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Strongly Proximinal Sets are Convex

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Abstract. The purpose of this paper is to show that a strongly proximinal set in a metric space is Chebyshev and the associated metric projection is continuous. As a consequence, we obtain that every strongly proximinal subset of a Hilbert space is convex thereby giving a partial answer to an outstanding open problem in Approximation theory: Whether every Chebyshev set in a Hilbert space is convex?

1 Introduction

Theory of nearest points (best approximation) and theory of farthest points (worst approximation) are the main branches of Approximation Theory. Two outstanding and hitherto unsolved problems in these two branches are:
(i) Must a Chebyshev set in a Hilbert space be convex? (n.p.p.)
(ii) Must a uniquely remotal set in a Banach space be singleton?(f.p.p.)

The two problems are so related to each other that a solution of one will lead to a solution of the other. This interesting connection was given in the Hilbert space setting by F.A. Ficken in 1951. The result of Ficken appeared in 1961 in an article by V.L. Klee [9]. Although, various partial answers of the two problems (by imposing additional conditions) are known in the literature but both the problems, in full generality, seem to be far from solutions. Surveys on the convexity of Chebyshev sets have been made by V.S. Balaganskii and L.P. Vlasov [2], Frank Deutsch [5], James Fletcher and Warren B. Moors [7], T.D. Narang [11], Ivan Singer [19], L.P. Vlasov [20] and others . Surveys on the singletonness of uniquely remotal sets have been made by T.D. Narang [10], [13] and others.

It is known that a Chebyshev subset of a Banach space or of an inner product space need not be convex (see [7]). It is only under various additional hypothesis that a Chebyshev set is convex. The problem of constructing a non convex Chebyshev subset of a Hilbert space is still open.

To discuss the f.p.p., Khalil and Matar [8] introduced and discussed strongly remotel sets (analogous to strongly proximinal sets) in Banach spaces. In this paper, we use strongly proximinal sets to discuss the n.p.p.. We show that a strongly proximinal set in a metric space is Chebyshev and the associated metric projection is continuous. As a consequence, we obtain that every strongly proximinal subset of a Hilbert space is convex thereby giving a partial answer to the n.p.p..

To start with, we recall a few definitions.

Let G be a non-empty subset of a metric space (X, d). The set-valued map P_G defined by $P_G(x) = \{g \in G : d(x, g) \leq d(x, y) \; for \; all \; y \in G\}$ is called the **metric projection** of X onto G. The set G is said to be **proximinal(uniuely proximinal or Chebyshev)** if for each $x \in X$, the set $P_G(x)$ contains at least one (exactly one) element. For Chebyshev sets, the map P_G is single-valued.

A stronger form of proximinality, called strong proximinality introduced in normed linear sapces (which can be easily extended to metric spaces) by Newman and Shapiro [14] was studied in detail by Bartelt and McLaughlin [3], Papini [15], Rao and her coworkers (see [16]-[18] and references cited therein) and many others. We say that the set G is **strongly proximinal** if for each $x \in X \backslash G$, there exist $g_0 \in G$ and an $r > 0$ such that $d(x, g) \geq d(x, g_0) + rd(g_0, g)$ for all $g \in G$. Such a g_0 is called a **strongly unique best approximation** to x from G.

When M is a Haar subspace of $C(X)$, the space of all real-valued continuous functions on a compact Hausdorff space X with the supremum norm, Newman and Shapiro [15] have shown that to every f in $C(X)$ there exist a strongly unique best approximation from M. When M is a finite dimensional subspace of X, but not a Haar subspace, there exist at least one f in $C(X)$ for which there is no strongly unique best approximation in M. For more such examples, one may refer to [3], [16]-[18].

A relationship between strongly proximinal and Chebyshev sets is given by the following lemma:

Lemma 1.1. *A strongly proximinal set in a metric space is Chebyshev.*

Proof. Let G be a strongly proximinal set in a metric space (X, d). Let $x \in X \backslash G$ be arbitrary. Since G is strongly proximinal, there exist $g_0 \in G$ and $r > 0$ such that

$$d(x, g_0) \leq d(x, g) - rd(g_0, g)$$

for all $g \in G$. This gives

$$d(x, g_0) < d(x, g)$$

for all $g \in G$, $g \neq g_0$ i.e. $g_0 \in G$ is the only nearest point from x. Hence G is uniquely proximinal (Chebyshev) in X.

Since strongly proximinal sets are Chebyshev, we see that the nearest point map P_G is single-valued for strongly proximinal sets G. The following result shows that the map P_G is continuous for such sets G.

Theorem 1.2. *Let G be a non-empty strongly proximinal subset of a metric space (X, d) then the nearest point map P_G is continuous.*

Proof. Let $x \in X$ be arbitrary. If $x \in G$, then the result is obvious. Suppose $x \in X \backslash G$. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. We show that $P_G(x_n) \to P_G(x)$. Since G is strongly proximinal, there exist $g_0 \in G$ and $r > 0$ such that

$$d(x, g_0) \leq d(x, g) - rd(g_0, g) \tag{1.1}$$

for all $g \in G$. Since G is Chebyshev (Lemma 1.1), $g_0 \in G$ is the nearest point to x i.e. $P_G(x) = \{g_0\}$ and so (1.1) gives

$$d(x, P_G(x)) \leq d(x, g) - rd(P_G(x), g)$$

for all $g \in G$. This implies

$$d(x, P_G(x)) \leq d(x, P_G(x_n)) - rd(P_G(x), P_G(x_n)) \; for \; all \; n$$

$$i.e. \; d(P_G(x), P_G(x_n)) \leq \frac{1}{r}[d(x, P_G(x_n)) - d(x, P_G(x))] \; for \; all \; n \tag{1.2}$$

Also

$$d(x_n, P_G(x)) \leq d(x_n, P_G(x_n)) \leq d(x_n, x) + d(x, P_G(x))$$

gives

$$d(x, P_G(x)) = lim \; d(x, P_G(x_n)) \tag{1.3}$$

So, (1.2) implies

$$lim \; d(P_G(x), P_G(x_n)) \leq \frac{1}{r}[lim \; d(x, P_G(x_n)) - d(x, P_G(x))]$$

Using (1.3), we obtain $P_G(x_n) \to P_G(x)$ and therefore P_G is continuous at x and hence on X.

□

It is well known (see e.g. [6]) that a closed convex subset of a Hilbert space is Chebyshev. What about its converse? Whether every Chebyshev set in a Hilbert space is convex? This is one of the oldest open problem in Approximation Theory. Several researchers have tried to solve this problem but only partial answers to this problem are available in the literature. One such partial answer was given by E. Asplund [1] who proved that if the metric projection P_G onto a Chebyshev set G in a Hilbert space is continuous then G is convex. It is also well known (see [19]) that for a convex Chebyshev set G in a Hilbert space, the metric projection P_G is continuous. Thus a Chebyshev set in a Hilbert space is convex if and only if P_G is continuous (see also [4]). The above proved results show that a strongly proximinal set in a Hilbert space is Chebyshev and the associated metric projection is continuous, therefore, we have the following partial answer to the n.p.p.:

Theorem 1.3. *A strongly proximinal set in a Hilbert space is convex.*

Open Problem.*Whether a Chebyshev set in a Hilbert space is strongly proximinal? An affirmative answer to this question will solve the n.p.p..*

□

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