Palestine Journal of Mathematics

Vol. 8(Special Issue: I, 2019) , 444–450

Analytic Odd Mean Labeling Of Some Standard Graphs

P. Jeyanthi, R.Gomathi and Gee-Choon Lau

Communicated by Gyula O.H. Katona

AMS Subject Classification(2010) : 05C78 .

Keywords and phrases: mean labeling; analytic mean labeling; analytic odd mean labeling; analytic odd mean graph.

Abstract Let $G = (V, E)$ be a graph with p vertices and q edges. A graph G is analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \ldots, 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2 - (f(u)+1)^2}{2} \right\rceil & \text{if } f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil & \text{if } f(u) = 0 \end{cases}$$

is injective. We say that f is an analytic odd mean labeling of G. In this paper we prove that path P_n, cycle C_n, complete graph K_n, Wheel graph W_n, complete bipartite graph $K_{m,n}$, flower graph Fl_n, ladder graph L_n, comb $P_n \odot K_1$, the graph $L_n \odot K_1$ and the graph $C_m \cup C_n$ are analytic odd mean graph.

1 Introduction

Throughout this paper we consider only finite, simple and undirected graph $G = (V, E)$ with p vertices and q edges and notations not defined here are used in the sense of Harary [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. An excellent survey of graph labeling is available in [2]. The concept of mean labeling was introduced in [3]. A graph G is called a mean graph if there is an injective function $f : V \to \{0, 1, 2, 3 \ldots, q\}$ with an induce edge labeling $f^* : E \to Z$ given by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is injective. The concept of analytic mean labeling was introduced in [4]. A graph G is analytic mean graph if it admits a bijection $f : V \to \{0, 1, 2, \ldots, p - 1\}$ such that the induced edge labeling $f^* : E \to Z$ given by $f^*(uv) = \left\lceil \frac{f(u)^2 - f(v)^2}{2} \right\rceil$ with $f(u) > f(v)$ is injective. Motivated by the results in [4], we introduced a new mean labeling called analytic odd mean labeling. A graph G is an analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \ldots, 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$, $f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2 - (f(u)+1)^2}{2} \right\rceil & \text{if } f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil & \text{if } f(u) = 0 \end{cases}$ is injective. We say that f is an analytic odd mean labeling of G.

We use the following definitions in the subsequent section to prove the results.

Definition 1.1. A graph G is complete if every pair of its vertices is adjacent. A complete graph on n vertices is denoted by K_n.

Definition 1.2. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 so that every edge of G has one end in V_1 and the other end in V_2; (V_1, V_2) is called a bipartition of G. If every vertex of V_1 is joined to all the vertices of V_2, G is called a complete bipartite graph. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

Definition 1.3. Let u and v be (not necessarily distinct) vertices of a graph G. A $u - v$ walk of G is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i$, $i = 1, 2, \ldots, n$. The number n is called the length of the walk. The walk is said to be open if u and v are distinct vertices; otherwise it is closed. A walk $u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_n$ of its vertices and hence we specify this walk by $(u_0, u_1, u_2, \ldots, u_n)$. A walk in which all the vertices are distinct is called a path. A closed walk $(u_0, u_1, u_2, \ldots, u_n = u_0)$ in which $u_0, u_1, u_2, \ldots, u_{n-1}$ are distinct is called a cycle. P_n denotes a path on n vertices and C_n denotes a cycle on n vertices.

Definition 1.4. A wheel graph W_n is obtained from a cycle C_n by adding a new vertex and joining it to all the vertices of the cycle by an edge. The new edges are called spokes of wheel.

Definition 1.5. The union of two graphs G_1 and G_2 is a graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 1.6. The ladder graph L_n is the Cartesian product $P_2 \times P_n$ of a path on two vertices and another path on n vertices.

Definition 1.7. The flower graph Fl_n is constructed from a wheel W_n by attaching a pendent edge at each vertex of the n- cycle and by joining each pendent vertex to the central vertex.

Definition 1.8. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} vertex of G_2.

