Palestine Journal of Mathematics

Vol. 8(Special Issue: I, 2019) , 426–435

UNIQUE FIXED POINT THEOREM FOR WEAKLY B-CONTRACTIVE MAPPINGS IN 2-METRIC SPACES

V. S. Bright and M. Marudai

Communicated by T. Tamizh Chelvam

MSC 2010 Classifications: Primary 47H10; Secondary 54H25.

Keywords and phrases: Fixed Points, 2-Metric Space, Weak B-contraction.

Abstract The aim of this paper is to present a unique fixed point result for weak B-contractions in 2-metric spaces and an example is also given to illustrate the result. Further, from the particular cases of weakly B- contractive mappings, one can obtain weakly C-contractive, weakly S-contractive mappings etc., and the theorem proved herein is true for these cases as well. In fact, weak B-contraction is a generalization of all these so-called weak contractions.

1 Introduction and preliminaries

V.S. Bright in [12], [2] and [3] introduced the notions of B-contraction and weak B-contraction.

Definition 1.1. [12]: A mapping $T : X \to X$, where (X, d) is a metric space, is called a B-contractive if there exist positive real numbers α, β, γ such that $0 \leq \alpha + 2\beta + 2\gamma < 1$ for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{1.1}$$

This notion was also generalized to a weak B- contraction by V.S. Bright in [12]

Definition 1.2. ([12], definition 1.7): A mapping $T : X \to X$, where (X, d) is a metric space, is said to be weakly B-contractive or a weak B- contraction if for all $x, y \in X$ such that

$$\begin{aligned} d(Tx, Ty) \leq{}& \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \\ &- \psi[d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)] \end{aligned} \tag{1.2}$$

where $\psi : [0, \infty)^5 \to [0, \infty)$ is a continuous mapping such that $0 < \alpha + 2\beta + 2\gamma \leq 1$ and α, β and γ are non-zero positive real numbers.
$\psi(x, y, z, u, v) = 0$ if and only if $x = y = z = u = v = 0$

Note 1. If we take $\psi(x, y, z, u, v) = \alpha_1 x + \beta_1(y + z) + \gamma_1(u + v)$,
where $0 < \alpha_1 + 2\beta_1 + 2\gamma_1 < 1$ with $\alpha > \alpha_1$, $\beta > \beta_1$ and $\gamma > \gamma_1$ and α_1, β_1 and γ_1 are non-zero positive real numbers, and that (1.2) reduces to (1.1).
That is, weak B-contraction is a generalization of B-contraction.

In [12], V.S. Bright et al, proved if X is a complete metric space, then every weak B-contraction has a unique fixed point, see ([12] theorem 2.1). This result was generalized to a complete, partially ordered metric space in [2]; see [[2], theorem 2.2, 2.3, 2.6 and 3.2]

There were some generalizations of a metric, namely, 2 - metric, a D-metric, a G-metric, a cone metric, a complex valued metric, dislocated metric space and dislocated quasic - metric space [1].

The notion of a 2-metric has been introduced by Gähler in [7]. Nevertheless, a 2-metric is not a continuous function of its variable as in the case of ordinary metric. This enabled Dhage to introduce the notion called a D-metric is [5]. However, in [13] Mustafa and Sims proved that most of topological properties of D-metric were not fulfilled. In [14] Mustafa and sims introduced the notion of G-metric to overcome certain flaws of a D-metric and therefore many fixed point theorems on G-metric spaces have been established.

Note that there was no easy relationship between results obtained in 2-metric spaces and metric spaces. In fact, the fixed point theorem on 2-metric spaces and metric spaces may be unrelated easily.
The purpose of the paper is to prove unique fixed point result for weak B - contraction in a complete 2 - metric space.A simple example is also given to illustrate the theorem followed by discussing some particular cases of weak B-contraction in 2-metric spaces, which in turn, are true to this theorem.

Now we recall some definitions and lemmas which are useful in what follows.

Definition 1.3. [7] Let X be a non empty set and $d : X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions:

(i) For every pair of distinct point $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(ii) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$

(iii) The symmetry: $d(x, y, z) = d(y, x, z) = d(y, z, x) = d(x, z, y) = d(z, x, y) = d(z, y, x)$.

