Palestine Journal of Mathematics

Vol. 6(Special Issue: I) (2017) , 80–94

Conformal maps, biharmonic maps and the warped product

Seddik Ouakkas

Communicated by Ali Wehbe

MSC 2010 Classifications: 51-XX, 53C43.

Keywords and phrases: Equidimensional manifolds, Non-harmonic biharmonic maps, The biharmonicity.

Abstract. In this paper we study some properties of a conformal maps between equidimensional manifolds, we construct new example of non-harmonic biharmonic maps and we characterize the biharmonicity of some maps on the warped product manifolds.

1 Introduction.

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g \tag{1.1}$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau(\phi) = Tr_g \nabla d\phi = 0, \tag{1.2}$$

$\tau(\phi)$ is called the tension field of ϕ. One can refer to [7-10] for background on harmonic maps. In the context of harmonic maps, the stress-energy tensor was studied in details by Baird and Eells in [2]. Indeed, the Euler-Lagrange equation associated to the energy is the vanishing of the tension field $\tau(\phi) = Tr_g \nabla d\phi$, and the stress-energy tensor for a map $\phi : (M^m, g) \longrightarrow (N^n, h)$ defined by

$$S(\phi) = e(\phi)g - \phi^* h.$$

The relation between $S(\phi)$ and $\tau(\phi)$ is given by

$$div S(\phi) = -h(\tau(\phi), d\phi).$$

The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional :

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g \tag{1.3}$$

Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0, \tag{1.4}$$

where ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on M, then

$$Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) = \left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i}e_i}\right)\tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ.
In analogy with harmonic maps, Jiang In [11] has constructed for a map ϕ the stress bi-energy tensor defined by

$$S_2(\phi) = \left(\frac{-1}{2}|\tau(\phi)|^2 + divh(\tau(\phi), d\phi)\right)g - 2symh(\nabla\tau(\phi), d\phi),$$

where

$$symh\left(\nabla\tau(\phi), d\phi\right)(X,Y) = \frac{1}{2}\left\{h\left(\nabla_X\tau(\phi), d\phi\left(Y\right)\right) + h\left(\nabla_Y\tau(\phi), d\phi\left(X\right)\right)\right\},$$

for any $X, Y \in \Gamma\left(TM\right)$. The stress bi-energy tensor of ϕ satisfies the following relationship

$$divS_2(\phi) = h\left(\tau_2(\phi), d\phi\right).$$

Clearly any harmonic map is biharmonic, therefore it is interesting to construct non-harmonic biharmonic maps. In [4] the authors found new examples of biharmonic maps by conformally deforming the domain metric of harmonic ones. While in [6] the author analyzed the behavior of the biharmonic equation under the conformal change the domain metric, he obtained metrics $\widetilde{g} = e^{2\gamma}$ such that the idendity map $Id : (M,g) \longrightarrow (M,\widetilde{g})$ is biharmonic non-harmonic. Moreover, in [14] the author gave some extensions of the result in [6] together with some further constructions of biharmonic maps. The author in [13] deform conformally the codomain metric in order to render a semi-conformal harmonic map biharmonic. In [5] the authors studied the case where $\phi : (M^n, g) \longrightarrow (N^n, h)$ is a conformal mapping between equidimensional manifolds where they show that a conformal mapping ϕ is biharmonic if and only if the gradient of its dilation satisfies a second order elliptic partial differential equation. We can refer the reader to [12], for a survey of biharmonic maps. In the first section of this paper, we present some properties for a conformal mapping $\phi : (M^n, g) \longrightarrow (N^n, h)$, we prove that the stress bi-energy tensor depend only on the dilation (Theorem 2.1) and we calculate the bitension field of ϕ (Theorem 2.2). In the last section we study the biharmonicity of some maps on the warped product (Theorem 3.1 and 3.2), with this setting we obtain new examples of biharmonic non-harmonic maps.

2 Some properties for conformal maps.

We study conformal maps between equidimensional manifolds of the same dimension $n \geq 3$. Note that by a result in [5], any such map can have no critical points and so is a local conformal diffeomorphism. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exist a C^∞ function $\lambda : M \to \mathbb{R}_+^*$ such that for any $X, Y \in \Gamma(TM)$:

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X,Y).$$

The function λ is called the dilation for the map ϕ. The tension field and the stress energy tensor for a conformal map are given by (see [1]):

Proposition 2.1. Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map of dilation λ, we have

$$(i)\ \ divS(\phi) = (n-2)\lambda^2 d\ln\lambda, \tag{2.1}$$

$$(ii)\ \ divh(\tau(\phi), d\phi) = (2-n)\left(2\lambda^2\left|grad\ln\lambda\right|^2 + \lambda^2\Delta\ln\lambda\right). \tag{2.2}$$

$$(iii)\ \ \tau(\phi) = (2-n)d\phi(grad\ln\lambda). \tag{2.3}$$

$$(iv)\ \ \left|\tau(\phi)\right|^2 = (2-n)^2\lambda^2\left|grad\ln\lambda\right|^2. \tag{2.4}$$

Note that the conformal map $\phi : (M^n, g) \to (N^n, h)$ of dilation λ is harmonic if and only if $n = 2$ or the dilation λ is constant.

In the first, wa calculate the stress bi-energy tensor for a conformal map ϕ when we prove that $S_2(\phi)$ depend only the dilation.

Theorem 2.1. Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then we have

$$S_2(\phi) = (2-n)\lambda^2\left\{\left(\frac{n-2}{2}\left|grad\ln\lambda\right|^2 + \Delta\ln\lambda\right)g - 2\nabla d\ln\lambda\right\}, \tag{2.5}$$

and the trace of $S_2(\phi)$ is given by

$$TrS_2(\phi) = -(2-n)^2\lambda^2\left\{\frac{n}{2}\left|grad\ln\lambda\right|^2 + \Delta\ln\lambda\right\}. \tag{2.6}$$

To prove Theorem 2.1, we need the following Lemma :

Lemma 2.1. Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then for any function $f \in C^\infty(M)$ and for any $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} h\left(\nabla_X d\phi\left(gradf\right), d\phi\left(Y\right)\right) &= \lambda^2\left(X\left(\ln\lambda\right)Y\left(f\right) - Y\left(\ln\lambda\right)X\left(f\right)\right) \\ &+ \lambda^2\nabla df\left(X,Y\right) + \lambda^2 d\ln\lambda\left(gradf\right)g\left(X,Y\right). \end{aligned} \tag{2.7}$$

Proof of Lemma 2.1. Let $f \in C^\infty(M)$, for any $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} h\left(\nabla_X d\phi\left(gradf\right), d\phi\left(Y\right)\right) &= X\left(\lambda^2 g\left(gradf, Y\right)\right) - h\left(d\phi\left(gradf\right), \nabla_X d\phi\left(Y\right)\right) \\ &= X\left(\lambda^2\right)g\left(gradf, Y\right) + \lambda^2 g\left(\nabla_X gradf, Y\right) + \lambda^2 g\left(gradf, \nabla_X Y\right) \\ &- h\left(d\phi\left(gradf\right), \nabla d\phi\left(X,Y\right)\right) - h\left(d\phi\left(gradf\right), d\phi\left(\nabla_X Y\right)\right) \\ &= X\left(\lambda^2\right)g\left(gradf, Y\right) + \lambda^2 g\left(\nabla_X gradf, Y\right) + \lambda^2 g\left(gradf, \nabla_X Y\right) \\ &- h\left(d\phi\left(gradf\right), \nabla d\phi\left(X,Y\right)\right) - \lambda^2 g\left(gradf, \nabla_X Y\right). \end{aligned}$$

Note that

$$g\left(\nabla_X gradf, Y\right) = \nabla df\left(X,Y\right),$$

then we obtain

$$h\left(\nabla_X d\phi\left(gradf\right), d\phi\left(Y\right)\right) = 2\lambda^2 X\left(\ln\lambda\right)Y\left(f\right) + \lambda^2\nabla df\left(X,Y\right) - h\left(d\phi\left(gradf\right), \nabla d\phi\left(X,Y\right)\right).$$

