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SOME RESULTS ON CLESS LATTICES

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Abstract We introduce the concept of a CLESS lattice which is a generalization of the concept of an extending lattice (or a CS lattice). We study relationship between various generalizations of the concept of an extending lattice namely, CLS lattice, CLESS lattice and CESS lattice. We also prove that, if a, b are direct summands of 1 which are CLESS elements then 1 is a CLESS element.

1 Introduction

The concept of an extending module (or a CS module) played an important role in module theory and also offered a rich topic of research. A module is called extending (or CS module, *i.e.*, complements are summands) if every submodule of it is essential in a direct summand. This concept was introduced by Chatters and Hajarnavis [4]. These modules and generalizations are studied by several researchers such as Harmanci and Smith [9], Dung et. al. [6], Akalan, Birkenmeir and Tercan [1], M$\ddot{u}$ller and Rizvi [12], Celik, Harmanci and Smith [3] and many others. Tercan [19] studied the concept of a CLS module. A module M is called a CLS module provided every closed submodule of M is a direct summand of M. Crivei and Sahinkaya [5] studied the concept of a CLESS module. A module M is called a CLESS module if every closed submodule N of M with essential socle is a direct summand of M.

Călugăreanu [2] used lattice theory in module theory and studied several concepts from module theory in lattice theory. Keskin [10] obtained some properties of extending modules using modular lattices. Nimbhorkar and Shroff [15], [16], [14] have studied, respectively ojective ideals, generalized extending ideals and Goldie extending elements in modular lattices. Nimbhorkar and Banswal [13] studied CESS lattices.

In the present paper, we introduce the concept of a CLESS lattice and obtain some properties of such lattices. We show that, if a, b are direct summands of 1 which are CLESS elements, then 1 is a CLESS element. We show a relationship between various generalizations of extending lattices such as a CLS lattice, a CLESS lattice and a CESS lattice. This work extends the results of Crevei and Sahinkaya [5] in the context of certain modular lattices.

Throughout in this paper L denotes a lattice with 0.

2 Priliminaries

We recall some terms from lattice theory. These and undefined terms can be found in Grätzer [7].

Definition 2.1. A lattice L is said to be a modular lattice if for $a, b, c \in L$ with $a \leq c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Călugăreanu [2] developed the concept of an essential element in a lattice with least element 0, see also Grzeszczuk and Puczylowski [8].

Definition 2.2. Let L be a lattice with 0. An element $a \in L$ is called an essential element if $a \wedge b \neq 0$, for any nonzero $b \in L$.

If a is essential in $[0, b]$, then we say that a is essential in b and write $a \leq_e b$ and call b as an essential extension of a.

If $a \leq_e b$ and there is no $c \in L$ such that $a \leq_e c$ and $b \leq c$, then we say that b is a maximal essential extension of a.

Definition 2.3. Let a, b be elements of a lattice L with 0. We say that a is closed (or essentially closed) in b if a does not have a proper essential extension in b. If a is closed in b then we write $a \leq_{cl} b$. If a does not have a proper essential extension in L, then we say that a is closed in L.

The concepts of a semicomplement and a maximal semicomplement are known in a lattice with 0, see Szāsz [18, p. 47]. Let $a, b \in L$, we say that a, b are semicomplements of each other if $a \wedge b = 0$.

Definition 2.4. If $a, b \in L$ and b is a maximal element in the set $\{x \in L \mid a \wedge x = 0\}$, then we say that b is a max-semicomplement of a in L.

In the theory of modules (e.g. Lam [11, Proposition 6.24], Chatters and Hajarnavis [4, Proposition 2.2] and others), it is known that if A, B, C are modules of a ring R with $A \subseteq B \subseteq C$ and if A is closed in B and B is closed in C then A is closed in C.

The following proposition is an analog of this result, the proof of which is due to Wehrung.

Proposition 2.5. *(Nimbhorkar and Shroff [16]). Let L be a modular lattice with 0. For $a, b, c \in L$, if $a \leq_{cl} b$ and $b \leq_{cl} c$, then $a \leq_{cl} c$.*

Definition 2.6. If $a, b, c \in L$ are such that $a \vee b = c$ and $a \wedge b = 0$, then we say that a, b are direct summands of c and we write $c = a \oplus b$. We say that c is a direct sum of a and b.

The set of all direct summands of an element $c \in L$ is denoted by $\mathfrak{D}(c)$. That is, for every $a \in \mathfrak{D}(c)$ there exists $b \in \mathfrak{D}(c)$ such that $c = a \oplus b$.

Definition 2.7. A lattice L with the least element 0 is said to be an atomic lattice if every nonzero element of L contains an atom.