(iv) The rectangle inequality: $d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$.

Then d is called 2-metric on X and (X, d) is called a 2-metric space which will be sometimes denoted by X if there is no confusion. For each element $x \in X$ is called a point in X.

Definition 1.4. [7] Let (X, d) be a 2-metric space and $a, b \in X$, $r \geq 0$. The set

$$\mathbb{B}(a, b, r) = \{x \in X : d(a, b, x) < r\}$$

is called a 2-ball centred at a and b with radius r. The topology generated by the collection of all 2-balls as a subbasis is called a 2-metric topology on X.

Definition 1.5. [9] Let $\{x_n\}$ be a sequence in a 2-metric space (X, d).

(i) $\{x_n\}$ is said to be convergent to x in (X, d), written $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.

(ii) $\{x_n\}$ is said to be Cauchy in X if for all $a \in X$, $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$, that is, for each $\epsilon > 0$, there exists n_0 such that $d(x_m, x_n, a) < \epsilon$ for all $m, n \geq n_0$.

(iii) (X, d) is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 1.6. ([11], definition 8) : A 2-metric space (X, d) is said to be compact if every sequence in X has a convergent subsequence.

Lemma 1.1. *([11], lemma 3): Every 2-metric space is a T_1 space.*

Lemma 1.2. *([11], lemma 4): $\lim_{n\to\infty} x_n = x$ in a 2-metric space (X, d) if and only if $\lim_{n\to\infty} x_n = x$ in the 2-metric topological space X.*

Lemma 1.3. *([11], lemma 5): If $T : X \to Y$ is a continuous map from 2-metric space X to a 2-metric space Y, then*
$\lim_{n\to\infty} x_n = x$ in X implies $\lim_{n\to\infty} Tx_n = Tx$ in Y.

Note 2. (i) It is straight forward from definition (1.3) that every 2-metric is non-negative and every 2 - metric space contains at least three distinct points.

(ii) A convergent sequence in a 2-metric spaces need not be a Cauchy sequence [[15], remark 1 and example 1].

(iii) In a 2-metric space (X, d), every convergent sequence is a Cauchy sequence if d is continuous [[15], remark 2].

(iv) There exists a 2-metric space (X, d) such that every convergent sequence is a Cauchy sequence but d is not continuous [[15], remark 2 and example 2].

2 Main results

We shall first introduce the notion of a Weak B-contraction in 2-metric space.

Definition 2.1. ([12], [2] and [3]) Let (X, d) be a 2-metric space and $T : X \to X$ be a map. Then T is called a Weak B-contraction if there exists $\psi : [0,\infty)^5 \to [0,\infty)$ which is continuous and $\psi(x, y, z, u, v) = 0$ if and only if $x = y = z = u = v = 0$ such that

$$\begin{aligned} d(Tx, Ty, a) \leq \alpha d(x, y, a) &+ \beta[d(x, Tx, a) + d(y, Ty, a)] \\ &+ \gamma[d(x, Ty, a) + d(y, Tx, a)] - \psi[d(x, y, a), \\ &\quad d(x, Tx, a), d(y, Ty, a), d(x, Ty, a), d(y, Tx, a)] \end{aligned} \tag{2.1}$$

for all $x, y, a \in X$ and $0 < \alpha + 2\beta + 2\gamma \leq 1$ and α, β and γ are non-zero positive real numbers. The following example gives an example of ψ in definition (2.1), see ([12], example 2.3) and other example see ([12], definition 1.7)

Example 2.1.

$$\psi(a, b, c, d, e) = \frac{1}{5} max\{a, b, c, d, e\}.$$

In this section we are going to prove in a complete 2-metric space a weak B-contraction has a unique fixed point and we support it by an example. Also some particular cases of B-contraction and weak B-contraction have been given, which had been proved by different authors, can easily be drawn as corollaries of weak B-contraction.