By similary, we have

$$h\left(\nabla_Y d\phi\left(gradf\right), d\phi\left(X\right)\right) = 2\lambda^2 Y\left(\ln\lambda\right)X\left(f\right) + \lambda^2\nabla df\left(X,Y\right) - h\left(d\phi\left(gradf\right), \nabla d\phi\left(X,Y\right)\right).$$

Then, we deduce that

$$\begin{aligned} h\left(\nabla_X d\phi\left(gradf\right), d\phi\left(Y\right)\right) &= h\left(d\phi\left(X\right), \nabla_Y d\phi\left(gradf\right)\right) \\ &+ 2\lambda^2\left(X\left(\ln\lambda\right)Y\left(f\right) - Y\left(\ln\lambda\right)X\left(f\right)\right). \end{aligned} \tag{2.8}$$

For the term $h\left(d\phi\left(X\right), \nabla_Y d\phi\left(gradf\right)\right)$, we have

$$\begin{aligned} h\left(\nabla_Y d\phi\left(gradf\right), d\phi\left(X\right)\right) &= h\left(\nabla d\phi\left(gradf, Y\right), d\phi\left(X\right)\right) + \lambda^2 g\left(\nabla_Y gradf, X\right) \\ &= h\left(\nabla_{gradf} d\phi\left(Y\right), d\phi\left(X\right)\right) - \lambda^2 g\left(\nabla_{gradf} Y, X\right) \\ &+ \lambda^2 g\left(\nabla_Y gradf, X\right) \\ &= gradf\left(\lambda^2 g\left(X,Y\right)\right) - h\left(\nabla_{gradf} d\phi\left(X\right), d\phi\left(Y\right)\right) \\ &- \lambda^2 g\left(\nabla_{gradf} Y, X\right) + \lambda^2 g\left(\nabla_Y gradf, X\right) \\ &= 2\lambda^2 d\ln\lambda\left(gradf\right)g\left(X,Y\right) - h\left(\nabla d\phi\left(X, gradf\right), d\phi\left(Y\right)\right) \\ &+ \lambda^2 g\left(\nabla_Y gradf, X\right). \end{aligned}$$

We deduce that

$$\begin{aligned} h\left(\nabla_Y d\phi\left(gradf\right), d\phi\left(X\right)\right) &= -h\left(\nabla_X d\phi\left(gradf\right), d\phi\left(Y\right)\right) + 2\lambda^2\nabla df\left(X,Y\right) \\ &+ 2\lambda^2 d\ln\lambda\left(gradf\right)g\left(X,Y\right). \end{aligned} \tag{2.9}$$

Finally, if we replace (2.9) in (2.8), we obtain

$$\begin{aligned} h\left(\nabla_X d\phi\left(gradf\right), d\phi\left(Y\right)\right) &= \lambda^2\left(X\left(\ln\lambda\right)Y\left(f\right) - Y\left(\ln\lambda\right)X\left(f\right)\right) \\ &+ \lambda^2\nabla df\left(X,Y\right) + \lambda^2 d\ln\lambda\left(gradf\right)g\left(X,Y\right). \end{aligned}$$

This completes the proof of Lemma 2.1.

Remark 2.1. Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then if we consider $f = \ln\lambda$, the equation (2.7) gives

$$h\left(\nabla_X d\phi\left(grad \ln\lambda\right), d\phi\left(Y\right)\right) = \lambda^2\left(\nabla d\ln\lambda\left(X,Y\right) + \left|grad \ln\lambda\right|^2 g\left(X,Y\right)\right).$$

Proof of Theorem 2.1. By definition, the stress bi-energy tensor is given by :

$$S_2(\phi) = \left(-\frac{1}{2}\left|\tau(\phi)\right|^2 + divh\left(\tau(\phi), d\phi\right)\right) g - 2symh\left(\nabla\tau(\phi), d\phi\right). \tag{2.10}$$

Using the equations (1.2) et (1.4) for the Proposition 2.1, we have

$$-\frac{1}{2}\left|\tau(\phi)\right|^2 + divh\left(\tau(\phi), d\phi\right) = (2-n)\lambda^2\left(\frac{n+2}{2}\left|grad \ln\lambda\right|^2 + \Delta\ln\lambda\right). \tag{2.11}$$

Calculate now $symh\left(\nabla\tau(\phi), d\phi\right)$, we have by definition for any $X, Y \in \Gamma\left(TM\right)$

$$\begin{aligned} symh\left(\nabla\tau(\phi), d\phi\right)\left(X,Y\right) =& \frac{1}{2}\left(h\left(\nabla_X\tau\left(\phi\right), d\phi\left(Y\right)\right) + h\left(\nabla_Y\tau\left(\phi\right), d\phi\left(X\right)\right)\right) \\ =& \frac{2-n}{2} h\left(\nabla_X d\phi\left(grad\ln\lambda\right), d\phi\left(Y\right)\right) \\ &+ \frac{2-n}{2} h\left(\nabla_Y\left(grad\ln\lambda\right), d\phi\left(X\right)\right). \end{aligned}$$

By Lemma 2.1, we have

$$h\left(\nabla_X d\phi\left(grad \ln\lambda\right), d\phi\left(Y\right)\right) = \lambda^2\left(\nabla d\ln\lambda\left(X,Y\right) + \left|grad \ln\lambda\right|^2 g\left(X,Y\right)\right)$$

and

$$h\left(\nabla_Y d\phi\left(grad \ln\lambda\right), d\phi\left(X\right)\right) = \lambda^2\left(\nabla d\ln\lambda\left(X,Y\right) + \left|grad \ln\lambda\right|^2 g\left(X,Y\right)\right),$$

then

$$symh\left(\nabla\tau(\phi), d\phi\right)\left(X,Y\right) = \left(2-n\right)\lambda^2\left(\nabla d\ln\lambda\left(X,Y\right) + \left|grad \ln\lambda\right|^2 g\left(X,Y\right)\right). \tag{2.12}$$

If we substitute (2.11) and (2.12) in (2.10), we conclude that

$$S_2(\phi) = \left(2-n\right)\lambda^2\left\{\left(\frac{n-2}{2}\left|grad \ln\lambda\right|^2 + \Delta\ln\lambda\right) g - 2\nabla d\ln\lambda\right\}$$

Calculate now the trace of stress bi-energy tensor. Let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame on M, we have

$$\begin{aligned} Tr_g S_2(\phi) = S_2(\phi)(e_i, e_i) \\ &= \left(2-n\right)\lambda^2\left(\frac{n-2}{2}\left|grad \ln\lambda\right|^2 + \Delta\ln\lambda\right) g\left(e_i, e_i\right) \\ &- 2\left(2-n\right)\lambda^2\nabla d\ln\lambda\left(e_i, e_i\right) \\ &= \left(2-n\right)n\lambda^2\left(\frac{n-2}{2}\left|grad \ln\lambda\right|^2 + \Delta\ln\lambda\right) \\ &- 2\left(2-n\right)\lambda^2\left(\Delta\ln\lambda\right) \\ &= \left(2-n\right)\lambda^2\left\{\frac{n\left(n-2\right)}{2}\left|grad \ln\lambda\right|^2 + \left(n-2\right)\Delta\ln\lambda\right\}. \end{aligned}$$

Then

$$TrS_2(\phi) = -\left(2-n\right)^2\lambda^2\left\{\frac{n}{2}\left|grad \ln\lambda\right|^2 + \Delta\ln\lambda\right\}.$$

By calculating the Laplacian of the function $\lambda^{\frac{n}{2}}$ and by using

$$\Delta\lambda^{\frac{n}{2}} = \frac{n}{2}\lambda^{\frac{n}{2}}\left(\frac{n}{2}\left|grad \ln\lambda\right|^2 + \Delta\ln\lambda\right),$$

we obtain immediately the following corollary

Corollary 2.1. Let $\phi : (M^n, g) \to (N^n, h)$, $(n \neq 2)$ to be a conformal map of dilation λ, then the trace of $S_2(\phi)$ is zero if and only if the function $\lambda^{\frac{n}{2}}$ is harmonic.