Definition 2.8. [2, p. 47] The join of all atoms in L, denoted by $Soc(L)$, is called the socle of the lattice L.

For $a \in L$, $Soc(a)$ is the socle of the lattice $[0, a]$.

Throughout in this paper, wherever necessary, we assume that L satisfies one or more of the following conditions.

Condition (1): For any $a \leq b$ in L, there exists a maximal essential extension of a in b.

Condition (2): For any $a \leq b$ and for any $c \leq b$ in L with $a \wedge c = 0$, there exists a max-semicomplement $d \geq c$ of a in b.

Condition (3): If the socle is involved, $Soc(a)$ exists for any $a \in L$.

Condition (4): If $a \in L$ is a join of atoms in L then any $b \leq a$ is a join of atoms.

The following result is from Nimbhorkar and Banswal [13].

Lemma 2.9. *Let L be a lattice satisfying the conditions (1) and (3). Then every CS lattice is a CESS lattice and every CESS lattice is a weak CS lattice.*

The following results are from Nimbhorkar and Shroff [15].

Theorem 2.10. *Let L be a modular lattice and $a, b \in L$ be such that $a \wedge b = 0$. Then a is a max-semicomplement of b in L if and only if a is closed in L and $a \oplus b$ is essential in L.*

Theorem 2.11. *Let L be a modular lattice and $a, b, c, d, e \in L$ be such that $e = c \oplus d$ and $a, b \leq c$. Then following statements are equivalent:*
(1) b is a max-semicomplement of a in c.
(2) $b \oplus d$ is a max-semicomplement of a in e.
(3) b is a max-semicomplement of $a \oplus d$ in e.

The following results are from Nimbhorkar and Shroff [16].

Lemma 2.12. *In a modular lattice L, if $a, b, c \in L$ are such that $c = a \oplus b$ then a is a max-semicomplement of b in c.*

Lemma 2.13. *Let L be a modular lattice satisfying the condition (2). Let $a, b \in L$ and $a \leq b$. Then a is closed in b if and only if a is a max-semicomplement of some $c \leq b$.*

The following results are from Nimbhorkar and Shroff [14].

Lemma 2.14. *Every max-semicomplement in L is closed in L.*

Lemma 2.15. *In a lattice L the following statement hold.*
(1) Let $a, b \in L$. Then $a \leq_e b$, if and only if for any $c \in L$, $a \wedge c = 0$ implies that $b \wedge c = 0$.
(2) If $a, b, c \in L$, then $a \leq_e b$ implies $a \wedge c \leq_e b \wedge c$.
(3) If $a \leq b \leq c$, then $a \leq_e b$, $b \leq_e c$ if and only if $a \leq_e c$.

The following lemma is from Grzeszczuk and Puczylowski [8, Lemma 3].

Lemma 2.16. *Let L be a modular lattice. Suppose that $a, b, c, d \in L$ are such that $a \leq b$, $c \leq d$ and $b \wedge d = 0$. Then $a \leq_e b$, $c \leq_e d$ if and only if $a \oplus c \leq_e b \oplus d$.*

The following lemma is from Nimbhorkar and Shroff [15, Lemma 2.1].

Lemma 2.17. *Let L be a modular lattice and $a, b, c \in L$ be such that $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$ then $a \wedge (b \vee c) = 0$.*

The following lemma is from Nimbhorkar and Banswal [13].

Lemma 2.18. *Let L be a lattice satisfying the conditions (3) and (4). If $a \leq b$, then $Soc(a) = a \wedge Soc(b)$.*

The following definitions is from Nimbhorkar and Banswal [13].

Definition 2.19. Let L be a lattice with 0 and $a, b \in L$. If b is a maximal element in the set $\{x \mid x \in L \text{ and } a \leq_e x\}$, then we say that b is an essential closure of a in L.

Definition 2.20. A lattice L is called a UC-lattice if each of its nonzero element has a unique essential closure in L.

The following theorem is from Nimbhorkar and Banswal [13].

Theorem 2.21. *Let L be a lattice satisfying the conditions (1) and (2). A lattice L is a UC-lattice if and only if for any closed element a in L and for any $b \in L$, $a \wedge b$ is closed in b.*

3 CLESS lattices

Definition 3.1. A nonzero element $a \in L$ is called CS or extending if, every max-semicomplement in a is a direct summand of a.

A bounded lattice L is called CS or extending if every max-semicomplement in L is a direct summand of 1.