2 Main Results

In this section we prove that path P_n, cycle C_n, complete graph K_n, complete bipartite graph $K_{m,n}$, wheel graph W_n, flower graph Fl_n, ladder graph L_n, comb $P_n \odot K_1$, graph $L_n \odot K_1$ and union of two cycles are analytic odd mean graphs.

Theorem 2.1. *Every path P_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of path be $V(P_n) = \{u_i : 0 \leq i \leq n-1\}$ and $E(P_n) = \{u_i u_{i+1} : 0 \leq i \leq n-2\}$
Now $|V(P_n)| = n$ and$|E(G)| = n-1$.
We define an injective map $f : V(S_n) \to \{0, 1, 3, 5, \ldots, 2n-3\}$ by
$f(u_0) = 0$ and $f(u_i) = 2i - 1$ for $1 \leq i \leq n-1$.
The induced edge labeling f^* is defined as follows:
$f^*(u_{i-1}u_i) = 2i - 1$ for $1 \leq i \leq n-1$.
We observe that the edge labels are distinct and odd . Hence P_n admits an analytic odd mean labeling. An analytic odd mean labeling of P_6 is shown in Figure 1.

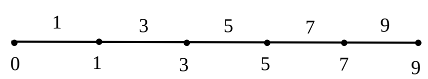

Figure-1

Theorem 2.2. *The cycle C_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of cycle be $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$.
Now $|V(C_n)| = n = |E(C_n)|$.
We define an injective map $f : V(C_n) \to \{0, 1, 3, 5, \ldots, 2n-1\}$ by
$f(v_i) = 2i - 1$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(v_i v_{i+1}) = 2i + 1$ for $1 \leq i \leq n-1$
$f^*(v_n v_1) = 2n^2 - 2n - 1$.
We observe that the edge labels are distinct and odd. Hence C_n admits an analytic odd mean labeling.
An analytic odd mean labeling of C_8 is shown in Figure 2.

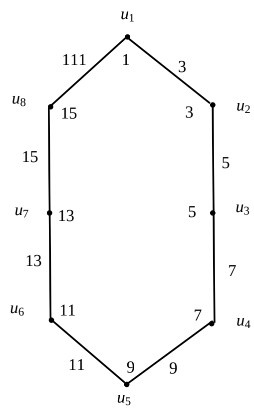

Figure-2

Theorem 2.3. *The complete graph K_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of complete graph be $V(K_n) = \{v_i : 0 \le i \le n-1\}$ and $E(K_n) = \{v_iv_j : 0 \le i < j \le n-1\}$.
Now $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.
We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, n^2 - n - 1\}$ by
$f(v_0) = 0$ and $f(v_i) = 4i - 3$ for $1 \le i \le n-1$.
The induced edge labeling f^* is defined as follows:
$f^*(v_0v_i) = 8i^2 - 12i + 5$ for $1 \le i \le n-1$
and $f^*(v_iv_j) = 4j(2j-3) - 8i(i-1) + 3$ for $1 \le i \le n-2$ and $i+1 \le j \le n-1$.
We observe that the edge labels are odd and distinct. Hence K_n admits an analytic odd mean labeling.
An analytic odd mean labeling of K_6 is shown in Figure 3.