The bi-tension field of the conformal map is given by

Theorem 2.2. Let $\phi : (M^n, g) \to (N^n, h)$, $(n \geq 3)$ to be a conformal map of dilation λ, then bi-tension field of ϕ is given by :

$$\tau_2(\phi) = (n-2)\, d\phi\,(H)$$

where

$$\begin{aligned} H = {} & grad\Delta \ln\lambda - \frac{(n-6)}{2} grad\left(|grad \ln\lambda|^2\right) + 2Ricci^M\,(grad \ln\lambda) \\ & - \left(2\,(\Delta \ln\lambda) + (n-2)\,|grad \ln\lambda|^2\right) grad \ln\lambda. \end{aligned} \tag{2.13}$$

To prove the Theorem 2.2, we need two Lemmas. In the first Lemma, we give a simple formula of the term $Tr_g\left(\nabla^\phi\right)^2 d\phi\,(grad\gamma)$ for a conformal map $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ of dilation λ and for any function $\gamma \in C^\infty(M)$.

Lemma 2.2. Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ, then for any function $\gamma \in C^\infty(M)$, we have

$$\begin{aligned} Tr_g\left(\nabla^\phi\right)^2 d\phi\,(grad\gamma) = {} & d\phi\,(grad\Delta\gamma) + 4d\phi\,(\nabla_{grad \ln\lambda} grad\gamma) + d\phi\left(Ricci^M\,(grad\gamma)\right) \\ & + (\Delta \ln\lambda)\, d\phi\,(grad\gamma) - 2\,(\Delta\gamma)\, d\phi\,(grad \ln\lambda) \\ & - (n-2)\, d\ln\lambda\,(grad\gamma)\, d\phi\,(grad \ln\lambda)\,. \end{aligned} \tag{2.14}$$

Proof of Lemma 2.2. Let $\gamma \in C^\infty(M)$, by definition, we have

$$Tr_g\left(\nabla^\phi\right)^2 d\phi\,(grad\gamma) = \nabla^\phi_{e_i}\nabla^\phi_{e_i} d\phi\,(grad\gamma) - \nabla^\phi_{\nabla_{e_i}e_i} d\phi\,(grad\gamma)\,. \tag{2.15}$$

(Here henceforth we sum over repeated indices.) Let us start with the calculation of the term $\nabla^\phi_{e_i}\nabla^\phi_{e_i} d\phi\,(grad\gamma)$, we have

$$\nabla^\phi_{e_i} d\phi\,(grad\gamma) = \nabla d\phi\,(e_i, grad\gamma) + d\phi\,(\nabla_{e_i} grad\gamma)\,.$$

It is known that (see [3])

$$\nabla d\phi\,(e_i, grad\gamma) = e_i\,(\ln\lambda)\, d\phi\,(grad\gamma) + d\ln\lambda\,(grad\gamma)\, d\phi\,(e_i) - e_i\,(\gamma)\, d\phi\,(grad \ln\lambda)\,,$$

then

$$\begin{aligned} \nabla^\phi_{e_i} d\phi\,(grad\gamma) = {} & e_i\,(\ln\lambda)\, d\phi\,(grad\gamma) + d\ln\lambda\,(grad\gamma)\, d\phi\,(e_i) \\ & - e_i\,(\gamma)\, d\phi\,(grad \ln\lambda) + d\phi\,(\nabla_{e_i} grad\gamma)\,. \end{aligned} \tag{2.16}$$

It follows that

$$\begin{aligned} \nabla^\phi_{e_i}\nabla^\phi_{e_i} d\phi\,(grad\gamma) = {} & \nabla^\phi_{e_i}\left\{e_i\,(\ln\lambda)\, d\phi\,(grad\gamma)\right\} + \nabla^\phi_{e_i}\left\{d\ln\lambda\,(grad\gamma)\, d\phi\,(e_i)\right\} \\ & - \nabla^\phi_{e_i}\left\{e_i\,(\gamma)\, d\phi\,(grad \ln\lambda)\right\} + \nabla^\phi_{e_i} d\phi\,(\nabla_{e_i} grad\gamma)\,. \end{aligned} \tag{2.17}$$

We will study term by term the right-hand of this expression. For the first term $\nabla^\phi_{e_i}\left\{e_i\,(\ln\lambda)\, d\phi\,(grad\gamma)\right\}$, we have

$$\nabla^\phi_{e_i}\left\{e_i\,(\ln\lambda)\, d\phi\,(grad\gamma)\right\} = e_i\,(\ln\lambda)\,\nabla^\phi_{e_i} d\phi\,(grad\gamma) + e_i\,(e_i\,(\ln\lambda))\, d\phi\,(grad\gamma)\,.$$

By using the equation (2.16), we deduce that

$$\begin{aligned} \nabla^\phi_{e_i}\left\{e_i\,(\ln\lambda)\, d\phi\,(grad\gamma)\right\} = {} & e_i\,(\ln\lambda)\, e_i\,(\ln\lambda)\, d\phi\,(grad\gamma) + e_i\,(\ln\lambda)\, d\ln\lambda\,(grad\gamma)\, d\phi\,(e_i) \\ & - e_i\,(\ln\lambda)\, e_i\,(\gamma)\, d\phi\,(grad \ln\lambda) + e_i\,(\ln\lambda)\, d\phi\,(\nabla_{e_i} grad\gamma) \\ & + e_i\,(e_i\,(\ln\lambda))\, d\phi\,(grad\gamma)\,, \end{aligned}$$

then, we obtain

$$\nabla^{\phi}_{e_i}\left\{e_i\left(\ln\lambda\right)d\phi\left(grad\gamma\right)\right\} = \left|grad\ln\lambda\right|^2 d\phi\left(grad\gamma\right) + d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right) + e_i\left(e_i\left(\ln\lambda\right)\right)d\phi\left(grad\gamma\right). \tag{2.18}$$

For the second term $\nabla^{\phi}_{e_i}\left\{d\ln\lambda\left(grad\gamma\right)d\phi\left(e_i\right)\right\}$, a similar calculation gives

$$\begin{aligned}\nabla^{\phi}_{e_i}\left\{d\ln\lambda\left(grad\gamma\right)d\phi\left(e_i\right)\right\} &= d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i}d\phi\left(e_i\right) + e_i\left\{g\left(grad\ln\lambda, grad\gamma\right)\right\}d\phi\left(e_i\right)\\ &= d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i}d\phi\left(e_i\right) + g\left(\nabla_{e_i}grad\ln\lambda, grad\gamma\right)d\phi\left(e_i\right)\\ &+ g\left(grad\ln\lambda, \nabla_{e_i}grad\gamma\right)d\phi\left(e_i\right)\\ &= d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i}d\phi\left(e_i\right) + g\left(\nabla_{grad\gamma}grad\ln\lambda, e_i\right)d\phi\left(e_i\right)\\ &+ g\left(\nabla_{grad\ln\lambda}grad\gamma, e_i\right)d\phi\left(e_i\right),\end{aligned}$$

it follows that

$$\nabla^{\phi}_{e_i}\left\{d\ln\lambda\left(grad\gamma\right)d\phi\left(e_i\right)\right\} = d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i}d\phi\left(e_i\right) + d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right) + d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right). \tag{2.19}$$

