and C_j to w_j. We want to find $A_1, A_2, \ldots, A_l$ to allocate the vertices of U. Consider the subgraph $K_{m,m}$ induced by $\{v_1, \ldots, v_m, w_1, \ldots, w_m\}$. By theorem 2.8, it contains m edge disjoint perfect matchings. We choose any l edge-disjoint perfect matchings, say, $A'_1, A'_2, \ldots, A'_l$. Now we choose one edge from C_j for $m+1 \leq j \leq n$ and adjoin these edges to A'_1 to get A_1. Note that $|C_j| = m$. So by choosing edges from the remaining, it is possible to form the pair wise disjoint sets $A_2, \ldots, A_l$. Then each A_k $(1 \leq k \leq l)$ contains one element from each B_i and C_j $(1 \leq i \leq m, 1 \leq j \leq n)$. Thus $\omega(K_{l,m,n}) = mn$. □

Remark 3.2. $\omega(K_{1,1,n}) = n + 1$ when $n > 1$.

4 Intersection Number of $\Omega(R)$

In this section we find the intersection number of $\Omega(R)$ for some classes of R. First we consider the class of rings which are direct product of two finite fields. In what follows, $R^* = R - \{0\}$.

Theorem 4.1. *Let F and K be two finite fields and $|F| = m$ and $|K| = n$ $(m > 2, n > 2)$. Then,* $\omega(\Omega(F \times K)) = (m-1)(n-1) + 1$.

Proof. First we identify the structure of $\Omega(F \times K)$ as in figure 1.

Let $S = \{1, 2, \ldots, (m-1)(n-1)+1\}$ and let V_1 and V_2 be the partition set of $V(K_{m-1,n-1})$ with $|V_1| = m-1$ and $|V_2| = n-1$. That is, $V_1 = \{(\alpha, 0) : \alpha \in F^*\}, V_2 = \{(0, \beta) : \beta \in K^*\}$.

We partition the set $S - \{(m-1)(n-1)+1\}$ into $(m-1)$ disjoint subsets of size $(n-1)$ and allocate them to the vertices of V_1. We fix the order of the elements in the allocated subsets. The collection of the first elements in the allocated subsets will form the subset to be allocated to the first vertex in V_2; the collection of second elements in the allocated subsets will form the subset to be allocated to the second vertex in V_2 and so on. The zero element can be allocated with the

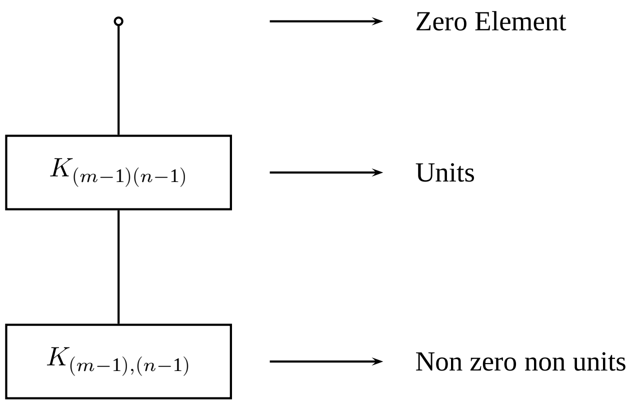

Figure 1. $\Omega(F \times K)$

symbol $(m-1)(n-1)+1$. We allocate the sets $S-\{1\}, S-\{2\}\ldots, S-\{(m-1)(n-1)\}$ to the $(m-1)(n-1)$ units. Thus, we get a proper allocation. Therefore,

$$\omega(\Omega(F \times K)) \leq (m-1)(n-1)+1.$$

But, since $K_{m-1,n-1}$ is an induced subgraph and the zero element is not adjacent to any of the vertices of this induced subgraph, we get

$$\omega(\Omega(F \times K)) \geq (m-1)(n-1)+1.$$

Hence,

$$\omega(\Omega(F \times K)) = (m-1)(n-1)+1. \qquad \square$$

Corollary 4.2. $\omega(\Omega(\mathbb{Z}_{pq})) = (p-1)(q-1)+1$, *if* p *and* q *are distinct odd primes.*

In the above theorem we assumed that the two fields must have more than two elements. Now we investigate the case when $m=2$ or $n=2$.

Theorem 4.3. *Let* F *be a finite field with* $|F| = n (n > 2)$. *Then* $\omega(\Omega(\mathbb{Z}_2 \times F)) = n + \lceil \log_2(n-1) \rceil$.

Proof. First we identify the structure of the graph as in figure 2.