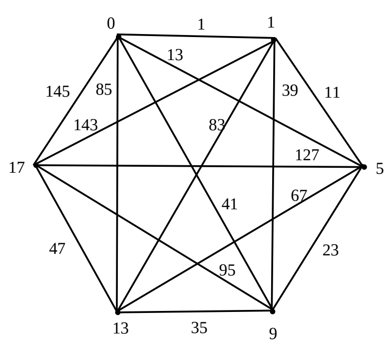

Figure-3

Theorem 2.4. *The wheel graph W_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of wheel graph be $V(W_n) = \{v_i : 0 \le i \le n\}$ and $E(W_n) = \{v_0v_i : 1 \le i \le n\} \cup \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_nv_1\}$.
Now $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$.
We define an injective map $f : V(W_n) \to \{0, 1, 3, 5, \ldots, 4n - 1\}$ by
$f(v_0) = 0$ and $f(v_i) = 4i - 3$ for $1 \le i \le n$.
The induced edge labeling f^* is defined as follows:
$f^*(v_0v_i) = 8i^2 - 12i + 5$ for $1 \le i \le n$
$f^*(v_iv_{i+1}) = 12i - 1$ for $1 \le i \le n-1$
$f^*(v_nv_1) = 8n^2 - 12n + 3$.
We observe that the edge labels are odd and distinct. Hence W_n admits an analytic odd mean labeling.
An analytic odd mean labeling of W_8 is shown in Figure 4.

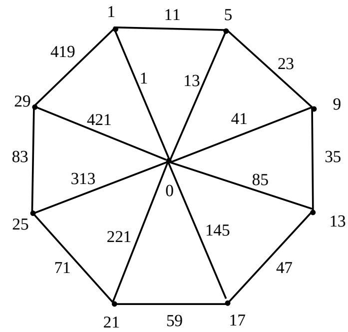

Figure-4

Theorem 2.5. *Every complete bipartite graph $K_{m,n}$ for any integer $n \geq m \geq 1$ is an analytic odd mean graph .*

Proof. Let the vertex set and edge set of complete bipartite graph be $V(K_{m,n}) = \{u_i, u_{n+j} : 0 \leq i \leq n-1 \text{ and } 1 \leq j \leq m\}$ and $E(K_{m,n}) = \{u_i u_{n+j} : 0 \leq i \leq n-1 \text{ and } 1 \leq j \leq m\}$.
Now $|V(G)| = m + n$ and $|E(G)| = mn$.
We define an injective map $f : V(K_{m,n}) \to \{0, 1, 3, 5, \ldots, 2mn-1\}$ by $f(u_0) = 0$ and $f(u_i) = 2i-1$ for $1 \leq i \leq n-1$ and label m vertices by $2mn-2m+1, 2mn-2m+3, 2mn-2m+5, \ldots, 2mn-1$.
That is $f(u_{n+j}) = 2mn - 2m + 2j - 1$ for $1 \leq j \leq m$.
The induced edge labeling f^* is defined as follows:
$f^*(u_{n+j}u_0) = \frac{(2mn-2m+2j-1)^2+1}{2}$ for $1 \leq j \leq m$ and
$f^*(u_{n+j}u_i) = \frac{(2mn-2m+2j-1)^2-(2i)^2+1}{2} = \frac{(2mn-2m-1)^2+1}{2} + 2j(2mn-2m-1) + 2j^2 - 2i^2$ for $1 \leq i \leq n-1$ and $1 \leq j \leq m$.
We observe that for fix j, the edge labels decreases by $2, 6, 10, 14, \ldots,$ as i increases from 0 to n-1. As j increases to $j+1$, $f(u_{n+j}u_0) < f(u_{n+j+1}u_{n-1})$. So all the edge labels are distinct and odd. Hence the complete bipartite graph $K_{m,n}$ admits an analytic odd mean labeling.
An analytic odd mean labeling of complete bipartite graph $K_{3,4}$ is shown in Figure 5.