For the third term $\nabla^{\phi}_{e_i}\left\{e_i\left(\gamma\right)d\phi\left(grad\ln\lambda\right)\right\}$, by using the same calculation method and the equation (2.16), we have

$$\begin{aligned}\nabla^{\phi}_{e_i}\left\{e_i\left(\gamma\right)d\phi\left(grad\ln\lambda\right)\right\} &= e_i\left(\gamma\right)\nabla^{\phi}_{e_i}d\phi\left(grad\ln\lambda\right) + e_i\left(e_i\left(\gamma\right)\right)d\phi\left(grad\ln\lambda\right)\\ &= e_i\left(\gamma\right)e_i\left(\ln\lambda\right)d\phi\left(grad\ln\lambda\right) + e_i\left(\gamma\right)d\ln\lambda\left(grad\ln\lambda\right)d\phi\left(e_i\right)\\ &- e_i\left(\gamma\right)e_i\left(\ln\lambda\right)d\phi\left(grad\ln\lambda\right) + e_i\left(\gamma\right)d\phi\left(\nabla_{e_i}grad\ln\lambda\right)\\ &+ e_i\left(e_i\left(\gamma\right)\right)d\phi\left(grad\ln\lambda\right),\end{aligned}$$

which gives us

$$\nabla^{\phi}_{e_i}\left\{e_i\left(\gamma\right)d\phi\left(grad\ln\lambda\right)\right\} = \left|grad\ln\lambda\right|^2 d\phi\left(grad\gamma\right) + d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right) + e_i\left(e_i\left(\gamma\right)\right)d\phi\left(grad\ln\lambda\right). \tag{2.20}$$

Now let us look at the last term $\nabla^{\phi}_{e_i}d\phi\left(\nabla_{e_i}grad\gamma\right)$, a simple calculation gives

$$\begin{aligned}\nabla^{\phi}_{e_i}d\phi\left(\nabla_{e_i}grad\gamma\right) &= e_i\left(\ln\lambda\right)d\phi\left(\nabla_{e_i}grad\gamma\right) + d\ln\lambda\left(\nabla_{e_i}grad\gamma\right)d\phi\left(e_i\right)\\ &- g\left(e_i, \nabla_{e_i}grad\gamma\right)d\phi\left(grad\ln\lambda\right) + d\phi\left(\nabla_{e_i}\nabla_{e_i}grad\gamma\right)\\ &= 2d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right) - \left(\Delta\gamma\right)d\phi\left(grad\ln\lambda\right)\\ &+ d\phi\left(\nabla_{e_i}\nabla_{e_i}grad\gamma\right),\end{aligned}$$

then

$$\nabla^{\phi}_{e_i}d\phi\left(\nabla_{e_i}grad\gamma\right) = d\phi\left(\nabla_{e_i}\nabla_{e_i}grad\gamma\right) + 2d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right) - \left(\Delta\gamma\right)d\phi\left(grad\ln\lambda\right). \tag{2.21}$$

If we replace (2.18), (2.19), (2.20) and (2.21) in (2.17), we obtain

$$\begin{aligned}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\gamma\right) &= 4d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right) + e_i\left(e_i\left(\ln\lambda\right)\right)d\phi\left(grad\gamma\right)\\ &+ d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i}d\phi\left(e_i\right) - e_i\left(e_i\left(\gamma\right)\right)d\phi\left(grad\ln\lambda\right)\\ &+ d\phi\left(\nabla_{e_i}\nabla_{e_i}grad\gamma\right) - \left(\Delta\gamma\right)d\phi\left(grad\ln\lambda\right).\end{aligned} \tag{2.22}$$

To complete the proof, it remains to investigate the term $\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(grad\gamma\right)$, we have

$$\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(grad\gamma\right) = \nabla d\phi\left(\nabla_{e_i}e_i, grad\gamma\right) + d\phi\left(\nabla_{\nabla_{e_i}e_i}grad\gamma\right),$$

Therefore, by using the equation (2.16), we obtain

$$\begin{aligned}\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(grad\gamma\right) &= \nabla_{e_i}e_i\left(\ln\lambda\right)d\phi\left(grad\gamma\right)+d\ln\lambda\left(grad\gamma\right)d\phi\left(\nabla_{e_i}e_i\right)\\ &\quad-\nabla_{e_i}e_i\left(\gamma\right)d\phi\left(grad\ln\lambda\right)+d\phi\left(\nabla_{\nabla_{e_i}e_i}grad\gamma\right).\end{aligned} \tag{2.23}$$

By substituting (2.22) and (2.23) in (2.15), we deduce

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\gamma\right) &= \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\gamma\right)-\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\left(grad\gamma\right)\\ &= d\phi\left(Tr_g\nabla^2 grad\gamma\right)+4d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right)\\ &\quad+\left(\Delta\ln\lambda\right)d\phi\left(grad\gamma\right)+d\ln\lambda\left(grad\gamma\right)\tau\left(\phi\right)\\ &\quad-2\left(\Delta\gamma\right)d\phi\left(grad\ln\lambda\right).\end{aligned}$$

Finally, using the fact that (see [13])

$$Tr_g\nabla^2 grad\gamma = grad\Delta\gamma + Ricci^M\left(grad\gamma\right)$$

and

$$\tau\left(\phi\right) = \left(2-n\right)d\phi\left(grad\ln\lambda\right),$$

we conclude that

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\gamma\right) &= d\phi\left(grad\Delta\gamma\right)+4d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right)+d\phi\left(Ricci^M\left(grad\gamma\right)\right)\\ &\quad+\left(\Delta\ln\lambda\right)d\phi\left(grad\gamma\right)-2\left(\Delta\gamma\right)d\phi\left(grad\ln\lambda\right)\\ &\quad-\left(n-2\right)d\ln\lambda\left(grad\gamma\right)d\phi\left(grad\ln\lambda\right).\end{aligned}$$

This completes the proof of Lemma 2.2. Now, in the second Lemma, we will calculate $Tr_g R^N\left(d\phi\left(grad\gamma\right),d\phi\right)d\phi$ for a conformal maps $\phi:(M^n,g)\to(N^n,h)$ $(n\geq 3)$ of dilation λ and for any function $\gamma\in C^{\infty}\left(M\right)$

Lemma 2.3. Let $\phi:(M^n,g)\to(N^n,h)$ $(n\geq 3)$ to be a conformal map of dilation λ, then for any function $\gamma\in C^{\infty}\left(M\right)$, we have

$$\begin{aligned}Tr_g R^N\left(d\phi\left(grad\gamma\right),d\phi\right)d\phi &= d\phi\left(Ricci^M\left(grad\gamma\right)\right)-\left(n-2\right)d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right)\\ &\quad-\left(\Delta\ln\lambda+\left(n-2\right)\left|grad\ln\lambda\right|^2\right)d\phi\left(grad\gamma\right)\\ &\quad+\left(n-2\right)d\ln\lambda\left(grad\gamma\right)d\phi\left(grad\ln\lambda\right)\end{aligned} \tag{2.24}$$

Proof of Lemma 2.3. Let $\gamma\in C^{\infty}\left(M\right)$, by definition we have

$$Tr_g R^N\left(d\phi\left(grad\gamma\right),d\phi\right)d\phi = R^N\left(d\phi\left(grad\gamma\right),d\phi\left(e_i\right)\right)d\phi\left(e_i\right) \tag{2.25}$$

but we know that (see [3])

$$\begin{aligned}Ric^N\left(d\phi\left(X\right),d\phi\left(Y\right)\right) &= Ric^M\left(X,Y\right)+\left(n-2\right)X\left(\ln\lambda\right)Y\left(\ln\lambda\right)\\ &\quad-\left(n-2\right)\left|grad\ln\lambda\right|^2 g\left(X,Y\right)\\ &\quad-\left(n-2\right)\nabla d\ln\lambda\left(X,Y\right)-\left(\Delta\ln\lambda\right)g\left(X,Y\right).\end{aligned}$$

Then

$$\begin{aligned}Ric^N\left(d\phi\left(grad\gamma\right),d\phi\left(e_i\right)\right) &= Ric^M\left(grad\gamma,e_i\right)+\left(n-2\right)grad\gamma\left(\ln\lambda\right)e_i\left(\ln\lambda\right)\\ &\quad-\left(n-2\right)\left|grad\ln\lambda\right|^2 g\left(grad\gamma,e_i\right)\\ &\quad-\left(n-2\right)\nabla d\ln\lambda\left(grad\gamma,e_i\right)-\left(\Delta\ln\lambda\right)g\left(grad\gamma,e_i\right)\end{aligned}$$

it follows that

$$\begin{aligned}Ric^N\left(d\phi\left(grad\gamma\right),d\phi\left(e_i\right)\right) &= Ric^M\left(grad\gamma,e_i\right)+\left(m-2\right)d\ln\lambda\left(grad\gamma\right)e_i\left(\ln\lambda\right)\\ &\quad-\left(n-2\right)\left|grad\ln\lambda\right|^2 e_i\left(\gamma\right)-\left(n-2\right)\nabla d\ln\lambda\left(grad\gamma,e_i\right)\\ &\quad-\left(\Delta\ln\lambda\right)e_i\left(\gamma\right).\end{aligned} \tag{2.26}$$