Theorem 2.1. *Let $(X, \leq, d)$ be a complete 2 - metric space and $T : X \to X$ be a weak B-contraction. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and $n \geq 1$

$$x_{n+1} = Tx_n \tag{2.2}$$

If $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T. So we assume $x_n \neq x_{n+1}$. Putting $x = x_{n-1}$ and $y = x_n$ in (2.1), we have for all $n = 1, 2, ...$

$$\begin{aligned} &d(x_{n+1}, x_n, a) \\ &= d(Tx_n, Tx_{n-1}, a) \\ &\leq \alpha d(x_n, x_{n-1}, a) + \beta[d(x_n, Tx_n, a) + d(x_{n-1}, Tx_{n-1}, a)] \\ &\quad + \gamma[d(x_n, Tx_{n-1}, a) + d(x_{n-1}, Tx_n, a)] - \psi[d(x_n, x_{n-1}, a), \\ &\qquad d(x_n, Tx_n, a), d(x_{n-1}, Tx_{n-1}, a), d(x_n, Tx_{n-1}, a), \\ &\qquad d(x_{n-1}, Tx_n, a)] \\ &= \alpha d(x_n, x_{n-1}, a) + \beta[d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] \\ &\quad + \gamma[d(x_n, x_n, a) + d(x_{n-1}, x_{n+1}, a)] - \psi[d(x_n, x_{n-1}, a), \\ &\qquad d(x_n, x_{n+1}, a), d(x_{n-1}, x_n, a), d(x_n, x_n, a), \\ &\qquad d(x_{n-1}, x_{n+1}, a)] \\ &= \alpha d(x_n, x_{n-1}, a) + \beta[d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] \\ &\quad + \gamma[0 + d(x_{n-1}, x_{n+1}, a)] - \psi[d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \\ &\qquad d(x_{n-1}, x_n, a), 0, d(x_{n-1}, x_{n+1}, a)] \qquad (2.3) \\ &\leq \alpha d(x_n, x_{n-1}, a) + \beta[d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] \\ &\quad + \gamma[d(x_{n-1}, x_{n+1}, a)] \qquad (2.4) \end{aligned}$$

for all $a \in X$. By choosing $a = x_{n-1}$ in (2.4), we have

$$d(x_{n+1}, x_n, x_{n-1}) \leq \beta d(x_n, x_{n+1}, x_{n-1}),$$

that is $d(x_{n+1}, x_n, x_{n-1}) \leq 0$
This implies that

$$d(x_{n+1}, x_n, x_{n-1}) = 0 \tag{2.5}$$

It follows from (2.4) by using (2.5) that

$$\begin{aligned} d(x_{n+1}, x_n, a) &\leq \alpha d(x_n, x_{n-1}, a) + \beta[d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] \\ &\quad + \gamma[d(x_{n-1}, x_{n+1}, x_n) + d(x_{n+1}, x_n, a) \\ &\quad\quad + d(x_n, x_{n-1}, a)] \\ &= \alpha d(x_n, x_{n-1}, a) + \beta[d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] \\ &\quad + \gamma[d(x_{n+1}, x_n, a) + d(x_n, x_{n-1}, a)]. \end{aligned} \tag{2.6}$$

It implies that

$$d(x_{n+1}, x_n, a) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right) d(x_n, x_{n-1}, a)$$

But

$$\begin{aligned} &\alpha + 2\beta + 2\gamma \leq 1 \\ &\Rightarrow \alpha + \beta + \gamma \leq 1 - \beta - \gamma \\ &\Rightarrow \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \leq 1 \end{aligned}$$

Then

$$d(x_n, x_{n+1}, a) \leq d(x_{n-1}, x_n, a) \tag{2.7}$$

Therefore $\{d(x_n, x_{n+1}, a)\}$ is a decreasing sequence of non-negative real numbers and hence it is convergent. Let

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = r \tag{2.8}$$

Taking the limit as $n \to \infty$ in (2.4),(2.6) and using (2.8), we get

$$\begin{aligned} r &\leq \alpha r + \beta 2r + \gamma \lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) \leq \alpha r + \beta 2r + \gamma 2r \\ \frac{(1 - \alpha - 2\beta)r}{\gamma} &\leq \lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) \leq 2r. \end{aligned}$$

But

$$\begin{aligned} &\alpha + 2\beta + 2\gamma \leq 1 \\ &\Rightarrow \frac{1 - \alpha - 2\beta}{\gamma} \geq 2 \\ \therefore\ 2r \leq &\lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) \leq 2r. \end{aligned}$$