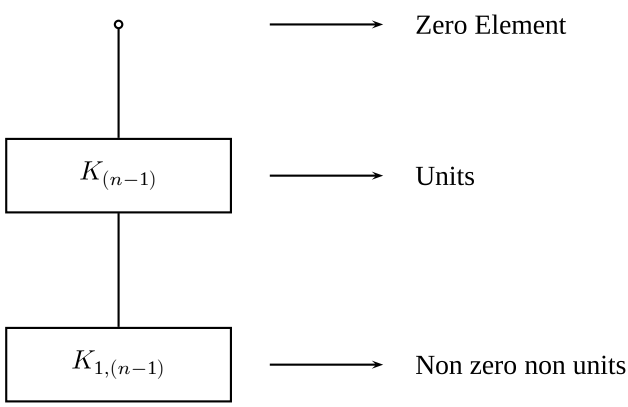

Figure 2. $\Omega(\mathbb{Z}_2 \times F)$

Let $S = \{a_1, \ldots, a_n, b_1, \ldots, b_t\}$; $t = \lceil \log_2(n-1) \rceil$. Let V_1 and V_2 with $|V_1| = 1$ and $|V_2| = n-1$ be the partition sets for $K_{1,n-1}$. That is, $V_1 = \{(1,0)\}$ and $V_2 = \{(0,\alpha) \,:\, \alpha \in F^*\}$.

The subset $\{a_1, \ldots, a_{n-1}\}$ can be allocated to the vertex in V_1 and the singletons $\{a_1\}, \ldots, \{a_{n-1}\}$ can be assigned to the vertices in V_2. The zero element can be allocated by $\{a_n\}$.

Now, the remaining elements in S are $\lceil \log_2(n-1) \rceil$ in number. Thus by allocating the sets formed by those $\lceil \log_2(n-1) \rceil$ elements to K_{n-1} and adjoining the elements $a_1, \ldots, a_n$ to each of these subsets, we get a proper allocation for $\Omega(\mathbb{Z}_2 \times F)$. Therefore,

$$\omega(\Omega(\mathbb{Z}_2 \times F)) \leq n + \lceil \log_2(n-1) \rceil.$$

But, $\omega(K_{1,n-1}) = n-1$ and the above discussed allocation is the unique allocation with $(n-1)$ symbols, say, $1, 2, \ldots, n-1$ for $K_{1,n-1}$. Any of these $(n-1)$ symbols cannot be allocated to the zero element. Therefore we need another symbol, say, n. Now, we observe that, each set allocated to the vertices of K_{n-1}, must be a superset of $\{1, 2, \ldots, n\}$. That is, we need a set of symbols with at least $(n-1)$ subsets. For this, we need at least $\lceil \log_2(n-1) \rceil$ symbols in that set. Therefore

$$\omega(\Omega(\mathbb{Z}_2 \times F)) \geq n + \lceil \log_2(n-1) \rceil$$

Hence $\omega(\Omega(\mathbb{Z}_2 \times F)) = n + \lceil \log_2(n-1) \rceil$. □

Corollary 4.4. $\omega(\Omega(\mathbb{Z}_{2p})) = p + \lceil \log_2(p-1) \rceil$ *where p is an odd prime.*

Theorem 4.5. $\omega(\Omega(\mathbb{Z}_{p^2})) = p + \lceil \log_2 p(p-1) \rceil$ *where p is a prime.*

Proof. We identify the structure of $\Omega(\mathbb{Z}_{p^2})$ as in figure 3.

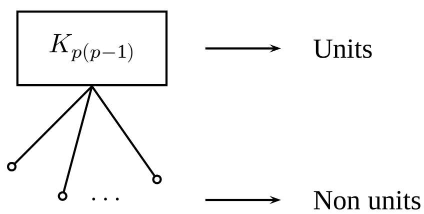

Figure 3. $\Omega(\mathbb{Z}_{p^2})$

The arguments of the proof are similar to that of the proof of theorem 4.3. □

Theorem 4.6. *Let F_i $(1 \leq i \leq 3)$ be finite fields with $|F_i| = n_i$ $\quad (n_1 \leq n_2 \leq n_3, n_3 \neq 2)$. Then,*

$$\omega(\Omega(F_1 \times F_2 \times F_3)) = (n_1-1)(n_2-1)(n_3-1)^2 + 3(n_1-1)(n_2-1)(n_3-1) + 1.$$

Proof. We identify the structure of $\Omega(F_1 \times F_2 \times F_3)$ as in figure 4.

In the figure 4, $I_n = \overline{K}_n$.

Since $K_{(n_1-1)(n_2-1),(n_1-1)(n_3-1),(n_2-1)(n_3-1)}$ is an induced subgraph, $\omega(\Omega(F_1 \times F_2 \times F_3)) \geq (n_1-1)(n_2-1)(n_3-1)^2$.