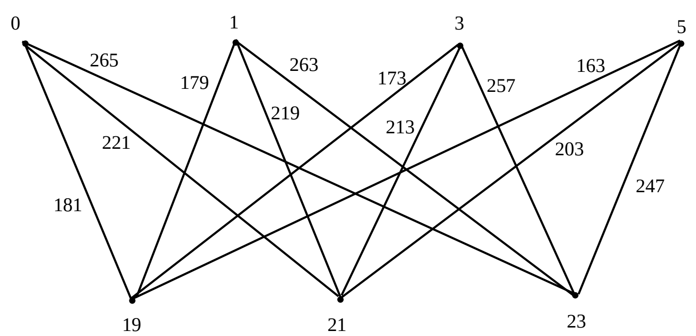

Figure-5

Theorem 2.6. *The flower graph Fl_n is an analytic odd mean graph.*

Proof. Let the vertex set and edge set be $V(Fl_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(Fl_n) = \{uu_i, uv_i, u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$.
Hence $|V| = 2n + 1$ and $|E| = 4n$.
We define an injective map $f : V(Fl_n) \to \{0, 1, 3, 5, \ldots, 8n-1\}$ by
$f(u) = 0$, $f(u_i) = 4i - 3$ for $1 \leq i \leq n$
$f(v_i) = 4(n+i) - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(uu_i) = 8i^2 - 12i + 5$ for $1 \leq i \leq n$
$f^*(uv_i) = 4(n+i)[2n + 2i - 3] + 5$ for $1 \leq i \leq n$
$f^*(u_iv_i) = 4n(2n-3) + 4i(4n-1) + 3$ for $1 \leq i \leq n$
$f^*(u_iu_{i+1}) = 12i - 1$ for $1 \leq i \leq n-1$
and $f^*(u_nu_1) = 8n^2 - 12n + 3$.
Clearly the edge labels are odd and distinct. Hence the flower graph Fl_n admits an analytic odd

mean labeling.
An analytic odd mean labeling of flower graph Fl_6 is shown in Figure 6.

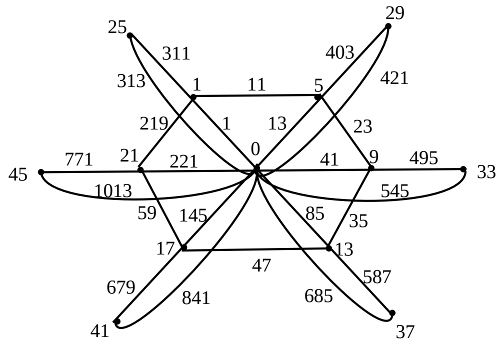

Figure-6

Theorem 2.7. *The ladder graph L_n for $n \geq 2$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set be $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$.
Hence $|V| = 2n$ and $|E| = 3n - 2$.
We define an injective map $f : V(L_n) \to \{0, 1, 3, 5, \ldots, 6n-5\}$ by
$f(u_i) = 2i - 1$ for $1 \leq i \leq n$
and $f(v_i) = 2n + 2i - 1$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_iu_{i+1}) = 2i + 1$ for $1 \leq i \leq n-1$
$f^*(v_iv_{i+1}) = 2n + 2i + 1$ for $1 \leq i \leq n-1$
and $f^*(v_iu_i) = 2n(n-1) + 2i(2n-1) + 1$ for $1 \leq i \leq n$.
Clearly the edge labels are odd and distinct. Hence the ladder L_n admits an analytic odd mean labeling.
An analytic odd mean labeling of ladder L_5 is shown in Figure 7.

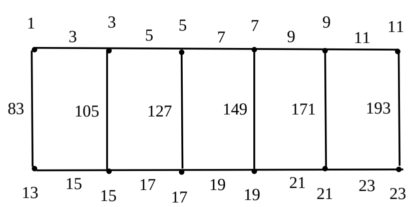

Figure-7

Theorem 2.8. *The n-bistar $B_{n,n}$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set be $V(B_{n,n}) = \{u_i, v_i : 0 \leq i \leq n\}$ and $E(B_{n,n}) = \{u_0v_0, u_iu_0, v_iv_0 : 1 \leq i \leq n\}$.
Hence $|V| = 2n + 2$ and $|E| = 2n + 1$.
We define an injective map $f : V(B_{n,n}) \to \{0, 1, 3, 5, \ldots, 4n+1\}$ by
$f(u_0) = 0$, $f(u_i) = 2i - 1$ for $1 \leq i \leq n$
$f(v_0) = 4n + 1$ and $f(v_i) = 2i + 2n - 1$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_0v_0) = 8n^2 + 4n + 1$
$f^*(u_iu_0) = 2i^2 - 2i + 1$ for $1 \leq i \leq n$
and $f^*(v_iv_0) = 6n^2 - 4n(i-1) - 2i^2 + 1$ for $1 \leq i \leq n$.
Clearly the edge labels are odd and distinct. Hence the n-bistar $B_{n,n}$ admits an analytic odd mean labeling.
An analytic odd mean labeling of 4-bistar $B_{4,4}$ is shown in Figure 8.