That is,

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) = 2r \tag{2.9}$$

Taking the limit as $n \to \infty$ in (2.3) and using (2.8) and (2.9), we get

$$\begin{aligned} r &\leq \alpha r + \beta 2r + \gamma 2r - \psi(r, r, r, 0, 2r) \\ &= (\alpha + 2\beta + 2\gamma)r - \psi(r, r, r, 0, 2r) \\ &\leq r - \psi(r, r, r, 0, 2r), \quad \mathit{since} \quad \alpha + 2\beta + 2\gamma \leq 1 \end{aligned}$$

$$r \leq r - \psi(r, r, r, 0, 2r) \leq r \quad \mathit{as} \quad \psi \geq 0$$

That is,

$$\psi(r,r,r,2r,0)=0$$
$$\Rightarrow r=0$$

Thus (2.8) becomes

$$\lim_{n\to\infty} d(x_{n+1},x_n,a)=0 \tag{2.10}$$

From (2.7),we have if $d(x_{n-1},x_n,a)=0$ then $d(x_n,x_{n+1},a)=0$.
Since $d(x_0,x_1,x_0)=0$, we have $d(x_n,x_{n+1},x_0)=0$ for all $n\in N$.
Since $d(x_{m-1},x_m,x_m)=0$, we have

$$\begin{aligned} d(x_n,x_m,x_m)&=0\\ d(x_n,x_{n+1},x_m)&=0 \quad \text{for all} \quad n\geq m-1 \end{aligned} \tag{2.11}$$

For $0\leq n<m-1$, noting that $m-1\geq n+1$,from (2.11) we have

$$d(x_{m-1},x_m,x_{n+1})=d(x_{m-1},x_m,x_n)=0$$

Now, by rectangle inequality, it follows that

$$\begin{aligned} d(x_n,x_{n+1},x_m)&\leq d(x_n,x_{n+1},x_{m-1})+d(x_{n+1},x_m,x_{m-1})\\ &\quad+d(x_m,x_n,x_{m-1})\\ &=d(x_n,x_{n+1},x_{m-1}) \end{aligned} \tag{2.12}$$

Since $d(x_n,x_{n+1},x_{m-1})=0$ for $m-1\geq n+1$, from (2.12) it follows that

$$d(x_n,x_{n+1},x_m)=0 \quad \text{for all} \quad 0\leq n\leq m-1 \tag{2.13}$$

From (2.11)and (2.13) it follows that $d(x_n,x_{n+1},x_m)=0$ for all $m,n\in N$.
Now for all $i,j,k\in N$ with $i<j$, we have $d(x_{j-1},x_j,x_i)=d(x_{j-1},x_j,x_k)=0$.
Now,

$$\begin{aligned} d(x_i,x_j,x_k)&\leq d(x_i,x_j,x_{j-1})+d(x_j,x_k,x_{j-1})+d(x_k,x_i,x_{j-1})\\ &\leq d(x_i,x_{j-1},x_k)\leq\ldots\ldots\leq d(x_i,x_i,x_k)=0 \end{aligned}$$

This show that for all $i,j,k\in N$

$$d(x_i,x_j,x_k)=0 \tag{2.14}$$

Next we show that $\{x_n\}$ is a cauchy sequence. Suppose to the contrary $\{x_n\}$ is not a cauchy sequence. Then there exists $\epsilon>0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$, where $n(k)$ is the smallest integer such that $n(k)>m(k)>k$ and

$$d(x_{n(k)},x_{m(k)},a)\geq\epsilon, \quad \text{for all} \quad k\in N \quad \text{and for all} \quad a\in X \tag{2.15}$$

Therefore

$$d(x_{n(k)-1},x_{m(k)},a)<\epsilon. \tag{2.16}$$

By using (2.14),(2.15)and (2.16),we have

$$\begin{aligned} \epsilon&\leq d(x_{n(k)},x_{m(k)},a)\\ &\leq d(x_{n(k)},x_{n(k)-1},a)+d(x_{n(k)-1},x_{m(k)},a)+d(x_{n(k)},x_{m(k)},x_{n(k)-1})\\ &\leq d(x_{n(k)},x_{n(k)-1},a)+d(x_{n(k)-1},x_{m(k)},a)\\ &<d(x_{n(k)},x_{n(k)-1},a)+\epsilon. \end{aligned} \tag{2.17}$$