If we replace (2.26) in (2.25), we deduce that

$$\begin{aligned} Tr_g R^N\left(d\phi\left(grad\gamma\right), d\phi\right) d\phi =& R^N\left(d\phi\left(grad\gamma\right), d\phi\left(e_i\right)\right) d\phi\left(e_i\right) \\ =& d\phi\left(Ricci^M\left(grad\gamma\right)\right) + (n-2)\, d\ln\lambda\left(grad\gamma\right) d\phi\left(grad\ln\lambda\right) \\ &- (n-2)\left|grad\ln\lambda\right|^2 d\phi\left(grad\gamma\right) - \left(\Delta\ln\lambda\right) d\phi\left(grad\gamma\right) \\ &- (n-2)\,\nabla d\ln\lambda\left(grad\gamma, e_i\right) d\phi\left(e_i\right). \end{aligned}$$

To complete the proof, we will simplify the term $\nabla d\ln\lambda\left(grad\gamma, e_i\right) d\phi\left(e_i\right)$, we obtain

$$\begin{aligned} \nabla d\ln\lambda\left(grad\gamma, e_i\right) d\phi\left(e_i\right) &= \left\{e_i\left(g\left(grad\ln\lambda, grad\gamma\right)\right) - d\ln\lambda\left(\nabla_{e_i} grad\gamma\right)\right\} d\phi\left(e_i\right) \\ &= g\left(\nabla_{e_i} grad\ln\lambda, grad\gamma\right) d\phi\left(e_i\right) \\ &= g\left(\nabla_{grad\gamma} grad\ln\lambda, e_i\right) d\phi\left(e_i\right) \\ &= d\phi\left(\nabla_{grad\gamma} grad\ln\lambda\right), \end{aligned}$$

which finally gives

$$\begin{aligned} Tr_g R^N\left(d\phi\left(grad\gamma\right), d\phi\right) d\phi =& d\phi\left(Ricci^M\left(grad\gamma\right)\right) - (n-2)\, d\phi\left(\nabla_{grad\gamma} grad\ln\lambda\right) \\ &- \left(\Delta\ln\lambda + (n-2)\left|grad\ln\lambda\right|^2\right) d\phi\left(grad\gamma\right) \\ &+ (n-2)\, d\ln\lambda\left(grad\gamma\right) d\phi\left(grad\ln\lambda\right). \end{aligned}$$

This completes the proof of Lemma 2.3. We are now able to prove Theorem 2.2.
Proof of Theorem 2.2. By definition, the bitension field is given by

$$\tau_2(\phi) = -Tr_g\left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N\left(\tau(\phi), d\phi\right) d\phi.$$

The tension field of the conformal map ϕ is given by

$$\tau(\phi) = (2-n)\, d\phi\left(grad\ln\lambda\right),$$

it follows that

$$\tau_2(\phi) = (n-2)\left(Tr_g\left(\nabla^\phi\right)^2 d\phi\left(grad\ln\lambda\right) + Tr_g R^N\left(d\phi\left(grad\ln\lambda\right), d\phi\right) d\phi\right). \tag{2.27}$$

By Lemma 2, we have

$$\begin{aligned} Tr_g\left(\nabla^\phi\right)^2 d\phi\left(grad\ln\lambda\right) =& d\phi\left(grad\Delta\ln\lambda\right) + 2d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right) \\ &- \left(\Delta\ln\lambda\right) d\phi\left(grad\ln\lambda\right) + d\phi\left(Ricci^M\left(grad\ln\lambda\right)\right) \\ &- (n-2)\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right). \end{aligned} \tag{2.28}$$

By using lemma 2.3 and the fact that $\nabla_{grad\ln\lambda} grad\ln\lambda = \frac{1}{2} grad\left(\left|grad\ln\lambda\right|^2\right)$

$$\begin{aligned} Tr_g R^N\left(d\phi\left(grad\ln\lambda\right), d\phi\right) d\phi =& d\phi\left(Ricci^M\left(grad\ln\lambda\right)\right) - \left(\Delta\ln\lambda\right) d\phi\left(grad\ln\lambda\right) \\ &- \frac{(n-2)}{2} d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right). \end{aligned} \tag{2.29}$$

If we replace (2.28) and (2.29) in (2.27), we deduce that

$$\begin{aligned} \tau_2(\phi) =& (n-2)\, d\phi\left(grad\Delta\ln\lambda\right) - \frac{(n-2)(n-6)}{2} d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right) \\ &- (n-2)\left(2\left(\Delta\ln\lambda\right) + (n-2)\left|grad\ln\lambda\right|^2\right) d\phi\left(grad\ln\lambda\right) \\ &+ 2(n-2)\, d\phi\left(Ricci^M\left(grad\ln\lambda\right)\right). \end{aligned}$$

Then the bi-tension field of ϕ is given by :

$$\tau_2(\phi) = (n-2)\, d\phi\,(H)$$

where

$$\begin{aligned} H &= grad\Delta \ln\lambda - \frac{(n-6)}{2} grad\left(|grad \ln\lambda|^2\right) + 2Ricci^M\,(grad\ln\lambda) \\ &- \left(2\,(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right) grad\ln\lambda. \end{aligned}$$

The proof of Theorem 2.2 is complete. By application of Theorem 2.2, we get the following result (see [5]).

Theorem 2.3. ([5]) Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ, then ϕ is biharmonic if and only if the dilation λ satisfies

$$\begin{aligned} & grad\,(\Delta\ln\lambda) - \left(2\,(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right) grad\ln\lambda \\ & + \frac{6-n}{2} grad\left(|grad\ln\lambda|^2\right) + 2Ricci^M(grad\ln\lambda) = 0. \end{aligned}$$

In particular, we prove that the biharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ where the dilation λ is radial $(\ln\lambda = \alpha\,(r)\,, r = |x|$ and $\alpha \in C^\infty\,(\mathbb{R},\mathbb{R}))$ is equivalent to an ordinary differential equation of the second order. More precisely, we have

Corollary 2.2. Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ when we suppose that $\ln\lambda$ is radial $(\ln\lambda = \alpha\,(r)\,, r = |x|$ and $\alpha \in C^\infty\,(\mathbb{R},\mathbb{R}))$. Then ϕ is biharmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation :

$$\beta'' - (n-4)\,\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{2\,(n-1)}{r}\beta^2 - (n-2)\,\beta^3 = 0. \tag{2.30}$$

Proof of Corollary 2.2 Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ such that $\ln\lambda = \alpha\,(r)$. By Theorem 2.3, ϕ is biharmonic if and only if the dilation λ satisfies

$$\begin{aligned} & grad\,(\Delta\ln\lambda) - \left(2\,(\Delta\ln\lambda) + (n-2)\,|grad\ln\lambda|^2\right) grad\ln\lambda \\ & + \frac{6-n}{2} grad\left(|grad\ln\lambda|^2\right) + 2Ricci^M(grad\ln\lambda) = 0. \end{aligned}$$

A direct calculation gives

$$grad\ln\lambda = \alpha'\frac{\partial}{\partial r},$$

$$|grad\ln\lambda|^2 = (\alpha')^2\,,$$

$$grad\left(|grad\ln\lambda|^2\right) = 2\alpha'\alpha''\frac{\partial}{\partial r},$$

$$\Delta\ln\lambda = \alpha'' + \frac{n-1}{r}\alpha'$$

and

$$grad\,(\Delta\ln\lambda) = \left(\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha'\right)\frac{\partial}{\partial r}.$$

Therefore ϕ is biharmonic if and only if the function α satisfies the following differential equation

$$\alpha''' - (n-4)\,\alpha'\alpha'' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha' - \frac{2\,(n-1)}{r}\,(\alpha')^2 - (n-2)\,(\alpha')^3 = 0.$$

If we denote $\beta = \alpha'$, the biharmonicity of ϕ is equivalent to the differential equation

$$\beta'' - (n-4)\,\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{2\,(n-1)}{r}\beta^2 - (n-2)\,\beta^3 = 0.$$

As a consequence of the Corollary 2.2, We will present some remarks which we give a particular solutions of the equation (2.30) that allows us to construct a biharmonic non-harmonic maps.