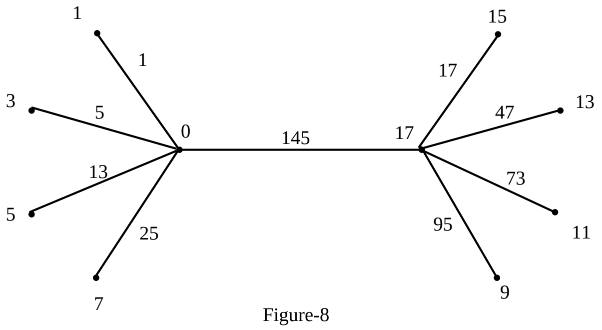

Figure-8

Theorem 2.9. *The graph $C_m \cup C_n$ for any $m \geq 3$ and $3 \leq n \leq m^2 - 2m - 1$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set of complete bipartite graph be $V(C_m \cup C_n) = \{u_i, v_j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(C_m \cup C_n) = \{u_iu_{i+1} : 1 \leq i \leq m-1\} \cup \{u_1u_m\} \cup \{v_jv_{j+1} : 1 \leq j \leq n-1\} \cup \{v_1v_n\}$.
Hence $|V| = m + n = |E|$.
We define an injective map $f : V(C_m \cup C_n) \to \{0, 1, 3, 5, \ldots, 2m + 2n - 1\}$ by
$f(u_i) = 2i - 1$ for $1 \leq i \leq m$
and $f(v_j) = 2m + 2j - 1$ for $1 \leq j \leq n$
The induced edge labeling f^* is defined as follows:
$f^*(u_iu_{i+1}) = 2i + 1$ for $1 \leq i \leq m - 1$
$f^*(u_1u_m) = 2m^2 - 2m - 1$
$f^*(v_jv_{j+1}) = 2m + 2j + 1$ for $1 \leq j \leq n - 1$
and $f^*(v_1v_n) = 2n(n - 1) + 2m(2n - 3) - 1$.
Clearly the edge labels are odd and distinct. Hence $C_m \cup C_n$ admits an analytic odd mean labeling.
An analytic odd mean labeling of $C_8 \cup C_6$ is shown in Figure 9.

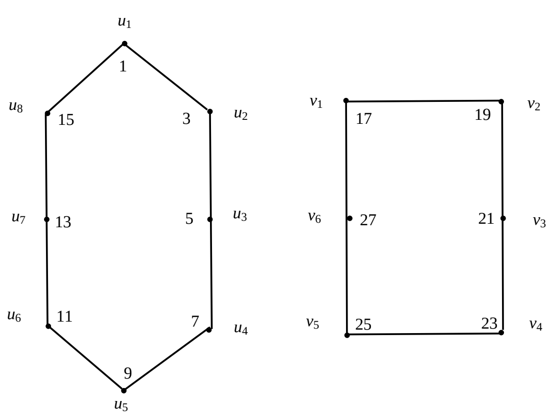

Figure-9

Theorem 2.10. *The comb $P_n \odot K_1$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set be $V(P_n \odot K_1) = \{u_0, u_1, \ldots u_{n-1}, v_1, v_2, \cdots v_n\}$ and $E(P_n \odot K_1) = \{u_{i-1}u_i : 1 \leq i \leq n - 1\} \cup \{u_{i-1}v_i : 1 \leq i \leq n\}$.
Hence $|V| = 2n$ and $|E| = 2n - 1$.
We define an injective map $f : V(P_n \odot K_1) \to \{0, 1, 3, 5, \ldots, 4n - 3\}$ by
$f(u_0) = 0$, $f(u_i) = 2i - 1$ for $1 \leq i \leq n - 1$
and $f(v_i) = 2n + 2i - 3$ for $1 \leq i \leq n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_{i-1}u_i) = 2i - 1$ for $1 \leq i \leq n - 1$
and $f^*(u_{i-1}v_i) = 2n^2 - 2n + 1 + 2(2n - 1)(i - 1)$ for $1 \leq i \leq n$.
Clearly the edge labels are odd and distinct. Hence the comb $P_n \odot K_1$ admits an analytic odd mean labeling.
An analytic odd mean labeling of $P_4 \odot K_1$ is shown in Figure 10.