Taking limit as $k \to \infty$ in (2.17)using (2.10) we have

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}, a) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}, a) = \varepsilon. \tag{2.18}$$

Also, using (2.14),we have

$$\begin{aligned} d(x_{m(k)}, x_{n(k)-1}, a) &\le d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) \\ &\quad + d(x_{m(k)}, x_{n(k)-1}, x_{m(k)-1}) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) \\ &\le d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)}) \\ &\quad + d(x_{n(k)-1}, x_{n(k)}, a) + d(x_{n(k)}, x_{m(k)-1}, a) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{n(k)-1}, x_{n(k)}, a) \\ &\quad + d(x_{n(k)}, x_{m(k)-1}, a) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} d(x_{m(k)-1}, x_{n(k)}, a) &\le d(x_{m(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, a) \\ &\quad + d(x_{m(k)-1}, x_{n(k)}, x_{m(k)}) \\ &= d(x_{m(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, a) \end{aligned} \tag{2.20}$$

Taking limit as $k \to \infty$ in (2.19),(2.20)using (2.10),(2.18), we get

$$\epsilon \le \lim_{k\to\infty} (x_{m(k)-1}, x_{n(k)}, a) \le \epsilon$$

Hence

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, a) = \epsilon. \tag{2.21}$$

Since $n(k) > m(k)$ and from (2.15) by using (2.1),we have

$$\begin{aligned} \epsilon &\le d(x_{n(k)}, x_{m(k)}, a) \\ &= d(Tx_{n(k)-1}, Tx_{m(k)-1}, a) \\ &\le \alpha d(x_{n(k)-1}, x_{m(k)-1}, a) + \beta[d(x_{n(k)-1}, x_{n(k)}, a) + d(x_{m(k)-1}, x_{m(k)}, a)] \\ &\quad + \gamma[d(x_{n(k)-1}, x_{m(k)}, a) + d(x_{m(k)-1}, x_{n(k)}, a)] - \psi[d(x_{n(k)-1}, x_{m(k)-1}, a), \\ &\qquad d(x_{n(k)-1}, x_{n(k)}, a), d(x_{m(k)-1}, x_{m(k)}, a), d(x_{n(k)-1}, x_{m(k)}, a), d(x_{m(k)-1}, x_{n(k)}, a)] \end{aligned} \tag{2.22}$$

Taking limit as $k \to \infty$ in (2.22) and using (2.21), (2.18),(2.10) and the continuity of ψ, we have

$$\begin{aligned} \epsilon &\le \alpha \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}, a) + \beta[\lim_{k\to\infty} d(x_{n(k)-1}, x_{n(k)}, a) \\ &\quad + \lim_{k\to\infty} d(x_{m(k)-1}, x_{m(k)}, a)] + \gamma 2\epsilon - \psi(\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}, a), 0, 0, \epsilon, \epsilon) \end{aligned} \tag{2.23}$$

Now we shall show

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}, a) = \epsilon$$

Now

$$\begin{aligned} d(x_{n(k)-1}, x_{m(k)}, a) &\le d(x_{n(k)-1}, x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)}, x_{m(k)-1}, a) \\ &\quad + d(x_{n(k)-1}, x_{m(k)-1}, a) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) \end{aligned}$$

Now from (2.18), we have

$$\begin{aligned}\epsilon &= \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}, a)\\ &\le \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}, a) + \lim_{k\to\infty} d(x_{m(k)-1}, x_{m(k)}, a)\\ &\le \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}, a)\\ &\le \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)}) + \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, a)\\ &\quad + \lim_{k\to\infty} d(x_{n(k)}, x_{n(k)-1}, a)\end{aligned} \tag{2.24}$$

using (2.21) and (2.14) in (2.24) we get

$$\begin{aligned}\epsilon \le \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}, a) \le \epsilon\\ \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}, a) = \epsilon\end{aligned} \tag{2.25}$$