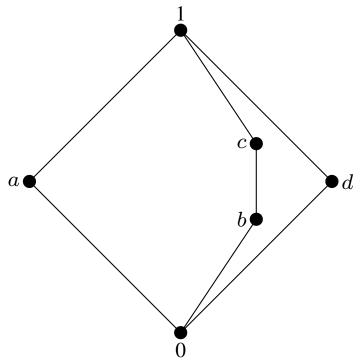

Figure 1

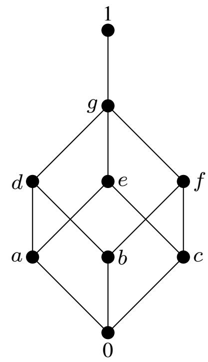

Figure 2

In the lattice shown in Figure 1, elements $a, c, d \in L$ are max-semicomplements in L which are direct summands of 1. Hence L is a CS lattice.

In the lattice L shown in Figure 2, the element $d \in L$ is a max-semicomplement in L which is not a direct summand of 1. Hence L is not extending.

The following definition is from Nimbhorkar and Banswal [13].

Definition 3.2. An element $a \in L$ is called a CESS element, if every max-semicomplement $b \leq a$ such that $Soc(b) \leq_e b$ is a direct summand of a.

A bounded lattice L is called a CESS-lattice if 1 is a CESS element.

In the lattice L shown in Figure 1, a, c, d are max-semicomplements in L such that $Soc(a) \leq_e a$, $Soc(c) \leq_e c$ and $Soc(d) \leq_e d$. Also, $1 = a \oplus b = a \oplus c = a \oplus d$. Hence L is a CESS lattice.

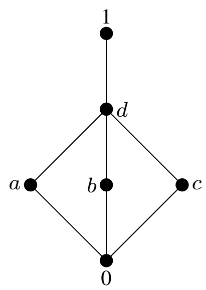

Figure 3

In the lattice L shown in Figure 3, a, b, c are max-semicomplements in L such that $Soc(a) \leq_e a$, $Soc(b) \leq_e b$ and $Soc(c) \leq_e c$. But none of a, b, c is a direct summand of 1. Hence L is not a CESS lattice. However, d is a CESS-element.

Tercan [19] defined the concept of a CLS-module which is a generalization of a CS-module. We introduce this concept in a lattice.

Definition 3.3. An element $a \in L$ is called a CLS element if every closed element $b \leq a$ is a direct summand of a.

A bounded lattice L is called CLS lattice if 1 is a CLS element.

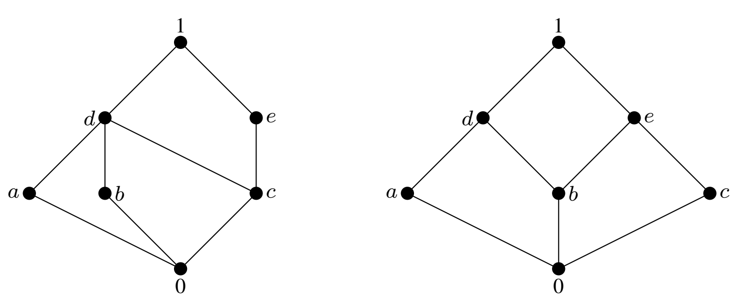

Figure 4 Figure 5

In the Lattice L shown in Figure 4, elements $a, b, e \in L$ are closed elements in L which are direct summands of 1. Hence L is a CLS lattice.

In the Lattice L shown in Figure 5, the element $b \in L$ is closed in L which is not a direct summand of 1. Hence L is not a CLS lattice.

Definition 3.4. An element $a \in L$ is called a CLESS element if every closed element $b \leq a$ such that $Soc(b) \leq_e b$ is a direct summand of a.

A bounded lattice L is called CLESS lattice if 1 is a CLESS element.

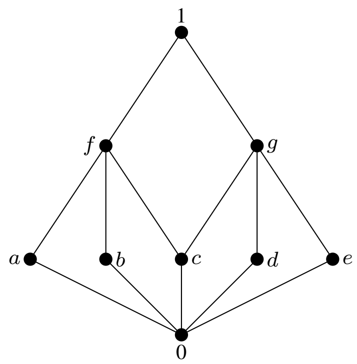

Figure 6

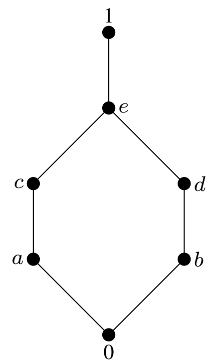

Figure 7

In the lattice L shown in Figure 6, all elements are closed in L with $Soc(x) \leq_e x$ for all $x \in L$ are direct summand of 1. Hence L is a CLESS lattice.