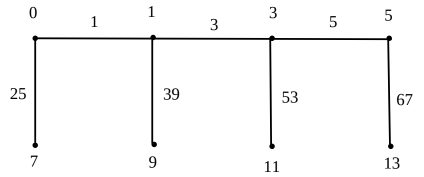

Figure-10

Theorem 2.11. *The graph $L_n \odot K_1$ is an analytic odd mean graph.*

Proof. Let the vertex set and edge set be $V(L_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq 2n\}$ and $E(L_n \odot K_1) = \{u_i u_{i+1} : 1 \leq i \leq 2n-1 \text{ and } i \neq n\} \cup \{u_i u_{n+i}, u_i v_i, u_{n+i} v_{n+i} : 1 \leq i \leq n\}$.
Hence $|V| = 4n$ and $|E| = 5n - 2$.
We define an injective map $f : V(L_n \odot K_1) \to \{0, 1, 3, 5, \ldots, 10n - 5\}$ by
$f(u_i) = 2i - 1$ for $1 \leq i \leq 2n$
and $f(v_i) = 4n + 2i - 1$ for $1 \leq i \leq 2n$.
The induced edge labeling f^* is defined as follows:
$f^*(u_i u_{i+1}) = 2i + 1$ for $1 \leq i \leq 2n - 1$ and $i \neq n$
$f^*(u_i u_{n+i}) = 2n(n-1) + 2i(2n-1) + 1$ for $1 \leq i \leq n$
and $f^*(u_i v_i) = 4n(2n-1) + 2i(4n-1) + 1$ for $1 \leq i \leq 2n$.
Clearly the edge labels are odd and distinct. Hence $L_n \odot K_1$ admits an analytic odd mean labeling.
An analytic odd mean labeling of $L_5 \odot K_1$is shown in Figure 11.

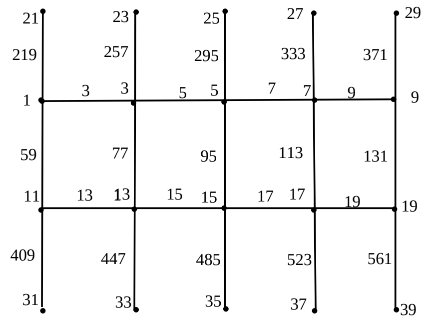

Figure-11

References

[1] F.Harary, *Graph Theory,Addison-Wesley* ,Reading,Massachusetts,1972.

[2] Joseph A. Gallian, *A Dynamic Survey of Graph Labeling*, The Electronic Journal of Combinatorics, (2016), # DS6.

[3] S.Somasundaram and R. Ponraj,*Mean labelings of graphs*,National Academy Science Letter,26 (2003),210-213.

[4] T.Tharmaraj and P.B.Sarasija, *Analytic mean labelled graphs*, International Journal of Mathematical Archive, 5(6)(2014), 136 -146.

Author information

P. Jeyanthi, Research Centre, Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur - 628215, Tamilnadu, India.
E-mail: jeyajeyanthi@rediffimail.com

R.Gomathi, Research scholar, Reg.No.11825, Research Centre, Govindammal Aditanar College for Women, Affliated to Manonmaniam Sundaranar University, Abisekappatti, Tirunelveli - 627012, Tamilnadu, India.
E-mail: gomathikarthi97@gmail.com

Gee-Choon Lau, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus) 85009, Johor, Malaysia.
E-mail: geeclau@yahoo.com