In (2.23), using (2.25) and the continuity of ψ, we get

$$\begin{aligned}\epsilon &\le \alpha\epsilon + 2\epsilon\gamma - \psi(\epsilon, 0, 0, \epsilon, \epsilon)\\ \epsilon &\le (\alpha + 2\gamma)\epsilon - \psi(\epsilon, 0, 0, \epsilon, \epsilon)\end{aligned}$$

since $\alpha + 2\beta + 2\gamma \le 1$ and since α, β, γ are non zero real numbers, it follows that $\alpha + 2\gamma < 1$. Therefore,
$\epsilon < \epsilon - \psi(\epsilon, 0, 0, \epsilon, \epsilon) \le \epsilon$, since $\psi \ge 0$

$$\Longrightarrow \psi(\epsilon, 0, 0, \epsilon, \epsilon) = 0 \Longrightarrow \epsilon = 0,$$

which is a contradiction. This proves that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Next, we show that $\lim_{n\to\infty} d(x_{n+1}, Tz, a) = d(Tz, z, a)$ for all $a \in X$.

By rectangle inequality, we have

$$d(x_{n+1}, Tz, a) \le d(x_{n+1}, Tz, z) + d(Tz, z, a) + d(x_{n+1}, z, a) \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{2.26}$$

Since $\lim_{n\to\infty} x_n = z$,$\lim_{n\to\infty} d(x_n, z, a) = 0$ for all $a \in X$.
Taking $n \to \infty$ in (2.26) we have

$$\lim_{n\to\infty} d(x_{n+1}, Tz, a) \le d(Tz, z, a) \tag{2.27}$$

Again,
$d(Tz, z, a) \le d(Tz, x_{n+1}, z) + d(z, a, x_{n+1}) + d(Tz, x_{n+1}, a)$ for $n = 0, 1, 2, \ldots$
Taking $n \to \infty$ in the above inequality, we have $d(Tz, z, a) \le \lim_{n\to\infty} d(Tz, x_{n+1}, a)$ That is,

$$d(Tz, z, a) \le \lim_{n\to\infty} d(x_{n+1}, Tz, a) \tag{2.28}$$

From (2.27) and (2.28) it follows that

$$\lim_{n\to\infty} d(x_{n+1}, Tz, a) = d(Tz, z, a) \text{for all} \quad a \in X. \tag{2.29}$$

Now in the following rectangle inequality and using (2.1) and (2.10), we have

$$\begin{aligned}d(z, Tz, a) &\le d(z, Tz, x_{n+1}) + d(z, x_{n+1}, a) + d(x_{n+1}, Tz, a)\\ &= d(z, Tz, x_{n+1}) + d(z, x_{n+1}, a) + d(Tx_n, Tz, a)\\ &\le d(z, Tz, x_{n+1}) + d(z, x_{n+1}, a) + \alpha d(x_n, z, a)\end{aligned}$$

$$+ \beta[d(z, Tx_n, a) + d(z, Tz, a)] + \gamma[d(x_n, Tz, a) + d(z, Tx_n, a)]$$
$$- \psi[d(x_n, z, a), d(x_n, Tx_n, a), d(z, Tz, a), d(x_n, Tz, a), d(z, Tx_n, a)]$$

Taking $n \to \infty$ on either side and using (2.29), we get

$$d(z, Tz, a) \leq \beta d(z, Tz, a) + \gamma d(z, Tz, a) - \psi[0, 0, d(z, Tz, a), d(z, Tz, a), 0]$$

That is,
$d(z, Tz, a) \leq (\beta + \gamma)d(z, Tz, a)$, since $\psi \geq 0$.
That is,
$[1 - (\beta + \gamma)]d(z, Tz, a) \leq 0$.
Since $1 - (\beta + \gamma) > 0$, we have $d(z, Tz, a) = 0$ for all $a \in X$
$\Rightarrow Tz = z$.
Next we shall establish the fixed point z is unique. Let z_1 and z_2 be two fixed points of T.