In the lattice L shown in Figure 7, $c \in L$ closed in L with $Soc(c) \leq_e c$ but not a direct summand of 1. Hence L is not a CLESS lattice.

Lemma 3.5. *Let L be a lattice satisfying the conditions (1) to (4). If L is a CLS lattice, then L is a CLESS lattice and if L is a CLESS lattice, then L is a CESS lattice.*

Proof. Let L be a CLS lattice. Let $a \in L$ be a closed element in L such that $Soc(a) \leq_e a$. Since L is a CLS lattice, a is a direct summand of 1. Hence L is a CLESS lattice.

Next, Suppose that L is a CLESS lattice. Let $a \in L$ be a max-semicomplement in L with $Soc(a) \leq_e a$. Then by Lemma 2.14, a is closed in L. Since L is a CLESS lattice, a is a direct summand of 1. □

Lemma 3.6. *Let L be a lattice satisfying the condition (2). If L is a CLS lattice, then L is a CS lattice and if L is a CS lattice then L is a CESS lattice.*

Proof. Let L be a CLS lattice. Let $a \in L$ be a max-semicomplement in L. Then by Lemma 2.14, a is closed in L. Since L is a CLS lattice, a is a direct summand of 1. The second part follows from Lemma 2.9. □

Lemma 3.7. *Let L be a modular lattice satisfying the conditions (1) to (4). If L is a CESS lattice, then L is a CLESS lattice.*

Proof. Let L be a CESS lattice. Let $a \in L$ be a closed element in L with $Soc(a) \leq_e a$. Then by Lemma 2.13, a is a max-semicomplement in L. Since L is a CESS lattice, a is a direct summand of 1. □

Lemma 3.8. *Let L be a modular lattice satisfying the condition (2). If L is a CS lattice, then L is a CLS lattice.*

Proof. Let L be a CS lattice. Let $a \in L$ be a closed element in L. Then by Lemma 2.13, a is a max-semicomplement in L. Since L is a CS lattice, a is a direct summand of 1. □

The proof of the following lemma follows from Lemma 3.5 and Lemma 3.8.

Lemma 3.9. *Let L be a modular lattice satisfying the conditions (1) to (4). If L is a CS lattice, then L is a CLESS lattice.*

Lemma 3.10. *Let L be a modular, atomic lattice satisfying the conditions (1) to (4). If L is a CLESS lattice, then L is a CLS lattice.*

Proof. Let $x \in L$. Then by the condition (1), there exist a maximal essential extension $y \in L$ such that $x \leq_e y$. Being a maximal essential extension, y is closed in L. To show that $Soc(y) \leq_e y$. Let $0 \neq a \leq y$. since L is an atomic lattice, there exist an atom $b \leq a$ and therefore, $Soc(y) \wedge b = b \neq 0$. Thus $Soc(y) \leq_e y$. Since L is a CLESS lattice, y is a direct summand of 1. Hence L is a CLS lattice. □

Theorem 3.11. *Let L be a modular lattice satisfying the conditions (1), (3) and (4). Suppose that L is a CLESS lattice. If $a \in L$ is closed in L, then a is a CLESS element.*

Proof. Let $b \in L, b \leq a$ be a closed element in a such that $Soc(b) \leq_e b$. To show that b is a direct summand of a. Since b is closed in a and a is closed in L, by Proposition 2.5, b is closed in L. Since L is a CLESS lattice, b is a direct summand of 1 that is $b \oplus c = 1$, for some $c \in L$. Now using modularity, as $b \leq a$, $b \vee (c \wedge a) = (b \vee c) \wedge a = 1 \wedge a = a$ and $b \wedge (c \wedge a) = 0$. Thus b is a direct summand of a. Hence a is a CLESS element. □

Theorem 3.12. *Let L be a modular atomic lattice satisfying the conditions (1), (3) and (4). Suppose that L is a CLESS lattice. Let $a, b \in L$ be such that $a \oplus b = 1$ and $Soc(a) \leq_e a$. Then b is a CLESS element.*

Proof. Let $c \in L$ be a closed element in b with $Soc(c) \leq_e c$. Then by Theorem 2.11, $a \oplus c$ is closed in L. To show that $Soc(a \oplus c) \leq_e a \oplus c$. Let $0 \neq x \leq (a \vee c)$. Since L is an atomic lattice there exists an atom $y \leq x$ and therefore $Soc(a \vee c) \wedge y = y \neq 0$. Thus $Soc(a \oplus c) \leq_e a \oplus c$. Since L is a CLESS lattice, $a \oplus c$ is a direct summand of 1, that is $(a \oplus c) \oplus d = 1$, for some $d \in L$. Now as $c \leq b$, using modularity, $c \vee [(a \oplus d) \wedge b] = (c \vee (a \vee d)) \wedge b = 1 \wedge b = b$. Here $a \wedge c = 0$ and $(a \vee c) \wedge d = 0$ then by Lemma 2.17, $c \wedge (a \vee d) \wedge b = c \wedge (a \vee d) = 0$. Thus c is a direct summand of b. Hence b is a CLESS element. □

Remark 3.13. Let $a, b \in L$ such that $Soc(a) \leq_e a$. Then $Soc(a \wedge b) \leq_e a \wedge b$.