Remark 2.2. . Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$. By (2.30), we deduce that $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ is biharmonic if and only if a is a solution of the algebraic equation

$$(n-2)\,a^2 + (n+2)\,a + 2n - 2 = 0.$$

This equation has real solutions if and only if $n \in \{3, 4\}$.

(i) If $n = 3$, we find $a = \frac{-5+\sqrt{17}}{2}$ or $a = \frac{-5-\sqrt{17}}{2}$, so $\lambda = Cr^{-\left(\frac{5-\sqrt{17}}{2}\right)}$ or $\lambda = Cr^{-\left(\frac{5+\sqrt{17}}{2}\right)}$ $(C \in \mathbb{R}_+^*)$. It follows that any conformal map $\phi : (\mathbb{R}^3, g) \to (N^3, h)$ of dilation $\lambda = Cr^{-\left(\frac{5-\sqrt{17}}{2}\right)}$ or $\lambda = Cr^{-\left(\frac{5+\sqrt{17}}{2}\right)}$ is biharmonic non-harmonic.

(ii) If $n = 4$, we find $a = -1$ or $a = -2$, so $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ $(C \in \mathbb{R}_+^*)$. Then, in this case any conformal map $\phi : (\mathbb{R}^4, g) \to (N^4, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ is biharmonic non-harmonic.

Remark 2.3. . Looking for particular solutions of type $\beta = \frac{ar}{1+r^2}$ $(a \in \mathbb{R}^*)$. By (2.30), $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ is biharmonic if and only we have

$$(n-2)\,a^2 + (n+2)\,a + 2n - 2 = 0$$

and

$$3\,(n-2)\,a + 2n + 4 = 0.$$

These two equations gives $a = -2$ and $n = 4$, it follows that the dilation is equal to $\lambda = \frac{Cr}{r^2+1}$ $(C \in \mathbb{R}_+^*)$. Then, all conformal maps $\phi : (\mathbb{R}^4, g) \to (N^4, h)$ of dilation $\lambda = \frac{Cr}{r^2+1}$ are biharmonic non-harmonic.

3 Biharmonic maps and the warped product

Let (M^m, g) and (N^n, h) two Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. The warped product $M \times_f N$ is the product manifolds $M \times N$ endowed with the Riemannian metric G_f defined, for $X, Y \in \Gamma(T(M \times N))$, by

$$G_f(X, Y) = g(d\pi(X), d\pi(Y)) + (f \circ \pi)^2\, h(d\eta(X), d\eta(Y)),$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. The function f is called the warping function of the warped product. Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$. The Levi-Civita connection $\widetilde{\nabla}$ of the warped product $M \times_f N$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + X_1(\ln f)(0, Y_2) + Y_1(\ln f)(0, X_2) - f^2 h(X_2, Y_2)(grad \ln f, 0). \tag{3.1}$$

In the first, we consider a smooth map $\phi : (M^m, g) \longrightarrow (P^p, k)$ and we defined the map $\widetilde{\phi} : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ by $\widetilde{\phi}(x, y) = \phi(x)$. We will study the biharmonicity of $\widetilde{\phi}$. By calculating the tension field of $\widetilde{\phi}$, we obtain the following result :

Proposition 3.1. Let $\phi : (M^m, g) \longrightarrow (P^p, k)$ be a smooth map. The tension field of the map $\widetilde{\phi} : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}(x, y) = \phi(x)$ is given by

$$\tau\left(\widetilde{\phi}\right) = \tau(\phi) + nd\phi(grad \ln f) \tag{3.2}$$

Proof of Proposition 3.1. Let us choose $\{e_i\}_{1\leq i\leq m}$ to be an orthonormal frame on M and $\{f_j\}_{1\leq j\leq n}$ to be an orthonormal frame on N. An orthonormal frame on $M\times_f N$ is given by $\left\{(e_i,0), \frac{1}{f}(0,f_j)\right\}$. Note that in this case we have $d\widetilde{\phi}(X,Y)=(d\phi(X),0)$ for any $X\in\Gamma(TM)$ and $Y\in\Gamma(TN)$. By definition to the tension field, we have

$$\begin{aligned}\tau\left(\widetilde{\phi}\right) &= Tr_{G_f}\nabla d\widetilde{\phi}\\ &= \nabla^{\widetilde{\phi}}_{(e_i,0)}d\widetilde{\phi}(e_i,0)+\frac{1}{f^2}\nabla^{\widetilde{\phi}}_{(0,f_j)}d\widetilde{\phi}(0,f_j)\\ &- d\widetilde{\phi}\left(\widetilde{\nabla}_{(e_i,0)}(e_i,0)\right)-\frac{1}{f^2}d\widetilde{\phi}\left(\widetilde{\nabla}_{(0,f_j)}(0,f_j)\right).\end{aligned}$$

A simple calculation gives

$$\nabla^{\widetilde{\phi}}_{(e_i,0)}d\widetilde{\phi}(e_i,0)=\nabla^{\phi}_{e_i}d\phi(e_i)$$

and

$$\nabla^{\widetilde{\phi}}_{(0,f_j)}d\widetilde{\phi}(0,f_j)=0,$$

By using the equation (3.1), we deduce that

$$\widetilde{\nabla}_{(e_i,0)}(e_i,0)=(\nabla_{e_i}e_i,0)$$

and

$$\widetilde{\nabla}_{(0,f_j)}(0,f_j)=(0,\nabla_{f_j}f_j)-nf^2(grad\ln f,0).$$

It follows that

$$\tau\left(\widetilde{\phi}\right)=\nabla^{\phi}_{e_i}d\phi(e_i)-d\phi\left(\nabla^M_{e_i}e_i\right)+nd\phi(grad\ln f),$$

then, we obtain

$$\tau\left(\widetilde{\phi}\right)=\tau(\phi)+nd\phi(grad\ln f).$$

Remark 3.1. If $\phi:(M^m,g)\longrightarrow(P^m,k)$ $(m\geq 3)$ is a conformal map with dilation λ, the tension field of $\widetilde{\phi}$ is given by

$$\tau\left(\widetilde{\phi}\right)=(2-m)\,d\phi(grad\ln\lambda)+nd\phi(grad\ln f)=d\phi\left(grad\ln\left(\lambda^{2-m}f^n\right)\right).$$

Then $\widetilde{\phi}$ is harmonic if and only if the function $\lambda^{2-m}f^n$ is constant.

We will now calculate the bitension field of the map $\widetilde{\phi}:(M^m\times_f N^n,G_f)\longrightarrow(P^p,k)$.