$$\begin{aligned}
d(z_1, z_2, a) &= d(Tz_1, Tz_2, a) \\
&\leq \alpha d(z_1, z_2, a) + \beta[d(z_1, Tz_1, a) + d(z_2, Tz_2, a)] \\
&\quad + \gamma[d(z_1, Tz_2, a) + d(z_2, Tz_1, a)] \\
&\quad - \psi[d(z_1, z_2, a), d(z_1, Tz_1, a), d(z_2, Tz_2, a), d(z_1, Tz_2, a), d(z_2, Tz_1, a)] \\
&= \alpha d(z_1, z_2, a) + \beta[d(z_1, z_1, a) + d(z_2, z_2, a)] \\
&\quad + \gamma[d(z_1, z_2, a) + d(z_2, z_1, a)]
\end{aligned}$$

$$\begin{aligned}
&\quad - \psi[d(z_1, z_2, a), d(z_1, z_1, a), d(z_2, z_2, a), d(z_1, z_2, a), d(z_2, z_1, a)] \\
&= (\alpha + 2\gamma)d(z_1, z_2, a) - \psi[d(z_1, z_2, a), 0, 0, d(z_1, z_2, a), d(z_2, z_1, a)] \\
&< d(z_1, z_2, a) - \psi[d(z_1, z_2, a), 0, 0, d(z_1, z_2, a), d(z_2, z_1, a)], \text{since} \quad (\alpha + 2\gamma) < 1 \\
&< d(z_1, z_2, a), \quad \text{since} \quad \psi \geq 0 \\
&\Rightarrow \psi[d(z_1, z_2, a), 0, 0, d(z_1, z_2, a), d(z_2, z_1, a)] = 0 \\
&\Rightarrow d(z_1, z_2, a) = 0 \quad \text{for all} \quad a \in X \\
&\Rightarrow z_1 = z_2.
\end{aligned}$$

This completes the proof. □

The above theorem can be illustrated by the following example.

Example 2.2. Let $X = \{a, b, c\}$. Let d be a 2-metric on X defined by a symmetry of all three variables and

$$d(x, y, z) = \begin{cases} 1, & \text{if } x \neq y \neq z \\ 0, & \text{otherwise} \end{cases}$$

Let $T : X \to X$ be define by $T(a) = c, T(b) = c, T(c) = c$. It is easy to see that X is complete and T is a weak B-contraction by defining ψ as in example 2.1 . Here T has a unique fixed point.

Note 3. In (2.1) if we take $\alpha = 0$, $\beta = 0$ then $2\gamma \leq 1 \Rightarrow \gamma \leq \frac{1}{2}$ Therefore, we have

$$d(Tx, Ty, a) \leq \gamma\{d(x, Ty, a) + d(y, Tx, a)\} - \psi\{d(x, Ty, a), d(y, Tx, a)\},$$

$\gamma \in (0, \frac{1}{2}]$ for all $x, y, a \in X$, where $\psi : [0, \infty)^2 \to [0, \infty)$ is continuous and $\psi(s, t) = 0$ if and only if $s = t = 0$ and in particular, if we take $\gamma = \frac{1}{2}$, we get,

$$d(Tx, Ty, a) \leq \frac{1}{2}\{d(x, Ty, a) + d(y, Tx, a)\} - \psi\{d(x, Ty, a), d(y, Tx, a)\}$$

for all $x, y, a \in X$, which is a weak C-contraction [4], [6]. In a similar manner, if we take in (2.1) and give $\beta = 0$ with the condition $0 < \alpha + 2\beta + 2\gamma \leq 1$ and $\alpha = \gamma$, we get $0 < 3\alpha \leq 1$. This implies $0 < \alpha \leq \frac{1}{3}$, ie, $\alpha \in (0, \frac{1}{3}]$ and the corresponding $\psi : [0, \infty)^3 \to [0, \infty)$ continuous and $\psi(s,t,u) = 0$ if and only if $s = t = u = 0$ and in particular, if we take $\alpha = \gamma = \frac{1}{3}$, the weak B-contraction (2.1) turns out to be

$$d(Tx,Ty,a) \leq \frac{1}{3}\{d(x,y,a) + d(x,Ty,a) + d(y,Tx,a)\} \\ - \psi\{d(x,y,a), d(x,Ty,a), d(y,Tx,a)\}$$