For allocating the vertices of $I_{(n_1-1)(n_3-1)}$ and $I_{(n_2-1)(n_3-1)}$ we have to use all the $(n_1-1)(n_2-1)(n_3-1)^2$ symbols. Therefore we cannot use any of these symbols to allocate the vertices of $I_{(n_3-1)}$. Also $I_{(n_3-1)}$ together with $I_{(n_1-1)(n_2-1)}$ forms $K_{(n_1-1)(n_2-1),(n_3-1)}$. Therefore we need $(n_1-1)(n_2-1)(n_3-1)$ new symbols. Similarly, for the vertices of $I_{(n_2-1)}$ we cannot use any of the symbols already used. Thus, considering $I_{(n_1-1)}$ also, we get,

$$\omega(\Omega(F_1 \times F_2 \times F_3)) \geq (n_1-1)(n_2-1)(n_3-1)^2 + 3(n_1-1)(n_2-1)(n_3-1).$$

We need a special symbol for the zero-element. Therefore,

$$\omega(\Omega(F_1 \times F_2 \times F_3)) \geq (n_1-1)(n_2-1)(n_3-1)^2 + 3(n_1-1)(n_2-1)(n_3-1) + 1.$$

From the above discussion it is clear that $(n_1-1)(n_2-1)(n_3-1)^2 + 3(n_1-1)(n_2-1)(n_3-1) + 1$ symbols are enough to allocate the non-units of $F_1 \times F_2 \times F_3$. Now it is enough to show that units of $F_1 \times F_2 \times F_3$ can be allocated by these symbols.

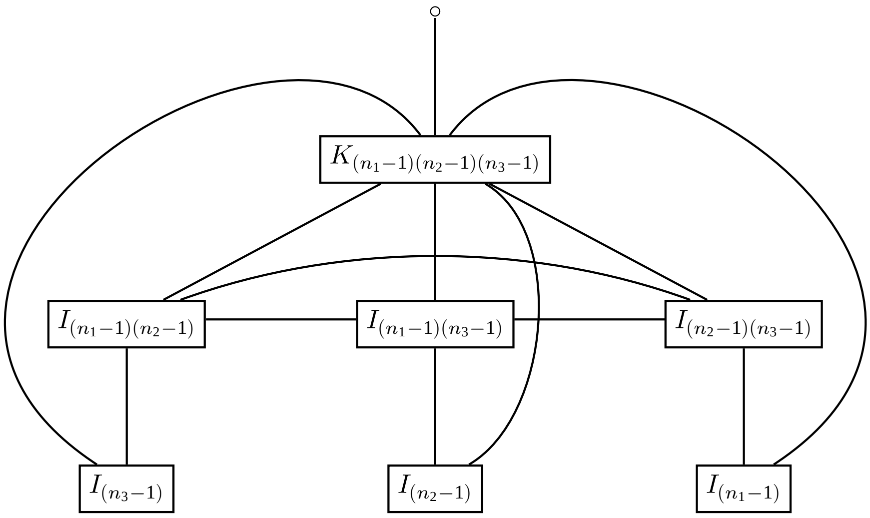

Figure 4. $\Omega(F_1 \times F_2 \times F_3)$

Let $S = \{1, 2, \ldots, (n_1-1)(n_2-1)(n_3-1)^2 + 3(n_1-1)(n_2-1)(n_3-1)+1\}$. Then $S-\{1\}, S-\{2\}, \ldots, S-\{(n_1-1)(n_2-1)(n_3-1)\}$ can be allocated to the units, assuming the zero element is not allocated with any of the symbols $1, 2, \ldots, (n_1-1)(n_2-1)(n_3-1)$. Thus we get a proper allocation. Hence,

$$\omega(\Omega(F_1 \times F_2 \times F_3)) = (n_1-1)(n_2-1)(n_3-1)^2 + 3(n_1-1)(n_2-1)(n_3-1)+1. \quad \square$$

In theorem 4.6 we insisted that $n_3 \neq 2$. In the following theorem we consider the case when $n_3 = 2$. That is, the case when $n_1 = n_2 = n_3 = 2$.

Theorem 4.7. $\omega(\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 5$.