Proof. Since $Soc(a) \leq_e a$ by Lemma 2.15, we have $Soc(a) \wedge b \leq_e a \wedge b$. Now $a \wedge b \leq a$, then by Lemma 2.18, we have $Soc(a \wedge b) = a \wedge b \wedge Soc(a) = b \wedge Soc(a)$. Since $Soc(a) \wedge b \leq_e a \wedge b$, we have $Soc(a \wedge b) \leq_e a \wedge b$. □

Theorem 3.14. *Let L be a modular UC-lattice satisfying the conditions (1), (3) and (4). Let $a, b \in L$ be such that $a \oplus b = 1$. Then L is a CLESS lattice if and only if every closed element c in L with $Soc(c) \leq_e c$ such that $c \wedge a = 0$ or $c \wedge b = 0$ is a direct summand of 1 and c is a CLESS element.*

Proof. Since L is a CLESS lattice, there exists a closed element c in L such that $Soc(c) \leq_e c$ and c is a direct summand of 1.

Conversely, Suppose that every closed element c in L with $Soc(c) \leq_e c$ such that $c \wedge a = 0$ or $c \wedge b = 0$ is a direct summand of 1.

To show that L is a CLESS lattice. By Theorem 2.21, $c \wedge a$ is closed in a and $Soc(c \wedge a) \leq_e c \wedge a$. Since $c \wedge a$ is closed in a and a is closed in L then by Proposition 2.5, $c \wedge a$ is closed in L. Since $c \wedge a \wedge b = 0$, by hypothesis $c \wedge a$ is a direct summand of 1, say $1 = (c \wedge a) \oplus d$ for some $d \in L$. By assumption, d is a CLESS element. Now by Theorem 2.21, $c \wedge d$ is closed in d and $Soc(c \wedge d) \leq_e c \wedge d$ follows that $c \wedge d$ is a direct summand of d that is $d = (c \wedge d) \oplus e$ for some $e \in L$. To show that c is a direct summand of 1. As $c \wedge a \leq c$, using modularity,

$$\begin{aligned} (c \wedge a) \vee d &= (c \wedge a) \vee (c \wedge d) \vee e \\ &= (c \wedge a) \vee (c \wedge d) \vee e = [[(c \wedge a) \vee d] \wedge c] \vee e \\ &= (1 \wedge c) \vee e = c \vee e = 1 \end{aligned}$$

and

$$c \wedge e = c \wedge e \wedge d = 0.$$

Thus c is a direct summand of 1. Hence L is a CLESS lattice. □

4 Direct Sum of CLESS Lattices

The following definition is from Nimbhorkar and Shroff [16].

Definition 4.1. Let $a, b, c \in L$ be such that $a = b \oplus c$. Then c is said to be $b - injective$ in a if for every $d \leq a$ with $d \wedge c = 0$, there exists $e \leq a$ such that $a = e \oplus c$ and $d \leq e$.

Theorem 4.2. *Let L be a modular atomic UC-lattice satisfying the conditions (1) to (4) and every closed elements in L are CLESS. Let $a, b \in L$ such that $1 = a \oplus b$ be a direct sum of CLESS elements and b is $a - injective$. Then 1 is a CLESS element.*

Proof. Let $c \in L$ be a closed element of L with $Soc(c) \leq_e c$. By Theorem 2.21, $c \wedge b$ is closed in b and $Soc(c \wedge b) \leq_e c \wedge b$. Since b is a CLESS element $c \wedge b$ is a direct summand of b that is $b = (c \wedge b) \oplus d$ for some $d \leq b$. Now,

$$\begin{aligned}(c \wedge b) \vee (a \vee d) &= a \vee (c \wedge b) \vee d \\ &= a \vee b = 1\end{aligned}$$

and

$$\begin{aligned}(c \wedge b) \wedge (a \vee d) &= (c \wedge b) \wedge (a \vee d) \wedge b \\ &= (c \wedge b) \wedge [d \vee (a \wedge b)] \\ &= (c \wedge b) \wedge d \\ &= 0.\end{aligned}$$