for all $x, y, a \in X$, which is known as weak S-contraction [18]. So also, if we take (2.1) and put $\beta = 0, \gamma = 0$ with the condition $0 < \alpha + 2\beta + 2\gamma \leq 1$ and this α in turn becomes $0 < \alpha \leq 1$. ie, $\alpha \in (0, 1]$ and in particular, if $\beta = 0, \gamma = 0$ and $\alpha = 1$ in (2.1), it becomes weakly contractive mappings [16].
Further, in (1.1) if we take $\alpha = 0$ and $\gamma = 0$ in the condition $0 < \alpha + 2\beta + 2\gamma < 1$ it becomes $0 \leq 2\beta < 1$. ie, $\beta \in [0, \frac{1}{2})$ and it becomes similar to Kannan type mappings [17] [19].
From the proof of theorem, it is clear that for all the aforesaid particular cases of weak B-contraction such as weak C-contraction [4], weak S-contraction [18] and Kannan contraction [17] are hold good for this theorem.
Also from note(1) it may be recalled that weak B-contraction becomes B-contraction and hence the above theorem is true for this case as well.
Hence weak B-contraction is a generalization of all the so-called contractions.

References

[1] Aage C.T and Salunke J.N, The results on fixed points in dislocated and dislocated quasi - metric space, *Appl. Math Sci.* **2(59)**, 2941-2948 (2008).

[2] Bright V.S and Marudai M, Fixed point theorem for weakly B-contractive mapping in ordered metric spaces, *Gen. Math. notes* **35(2)**, 1-14 (2016).

[3] Bright V.S and Marudai M, Best Proximity points for generalized proximal B-Contraction mappings in metric spaces with partial orders, *J. of the calcutta Math. Soc.* **12(1)**, 31-46 (2016).

[4] Choudhury B.S, Unique fixed point theorem for weakly C-Contractive mappings, *Kathmandu Univ. J.Sci. Eng. Technol.* **5**, 6-13 (2009).

[5] Dhage B.C, *A study of some fixed point theorems, Ph.D thesis*, Marathwada, Aurangabad, india (1984).

[6] Dung and Le Hang, Fixed point theorems for weak C-contractions in partially ordered 2-metric spaces, *Fixed point theory and appl.* **161**, 1-12 (2013).

[7] Gählar V.S, 2-metrische Räume und ihre topologische, *struktur. Math. Nachr.* **26**, 115-118(1963/64).

[8] Harjani J, Lopaz B and Sadarangani K, Fixed Point theorems for weakly C-Contractive mappings in ordered metric spaces, *Comput. Math. Appl.* **61**, 790-796 (2011).

[9] Iseki K, Fixed point theorems in 2-metric spaces, *Math-semi-Notes* **3**, 133-136 (1975).

[10] Kannan R, Some results on fixed points, *Bull. calcutta Math. soc.* **60**, 71-76 (1968).

[11] Lahiri B.K , Das P, Dey L.K, Cantor's theorem in 2.metric spaces and its applications to fixed point problems, *Taiwan J. Math* **15**, 337-352 (2011).

[12] Marudai M and Bright V.S, Unique fixed point theorem for weakly B-contractive mapping, *Far East j. of Math. sci. (FJMS)* **98(7)**, 897-914 (2015).

[13] Mustafa Z and Sims B, *Some remarks concerning D-metric Spaces*, In Pro. of the int. conf. on fixed point theory and Appl. Valencia, Spain (2003).

[14] Mustafa Z, Sims B, A new approach to generalized metric spaces, *J.Nonlinear Convex Anal.* **7(2)**, 289-297 (2006).

[15] Naidu S.V.R, Prasad J.R, Fixed point theorem in 2-metric spaces, *Indian J.Pure Appl. Math.* **17(8)**, 974-993 (1986).

[16] Rhoads B.E, Some theorems on weakly contractive maps, *Nonlinear Anal.* **47**, 2683-2693 (2001).

[17] Shioji N, Sazuki T and Takahashi W, *Contractive mappings, Kannan mappings and Metric Completeness*, **126**, Pro. Amer. Math. Soc. (1998).

[18] Shukla D.P and Shiv kant Tiwari, Unique fixed point theorem for weakly S-contractive mappings, *Gen. Math. Notes* **4(1)**, 28-34 (2011).

[19] Zamfirescu T, Fixed point theorems in metric spaces, *Arch. Math.* **23**, 292-298 (1972).

Author information

V. S. Bright, Department of Mathematics, Mary Matha college of Arts and Science, Periyakulam 625604, India.
E-mail: bright681957@gmail.com

M. Marudai, Bharathidasan University, Tiruchirappalli 620024, India.
E-mail: mmarudai@yahoo.co.in