Theorem 3.1. Let $\phi:(M^m,g)\longrightarrow(P^p,k)$ be a smooth map. The bitension field of the map $\widetilde{\phi}:(M^m\times_f N^n,G_f)\longrightarrow(P^p,k)$ defined by $\widetilde{\phi}(x,y)=\phi(x)$ is given by

$$\begin{aligned}\tau_2\left(\widetilde{\phi}\right) &= \tau_2(\phi)-n\left(Tr_g\nabla^2 d\phi(grad\ln f)+Tr_gR^p(d\phi(grad\ln f),d\phi)\,d\phi\right)\\ &- n\nabla_{grad\ln f}\tau(\phi)-n^2\nabla_{grad\ln f}d\phi(grad\ln f).\end{aligned}\tag{3.3}$$

Proof of Theorem 3.1. By definition of the bitension field, we have

$$\tau_2\left(\widetilde{\phi}\right)=-Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)-Tr_{G_f}R^P\left(\tau\left(\widetilde{\phi}\right),d\widetilde{\phi}\right)d\widetilde{\phi}\tag{3.4}$$

For the first term $Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)$, we have

$$\begin{aligned}Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right) &= \nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\tau\left(\widetilde{\phi}\right)+\frac{1}{f^2}\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}\tau\left(\widetilde{\phi}\right)\\ &- \nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\phi}\right)-\frac{1}{f^2}\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\phi}\right).\end{aligned}$$

We will study term by term the right-hand of this expression. A simple calculation gives

$$\begin{aligned}\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\tau\left(\widetilde{\phi}\right) &= \nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\tau\left(\phi\right) + n\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}d\phi\left(grad\ln f\right)\\ &= \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\tau\left(\phi\right) + n\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\ln f\right)\end{aligned}$$

and

$$\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}\tau\left(\widetilde{\phi}\right) = 0.$$

By using the equation (3.1), we obtain

$$\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\phi}\right) = \nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau\left(\phi\right) + n\nabla^{\phi}_{\nabla^M_{e_i}e_i}d\phi\left(grad\ln f\right),$$

and

$$\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\phi}\right) = -nf^2\nabla^{\phi}_{grad\ln f}\tau\left(\phi\right) - n^2f^2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln f\right).$$

Then, we deduce that

$$\begin{aligned}Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right) &= Tr_g\left(\nabla^{\phi}\right)^2\tau\left(\phi\right) + nTr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln f\right)\\ &\quad + n\nabla^{\phi}_{grad\ln f}\tau\left(\phi\right) + n^2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln f\right).\end{aligned} \tag{3.5}$$

To complete the proof, we will simplify the term $Tr_{G_f}R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right)d\widetilde{\phi}$, we have

$$\begin{aligned}Tr_{G_f}R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right)d\widetilde{\phi} &= R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\left(e_i,0\right)\right)d\widetilde{\phi}\left(e_i,0\right)\\ &\quad + \frac{1}{f^2}R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\left(0,f_j\right)\right)d\widetilde{\phi}\left(0,f_j\right)\\ &= R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\left(e_i,0\right)\right)d\widetilde{\phi}\left(e_i,0\right)\\ &= R^P\left(\tau\left(\phi\right), d\phi\left(e_i\right)\right)d\phi\left(e_i\right)\\ &\quad + nR^P\left(d\phi\left(grad\ln f\right), d\phi\left(e_i\right)\right)d\phi\left(e_i\right).\end{aligned}$$

It follows that

$$Tr_{G_f}R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right)d\widetilde{\phi} = Tr_gR^P\left(\tau\left(\phi\right), d\phi\right)d\phi + nTr_gR^P\left(d\phi\left(grad\ln f\right), d\phi\right)d\phi. \tag{3.6}$$

If we replace (3.5) and (3.6) in (3.4), we obtain

$$\begin{aligned}\tau_2\left(\widetilde{\phi}\right) &= \tau_2\left(\phi\right) - n\left(Tr_g\nabla^2 d\phi\left(grad\ln f\right) + Tr_gR^p\left(d\phi\left(grad\ln f\right), d\phi\right)d\phi\right)\\ &\quad - n\nabla_{grad\ln f}\tau\left(\phi\right) - n^2\nabla_{grad\ln f}d\phi\left(grad\ln f\right).\end{aligned}$$

The proof of Theorem 3.1 is complete. As a consequence, if ϕ is harmonic, we have

Corollary 3.1. Let $\phi : (M^m, g) \longrightarrow (P^p, k)$ a harmonic map. the map $\widetilde{\phi} : \left(M^m \times_{f^2} N^n, G_{f^2}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}\left(x,y\right) = \phi\left(x\right)$ is biharmonic if and only if

$$Tr_g\nabla^2 d\phi\left(grad\ln f\right) + Tr_gR^p\left(d\phi\left(grad\ln f\right), d\phi\right)d\phi + n\nabla_{grad\ln f}d\phi\left(grad\ln f\right) = 0.$$

In the following we shall present an example of biharmonic non-harmonic maps.

Example 3.1. Let $\widetilde{\varphi} : R^m \setminus \{0\} \times_f N^n \longrightarrow R^m \setminus \{0\}$ defined by $\widetilde{\varphi}\left(x,y\right) = \frac{x}{|x|^2}$ when we suppose that $\ln f$ is radial ($\ln f = \alpha\left(r\right)$). Then by Theorem 3.1, we deduce that the map $\widetilde{\varphi} : R^m \setminus \{0\} \times_f N^n \longrightarrow R^m \setminus \{0\}$ is biharmonic if and only if the function α satisfies the following differential equation

$$n\alpha''' + \frac{n\left(m-5\right)}{r}\alpha'' - \frac{3n\left(3m-7\right)}{r^2}\alpha' + n^2\alpha'\alpha'' - \frac{2n^2}{r}\left(\alpha'\right)^2 - \frac{8\left(m-2\right)\left(m-4\right)}{r^3} = 0.$$

Let $\beta = \alpha'$, this equation becomes

$$n\beta'' + \frac{n(m-5)}{r}\beta' - \frac{3n(3m-7)}{r^2}\beta + n^2\beta\beta' - \frac{2n^2}{r}\beta^2 - \frac{8(m-2)(m-4)}{r^3} = 0.$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, then $\widetilde{\varphi} : R^m \setminus \{0\} \times_f N^n \longrightarrow R^m \setminus \{0\}$ is biharmonic if and only if

$$3n^2a^2 + 2n(5m-14)a + 8(m-2)(m-4) = 0.$$

This equation has two solutions $a = \frac{4-2m}{n}$ and $a = \frac{4(4-m)}{3n}$.

(i) For $a = \frac{4-2m}{n}$, we obtain $f(r) = Cr^{\frac{4-2m}{n}}$ and in this case $\widetilde{\varphi} : R^m \setminus \{0\} \times_f N^n \longrightarrow R^m \setminus \{0\}$ is harmonic so biharmonic.

(ii) For $a = \frac{4(4-m)}{3n}$, we obtain $f(r) = Cr^{\frac{4(4-m)}{3n}}$ and in this case $\widetilde{\varphi} : R^m \setminus \{0\} \times_f N^n \longrightarrow R^m \setminus \{0\}$ is biharmonic non-harmonic.

Now, we consider a smooth map $\psi : (N^n, g) \longrightarrow (P^p, k)$ and we define the map $\widetilde{\psi} : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ by $\widetilde{\psi}(x, y) = \psi(y)$. We will study the biharmonicity of $\widetilde{\psi}$, we obtain the following result :

Theorem 3.2. Let $\psi : (N^n, h) \to (P^p, k)$ be a smooth map, we define $\widetilde{\psi} : \left(M^m \times_{f^2} N^n, G_{f^2}\right) \to (P^p, k)$ by $\widetilde{\psi}(x, y) = \psi(y)$. The tension field and the bitension field of $\widetilde{\psi}$ are given by

$$\tau\left(\widetilde{\psi}\right) = \frac{1}{f^2 \circ \pi}\tau(\psi) \tag{3.7}$$

and

$$\tau_2\left(\widetilde{\psi}\right) = \frac{1}{f^4 \circ \pi}\tau_2(\psi) - \frac{2}{f^2 \circ \pi}\left(\left(\Delta \ln f + (n-2)\left|grad \ln f\right|^2\right) \circ \pi\right)\tau(\psi). \tag{3.8}$$