Thus $c \wedge b$ is a direct summand of 1. As $c \wedge b \leq c$, using modularity we have,

$$\begin{aligned}(c \wedge b) \vee [(d \vee a) \wedge c] &= [(c \wedge b) \vee (d \vee a)] \wedge c \\ &= 1 \wedge c \\ &= c\end{aligned}$$

and

$$\begin{aligned}(c \wedge b) \wedge (d \vee a) \wedge c &= (c \wedge b) \wedge (d \vee a) \wedge b \\ &= (c \wedge b) \wedge [d \vee (a \wedge b)] \\ &= (c \wedge b) \wedge d \\ &= 0.\end{aligned}$$

Hence $c \wedge b$ is a direct summand of c that is $c = (c \wedge b) \oplus [(d \vee a) \wedge c]$. Putting $(d \vee a) \wedge c = e$, we have $c = (c \wedge b) \oplus e$. Here, $e \wedge b = 0$, since b is a - injective, there exists $f \in L$ such that $1 = f \oplus b$ and $e \leq f$. By assumption f is a CLESS element. Since e is closed in f and $Soc(e) \leq_e e$ implies e is a direct summand of f that is $f = e \oplus g$ for some $g \leq f$. We have

$$\begin{aligned}c \vee g \vee b &= e \vee g \vee (c \wedge b) \vee b \\ &= e \vee g \vee b \\ &= f \vee b \\ &= 1\end{aligned}$$

and

$$\begin{aligned}c \wedge g \wedge b &= c \wedge g \wedge b \wedge f \\ &= 0.\end{aligned}$$

Thus c is a direct summand of 1. Hence L is a CLESS lattice. □

The following definition is from Nimbhorkar and Shroff [14].

Definition 4.3. Let $a, b \in L$ be such that $a \oplus b = 1$. An element b is said to be $a - ojective$ if for any max-semicomplement $c \in L$ of b in L, 1 can be decomposed as $1 = a' \oplus b' \oplus c$ with $a' \leq a$ and $b' \leq b$.

The proof of the following theorem is the same as that of Proposition 4.1 from Nimbhorkar and Shroff [14].

Theorem 4.4. *Let L be a modular lattice satisfying the conditions (1) and (2). Let $a, b \in L$ be such that $1 = a \oplus b$. Let a_1 and b_1 be direct summands of a and b, respectively. If b is $a - ojective$, then*
(1) b_1 is $a - ojective$;
(2) b is $a_1 - ojective$;
(3) b_1 is $a_1 - ojective$.

Theorem 4.5. *Let L be a modular UC-lattice satisfying the conditions (1) to (4). Let $a, b \in L$ be such that $a \oplus b = 1$, a is a CLESS element and b is $a - ojective$. Then for every closed element $c \in L$ with $Soc(c) \leq_e c$ and $c \wedge b = 0$, $1 = c \oplus a' \oplus b'$ for some $a' \leq a$ and $b' \leq b$.*

Proof. Let $c \in L$ be a closed element in L with $Soc(c) \leq_e c$ and $c \wedge b = 0$. By Theorem 2.21, $c \wedge a$ is closed in a and by Remark 3.13, $Soc(c \wedge a) \leq_e c \wedge a$. since a is a CLESS element, it follows that $c \wedge a$ is a direct summand of a. Say $a = (c \wedge a) \oplus f_1$ for some $f_1 \leq a$. By Theorem 3.12, f_1 is a CLESS element. Now, let $k = (c \oplus b) \wedge a$. Let f_1' be the maximal essential extension of $k \wedge f_1$ in f_1 that is $k \wedge f_1 \leq_e f_1'$, $f_1' \leq f_1$.

Since f_1 is CLESS element, f_1' is a direct summand of f_1 say, $f_1 = f_1' \oplus f_1''$ for some $f_1'' \leq f_1$.

$$\begin{aligned} k \vee b &= [(c \oplus b) \wedge a] \vee b \\ &= [(c \vee b) \wedge a] \vee b \\ &= (c \vee b) \wedge (a \vee b) \quad \text{(using modularity, as } b \leq c \vee b) \\ &= (c \vee b) \wedge 1 \\ &= c \vee b \end{aligned}$$

and $k \wedge b = [(c \vee b) \wedge a] \wedge b = 0$. Thus $k \oplus b = c \oplus b$. Now,

$$\begin{aligned} k \oplus b &= [(c \oplus b) \wedge a] \oplus b \\ &= [(c \oplus b) \wedge [(c \wedge a) \oplus f_1]] \oplus b \\ &= [(c \vee b) \wedge [(c \wedge a) \vee f_1]] \vee b \\ &= [(c \wedge a) \vee [f_1 \wedge (c \vee b)]] \vee b \quad \text{(using modularity, as } c \wedge a \leq c \vee b) \\ &= [(c \wedge a) \vee [f_1 \wedge a \wedge (c \vee b)]] \vee b \\ &= (c \wedge a) \vee (k \wedge f_1) \vee b \\ &= (c \wedge a) \oplus (k \wedge f_1) \oplus b. \end{aligned}$$