Proof. The figure 5 gives $\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

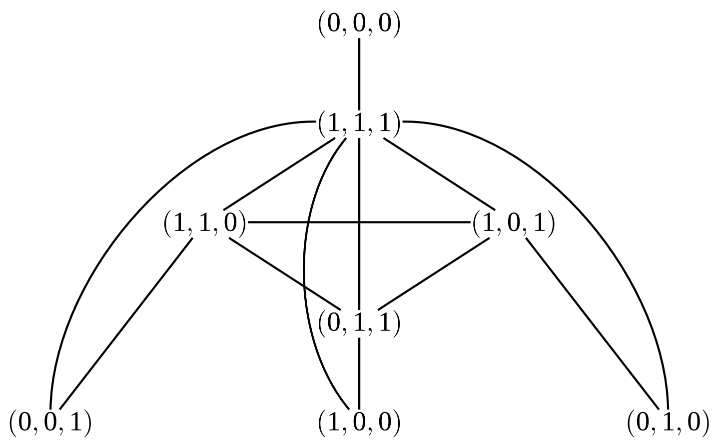

Figure 5. $\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

$\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ contains K_4 as an induced subgraph. So, $\omega(\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \geq \omega(K_4) = 3$. But these symbols are used to allocate $(1,0,0)$, $(1,0,1)$ and $(0,1,1)$. So, we need a new symbol for allocating $(0,0,0)$. Therefore, $\omega(\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \geq 4$. But by inspection, $\omega(\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \leq 5$.

Now it is enough to show that it is impossible to get a proper allocation with a set of 4 symbols.

Let $S = \{a, b, c, d\}$. One element of S, say a should be reserved for the zero element and a must be in the set allocated to $(1,1,1)$ and should not be in any other allocated sets. Then there are only 3 elements remaining in S. Thus we have only 7 subsets available, namely, $\{b,c,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{b\}, \{c\}$ and $\{d\}$. Also there are 7 vertices to be allocated. The graph structure demands that the 7 subsets to be allocated must contain 4 subsets which are pairwise not disjoint. The only possible such a collection is $\{\{b,c,d\}, \{b,c\}, \{b,d\}, \{c,d\}\}$. Then the remaining vertices need 3 pairwise disjoint subsets. The only possibility is $\{\{b\}, \{c\}, \{d\}\}$. Without loss of generality, assume that $\{b\}, \{c\}$ and $\{d\}$ are allocated to $(0,0,1), (0,1,0)$ and $(1,0,0)$ respectively. Note that $(0,0,1)$ is adjacent to only 2 vertices, but among the first 4 subsets, 3 subsets contain b. So, there doesn't exist a proper allocation with 4 elements.

Thus, $\omega(\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 5$. □

Note that the formula given in theorem 4.6 works here also.

Corollary 4.8. $\omega(\Omega(\mathbb{Z}_{pqr})) = (p-1)(q-1)(r-1)^2+3(p-1)(q-1)(r-1)+1$ *where* $p < q < r$ *are odd primes.*

Theorem 4.9. $\omega(\Omega(\mathbb{Z}_{4p})) = 4(p-1)+2$ *where* p *is an odd prime.*

Proof. We identify the structure of $\Omega(\mathbb{Z}_{4p})$ as in figure 6.

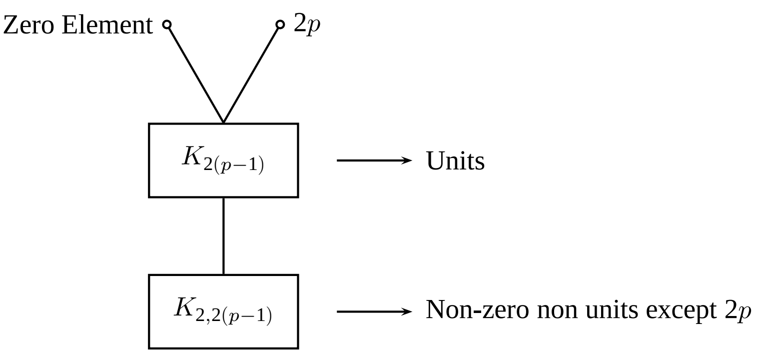

Figure 6. $\Omega(\mathbb{Z}_{4p})$

$\omega(\Omega(\mathbb{Z}_{4p})) \geq 4(p-1)$, since $K_{2,2(p-1)}$ is an induced subgraph. Also since the zero element and $2p$ are not adjacent to any of the vertices of $K_{2,2(p-1)}$, we need two more symbols. That is, $\omega(\Omega(\mathbb{Z}_{4p})) \geq 4(p-1)+2$.

Now let $S = \{1, 2, \ldots, 4(p-1), 4(p-1)+1, 4(p-1)+2\}$.

Using the symbols $1, 2, \ldots, 4(p-1)$ we can allocate the vertices of $K_{2,2(p-1)}$. Allocate $\{4(p-1)+1\}$ to the zero element and $\{4(p-1)+2\}$ to $2p$. Then we can allocate the sets $S-\{1\}, S-\{2\}, \ldots, S-\{2(p-1)\}$ to the vertices of $K_{2(p-1)}$ and get a proper allocation. Hence,

$$\omega(\Omega(\mathbb{Z}_{4p})) = 4(p-1)+2.$$ □

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