Proof of Theorem 3.2. In the first, we calculate the tension field of of $\widetilde{\psi}$. By definition, we have By definition to the tension field, we have

$$\begin{aligned}
\tau\left(\widetilde{\psi}\right) &= Tr_{G_f}\nabla d\widetilde{\psi} \\
&= \nabla^{\widetilde{\psi}}_{(e_i,0)}d\widetilde{\psi}(e_i, 0) + \frac{1}{f^2 \circ \pi}\nabla^{\widetilde{\psi}}_{(0,f_j)}d\widetilde{\psi}(0, f_j) \\
&\quad - d\widetilde{\psi}\left(\widetilde{\nabla}_{(e_i,0)}(e_i, 0)\right) - \frac{1}{f^2 \circ \pi}d\widetilde{\psi}\left(\widetilde{\nabla}_{(0,f_j)}(0, f_j)\right).
\end{aligned}$$

By using the equation (3.1), we obtain

$$\tau\left(\widetilde{\psi}\right) = \frac{1}{f^2 \circ \pi}\nabla^{\psi}_{f_j}d\psi(f_j) - \frac{1}{f^2 \circ \pi}d\psi\left(\nabla_{f_j}f_j\right) = \frac{1}{f^2 \circ \pi}\tau(\psi),$$

then

$$\tau\left(\widetilde{\psi}\right) = \frac{1}{f^2 \circ \pi}\tau(\psi).$$

By this expression, we deduce that $\widetilde{\psi}$ is harmonic if and only if ψ is harmonic. Now, we will calculate the bitension field of $\widetilde{\psi}$. By definition, we have

$$\tau_2\left(\widetilde{\psi}\right) = -Tr_{G_f}\left(\nabla^{\widetilde{\psi}}\right)^2\tau\left(\widetilde{\psi}\right) - Tr_{G_f}R^P\left(\tau\left(\widetilde{\psi}\right), d\widetilde{\psi}\right)d\widetilde{\psi}. \tag{3.9}$$

For the first term $Tr_{G_f}\left(\nabla^{\widetilde{\psi}}\right)^2\tau\left(\widetilde{\psi}\right)$, we have

$$\begin{aligned}
Tr_{G_f}\left(\nabla^{\widetilde{\psi}}\right)^2\tau\left(\widetilde{\psi}\right) &= \nabla^{\widetilde{\psi}}_{(e_i,0)}\nabla^{\widetilde{\psi}}_{(e_i,0)}\tau\left(\widetilde{\psi}\right) + \frac{1}{f^2 \circ \pi}\nabla^{\widetilde{\psi}}_{(0,f_j)}\nabla^{\widetilde{\psi}}_{(0,f_j)}\tau\left(\widetilde{\psi}\right) \\
&\quad - \nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\psi}\right) - \frac{1}{f^2 \circ \pi}\nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\psi}\right).
\end{aligned}$$

A long calculation gives

$$\nabla^{\widetilde{\psi}}_{(e_i,0)}\nabla^{\widetilde{\psi}}_{(e_i,0)}\tau\left(\widetilde{\psi}\right)=\frac{2}{f^2\circ\pi}\left(\left(2\left|grad\ln f\right|^2-e_i\left(e_i\left(\ln f\right)\right)\right)\circ\pi\right)\tau\left(\psi\right)$$

and

$$\frac{1}{f^2\circ\pi}\nabla^{\widetilde{\psi}}_{(0,f_j)}\nabla^{\widetilde{\psi}}_{(0,f_j)}\tau\left(\widetilde{\psi}\right)=\frac{1}{f^4\circ\pi}\nabla^{\psi}_{f_j}\nabla^{\psi}_{f_j}\tau\left(\psi\right).$$

Finally, by (3.1), we obtain

$$\nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\psi}\right)=\frac{2}{f^2\circ\pi}\left(\nabla_{e_i}e_i\left(\left(\ln f\right)\right)\circ\pi\right)\tau\left(\psi\right)$$

and

$$\frac{1}{f^2\circ\pi}\nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\psi}\right)=\frac{1}{f^4\circ\pi}\nabla^{\psi}_{\nabla_{f_j}f_j}\tau\left(\psi\right)+\frac{2n}{f^2\circ\pi}\left(\left(\left|grad\ln f\right|^2\right)\circ\pi\right)\tau\left(\psi\right).$$

Which gives us

$$Tr_{G_f}\left(\nabla^{\widetilde{\psi}}\right)^2\tau\left(\widetilde{\psi}\right)=\frac{1}{f^4\circ\pi}Tr_h\nabla^2\tau\left(\psi\right)-\frac{2}{f^2\circ\pi}\left(\left(\Delta\ln f+(n-2)\left|grad\ln f\right|^2\right)\circ\pi\right)\tau\left(\psi\right) \tag{3.10}$$

Finally for the first term $Tr_{G_f}R^P\left(\tau\left(\widetilde{\psi}\right),d\widetilde{\psi}\right)d\widetilde{\psi}$, it is easy to verify that

$$Tr_{G_f}R^P\left(\tau\left(\widetilde{\psi}\right),d\widetilde{\psi}\right)d\widetilde{\psi}=\frac{1}{f^4\circ\pi}Tr_hR^P\left(\tau\left(\psi\right),d\psi\right)d\psi. \tag{3.11}$$

If we substitute (3.11) and (3.11) in (3.9), we obtain

$$\tau_2\left(\widetilde{\psi}\right)=\frac{1}{f^4\circ\pi}\tau_2\left(\psi\right)-\frac{2}{f^2\circ\pi}\left(\left(\Delta\ln f+(n-2)\left|grad\ln f\right|^2\right)\circ\pi\right)\tau\left(\psi\right).$$

This completes the proof of Theorem 3.2. An immediate consequence of Theorem 3.2 is given by the following corollary :

Corollary 3.2. Let $\psi:(N^n,h)\longrightarrow(P^p,k)$ a biharmonic non-harmonic map. The map $\widetilde{\phi}:\left(M^m\times_f N^n,G_{f^2}\right)\longrightarrow(P^p,k)$ defined by $\widetilde{\psi}(x,y)=\psi(y)$ is biharmonic if and only if the function f^{n-2} is harmonic.

References

(i) P. Baird, *Harmonic maps with symmetry, harmonic morphisms and deformation of metrics*, Pitman Books Limited, (1983), 27-39.

(ii) P. Baird, J. Eells, *A conservation law for harmonic maps*, Lecture Notes in Math. 894, Springer (1981), 1-25.

(iii) P. Baird, J. Eells, *Harmonic morphisms between Riemannain manifolds*, Oxford Sciences Publications (2003).

(iv) P. Baird, D. Kamissoko, *On constructing biharmonic maps and metrics*, Annals of Global Analysis and Geometry 23, (2003), 65-75.

(v) Baird, P. Fardoun, A. Ouakkas, S. : *Conformal and semi-conformal biharmonic maps*, Ann. Glob Anal Geom 34, 403-414 (2008).

(vi) A. Balmus, *Biharmonic properties and conformal changes*, An. Stiint. Univ. Al.I. Cuza Iasi Mat. (N.S.) 50 (2004), 367-372

(vii) J. Eells, L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 16 (1978), 1-68.

(viii) J. Eells, L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. 20 (1988), 385-524.

(ix) J. Eells, L. Lemaire, *Selected topics in harmonic maps*, CNMS Regional Conference Series of the National Sciences Foundation, November 1981.

(x) J. Eells, A. Ratto, *Harmonic Maps and Minimal Immersions with Symmetries*, Princeton University Press 1993.

(xi) G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A 7(1986), 389-402.

(xii) S. Montaldo, C. Oniciuc, *A short survey of biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina, 47(2) (2006), 1-22.

(xiii) Ouakkas, S.: *Biharmonic maps, conformal deformations and the Hopf maps.*Diff. Geom. Appl, 26 (2008), 495-502.

(xiv) Y.-L. Ou, *p-harmonic morphisms, biharmonic morphisms, and non-harmonic biharmonic maps*, J. Geom. Phys. Volume 56, 3 (2006),358-374.

Author information

Seddik Ouakkas, Laboratory of Geometry, Analysis, Control and Applications, UniversitÃl' de Saida, BP138, En-Nasr, 20000 Saida, ALGERIA.
E-mail: ayman.kashmar@gmail.com

Received: December 24, 2015.
Accepted: March 17, 2016.