Now by Lemma 2.16, $(c \wedge a) \oplus (k \wedge f_1) \oplus b \leq_e (c \wedge a) \oplus f_1' \oplus b$ that is $c \oplus b = k \oplus b = (c \wedge a) \oplus (k \wedge f_1) \oplus b \leq_e (c \wedge a) \oplus f_1' \oplus b$. Now by Theorem 2.10, it follows that, c is a max-semicomplement of b in $f = (c \wedge a) \oplus f_1' \oplus b$. By Theorem 4.4 , b is $(c \wedge a) \oplus f_1' - ojective$, so $f = c \oplus a'' \oplus b'$ for some $a'' \leq (c \wedge a) \oplus f_1'$ and $b' \leq b$. Now $1 = f \oplus f_1'' = c \oplus a'' \oplus b' \oplus f_1'' = c \oplus a' \oplus b'$ with $a' = a'' \oplus f_1'' \leq a$, $b' \leq b$. □

The following definition is from Nimbhorkar and Shroff [16].

Definition 4.6. Let $a, b \in L$ be such that $1 = a \oplus b$. Then a is said to be $b - ejective$ in L, if for every $d \in L$ such that $d \wedge a = 0$ there exists an $f \in L$ such that $1 = a \oplus f$ and $d \wedge f \leq_e d$.

The following result is proved for modules by Wang and Wu [20]. We state and prove it in the context of lattices.

Theorem 4.7. *Let L be a modular lattice satisfying the condition (1). Let $a_1 \in L$ be a direct summand of $a \in L$ and $b_1 \in L$ be a direct summand of $b \in L$. If a is $b - ejective$ then a_1 is $b_1 - ejective$.*

Proof. Write $1 = a \oplus b$, $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ for some $a_1, a_2 \leq a$ and $b_1, b_2 \leq b$. To prove: a_1 is $b-ejective$. Let $c = a_1 \oplus b$. Let $x \leq c$ with $x \wedge a_1 = 0$. Then

$$\begin{aligned} x \wedge a &= x \wedge c \wedge a = x \wedge (a_1 \oplus b) \wedge a \\ &= x \wedge (a_1 \vee b) \wedge a = x \wedge [a_1 \vee (b \wedge a)] \\ &= x \wedge a_1 = 0. \end{aligned}$$

Since a is $b-ejective$ there exist an element $d \in L$ such that $1 = a \oplus d$ and $x \wedge d \leq_e x$. As $a_1 \leq c$, using modularity,

$$\begin{aligned} a_1 \vee [c \wedge (a_2 \vee d)] &= [a_1 \vee (a_2 \vee d)] \wedge c \\ &= (a_1 \vee a_2 \vee d) \wedge c \\ &= (a \vee d) \wedge c \\ &= 1 \wedge c \\ &= c \end{aligned}$$

and

$$\begin{aligned} a_1 \wedge [c \wedge (a_2 \oplus d)] &= a_1 \wedge c \wedge (a_2 \vee d) \wedge a \\ &= a_1 \wedge c \wedge [a_2 \vee (d \wedge a)] \\ &= a_1 \wedge c \wedge a_2 \\ &= 0. \end{aligned}$$

Thus $c = a_1 \oplus [c \wedge (a_2 \oplus d)]$. Also $x \wedge [c \wedge (a_2 \oplus d)] \leq_e x$. Hence a_1 is $b-ejective$.
To prove : a is $b_1-ejective$. Write $c' = a \oplus b_1$. Let $x' \leq c'$ with $x' \wedge a = 0$. Since a is $b-ejective$ there exist an element $d' \in L$ such that $1 = a \oplus d'$ and $d' \wedge x' \leq_e x'$. As $a \leq c'$, using modularity,

$$\begin{aligned} a \vee [c' \wedge d'] &= [a \vee d'] \wedge c \\ &= 1 \wedge c' \\ &= c'. \end{aligned}$$

And $a \wedge (c' \wedge d') = 0$. Thus $c' = a \oplus (c' \wedge d')$ and $x' \wedge (c' \wedge d') = x' \wedge d' \leq_e x'$. Thus a is $b_1-ejective$.
Hence a_1 is $b_1-ejective$. □

The following theorem is a lattice theoretic analogue of Corollary 2.8 from Akalan et. al.[1].

Theorem 4.8. *Let L be a modular lattice satisfying the condition (1). Let $a, b \in L$ be such that $1 = a \oplus b$. Then a is $b-injective$ if and only if a is $b-ejective$.*

Proof. First suppose that a is $b-injective$. To show that a is $b-ejective$. Let $c \in L$ be such that $c \wedge a = 0$. Since a is $b-injective$ there exist $d \in L$ such that $c \leq d$ and $1 = a \oplus d$ and also $d \wedge c = c \leq_e c$. Hence a is $b-ejective$.

Conversely, Suppose that a is $b-ejective$. To show that a is $b-injective$. Let $c \in L$ such that $c \wedge a = 0$. Since a is $b-ejective$, there exists $d \in L$ such that $1 = a \oplus d$ and $c \wedge d \leq_e c$. To show that $c \leq d$. Since c is a maximal essential extension in L, c is closed in L. As d is a direct summand, d is closed in L by Lemma 2.12. Now by Theorem 2.21, $c \wedge d$ is closed in d. Since $c \wedge d \leq_{cl} d \leq_{cl} L$, by Proposition 2.5, $c \wedge d \leq_{cl} L$. Thus $c \wedge d = c$ and $c = c \wedge d \leq d$. Thus $c \leq d$. Hence a is $b-injective$. □

Theorem 4.9. *Let L be a modular lattice satisfying the conditions (1), (3), (4) and let $a, b \in L$. Let $1 = a \oplus b$ be a direct sum of CLESS elements such that b is $a-ejective$. Then 1 is a $CLESS$ element.*

Proof. Let $c \in L$ be a closed element in L with $Soc(c) \leq_e c$. Then by Theorem 2.21, $c \wedge a$ is closed in a and $Soc(c \wedge a) \leq_e c \wedge a$. Since a is a CLESS element, $c \wedge a$ is a direct summand of a. Say $a = (c \wedge a) \oplus d_1$ for some $d_1 \leq a$. Then by Theorem 3.12, d_1 is a CLESS element. Similarly, we may write $b = (c \wedge b) \oplus d_2$ for some CLESS element $d_2 \leq b$. Now by Theorem 4.7, d_2 is $d_1 - ejective$. Also by Theorem 4.8,
d_2 is $d_1 - injective$. Also by Theorem 4.2, $d_1 \oplus d_2$ is a CLESS element. Now let

$$\begin{aligned}(c \wedge a) \vee (c \wedge b) \vee [c \wedge (d_1 \oplus d_2)] &= [c \wedge (a \vee b)] \vee [c \wedge (d_1 \vee d_2)] \\ &= (c \wedge 1) \vee [c \wedge (d_1 \vee d_2)] \\ &= c \vee [c \wedge (d_1 \vee d_2)] \\ &= c\end{aligned}$$

and $(c \wedge a) \wedge (c \wedge b) \wedge [c \wedge (d_1 \oplus d_2)] = 0$. Thus $c = (c \wedge a) \oplus (c \wedge b) \oplus [c \wedge (d_1 \oplus d_2)]$. Let $[c \wedge (d_1 \oplus d_2)]$ be a closed element in $d_1 \oplus d_2$ and $Soc([c \wedge (d_1 \oplus d_2)]) \leq_e [c \wedge (d_1 \oplus d_2)]$. Since $d_1 \oplus d_2$ is a CLESS element, $[c \wedge (d_1 \oplus d_2)]$ is a direct summand of $d_1 \oplus d_2$ that is $d_1 \oplus d_2 = [c \wedge (d_1 \oplus d_2)] \oplus [d_1' \oplus d_2']$ for some $d_1' \oplus d_2' \leq d_1 \oplus d_2$, $d_1' \leq d_1$ and $d_2' \leq d_2$. Now,

$$\begin{aligned}1 = a \oplus b &= (c \wedge a) \oplus (c \wedge b) \oplus d_1 \oplus d_2 \\ &= (c \wedge a) \oplus (c \wedge b) \oplus [c \wedge (d_1 \oplus d_2)] \oplus [d_1' \oplus d_2'] \\ &= c \oplus d_1' \oplus d_2'\end{aligned}$$

and $c \wedge d_1' \wedge d_2' = 0$. Thus c is a direct summand of 1. Hence 1 is a CLESS element. □
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