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ON SUBCLASSES OF BI-CLOSE-TO-CONVEX FUNCTIONS RELATED TO THE ODD-STARLIKE FUNCTIONS

Bilal ŞEKER and S. SÜMER EKER

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Abstract. In this paper, we introduce interesting new subclasses of bi-close-to-convex functions in the open unit disk. For functions in each of these subclasses, we determine initial coefficient estimates.

1 Introduction

We will denote the class of functions of the form as $\mathcal{A}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and provide the normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ symbolize the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [5])

Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the usual subclasses of $\mathcal{S}$ whose members are *close-to-convex* and *starlike* in $\mathbb{U}$, respectively. We also denote by $\mathcal{S}^*(\alpha)$ the class of *starlike functions of order* α $(0 \leq \alpha < 1)$.

For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ *is subordinate to* $g(z)$ in $\mathbb{U}$, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \ ,$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if and only if

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Sakaguchi [11] introduced and investigated the class $\mathcal{S}_s^*$ *starlike functions with respect to symmetric points* in $\mathbb{U}$, consisting functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0.$$

Following him, Gao and Zhou [7] discussed a class $\mathcal{K}_s$ of analytic functions related to the starlike functions. A function $f(z) \in \mathcal{S}$ is said to be in the class $\mathcal{K}_s$ if there exists a function $g(z) = z + b_2 z^2 + ... \in \mathcal{S}^*(1/2)$ such that

$$\text{Re}\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0 \qquad (\ z \in \mathbb{U}). \tag{1.2}$$

They proved that if $G(z)$ defined by

$$G(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1}, \tag{1.3}$$

where for $n = 2, 3, ...,$

$$B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + ... + (-1)^n 2b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2, \tag{1.4}$$

then $G \in \mathcal{S}^*$. Also, they showed that the class $\mathcal{K}_s$ is a subclass of the class $\mathcal{K}$ of close-to-convex functions. $G(z)$ is an *odd starlike function*, so $|B_{2n-1}| \leq 1$ for $n \geq 2$ (see [7]).

It is known that every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \cdots . \tag{1.5}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ stated by Taylor-Maclaurin series expansion 1.1. Similarly, a function $f \in \mathcal{A}$ is said to be *bi-close-to-convex* in $\mathbb{U}$ if both $f(z)$ and $F = f^{-1}$ are close-to-convex in $\mathbb{U}$.

If there exists a function $h(w) = w + \sum_{n=2}^{\infty} c_n w^n \in \mathcal{S}^*(1/2)$ such that

$$\operatorname{Re}\left(\frac{-w^2 F'(w)}{h(w)h(-w)}\right) > 0 \qquad (w \in \mathbb{U}), \tag{1.6}$$

then the inverse map $F = f^{-1}$ is close-to-convex in $\mathbb{U}$. Similar to the definition of the function $G(z)$, if we define $H(w)$,

$$H(w) = \frac{-h(w)h(-w)}{w} = w + \sum_{n=2}^{\infty} C_{2n-1} w^{2n-1}, \tag{1.7}$$

where for $n = 2, 3, ...,$

$$C_{2n-1} = 2c_{2n-1} - 2c_2 c_{2n-2} + ... + (-1)^n 2c_{n-1} c_{n+1} + (-1)^{n+1} c_n^2, \tag{1.8}$$

then $H \in \mathcal{S}^*$. $H(z)$ is also an odd starlike function and it is clear that $|C_{2n-1}| \leq 1$ for $n \geq 2$.

For a brief history of functions in the class Σ, see [14] (see also [4], [10], [12] and [16]). Coefficient bounds for various subclasses of bi-univalent functions were obtained by several authors including Akın and Sümer Eker [1], Ali *et al.* [2], Altınkaya and Yalçın [3], Frasin [6], Jahangiri and Hamidi [8], Jahangiri *et al.* [9], Srivastava *et al.* [13], Srivastava *et al.* [15], and Xu *et al.* [17, 18].

In this study, we give new subclasses of the bi-close-to-convex functions using odd starlike functions. Moreover, we obtain initial coefficient for the functions belonging these classes. These new classes will be able to described depending on this class for further studies.

We should remember here the following lemma here so as to derive our basic results:

Lemma 1.1. *[5] If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of functions p analytic in $\mathbb{U}$ for which $Re\{p(z)\} > 0, p(z) = 1 + p_1 z + p_2 z^2 + ...$ for $z \in \mathbb{U}$.*

2 Coefficient bounds for the function class $\mathcal{K}^s_\Sigma(\alpha)$

We begin by introducing the function class $\mathcal{K}^s_\Sigma(\alpha)$ by means of the following definition.

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{K}^s_\Sigma(\alpha)$ if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$, $h(w) = w + \sum_{n=2}^{\infty} c_n w^n \in \mathcal{S}^*(1/2)$ and the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left| arg\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1; z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(\frac{-w^2 F'(w)}{h(w)h(-w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1; z \in \mathbb{U}) \tag{2.2}$$

where the function $F(w)$ is given by

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2.3}$$

We start by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{K}^s_\Sigma(\alpha)$.

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{K}^s_\Sigma(\alpha)$ $(0 < \alpha \le 1\,;\, z \in \mathbb{U})$, then*

$$|a_2| \le \sqrt{\frac{\alpha(1+2\alpha)}{2+\alpha}} \qquad \textit{and} \qquad |a_3| \le \frac{\alpha(3\alpha+2)+1}{3}. \tag{2.4}$$

Proof. It can be written that the inequalities (2.1) and (2.2) are equivalent to

$$\frac{-z^2 f'(z)}{g(z)g(-z)} = \frac{zf'(z)}{\frac{-g(z)g(-z)}{z}} = \frac{zf'(z)}{G(z)} = [p(z)]^\alpha \tag{2.5}$$

and

$$\frac{-w^2 F'(w)}{h(w)h(-w)} = \frac{wF'(w)}{\frac{-h(w)h(-w)}{w}} = \frac{wF'(w)}{H(w)} = [q(w)]^\alpha \tag{2.6}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.8}$$

Now, equating the coefficients in (2.5) and (2.6), we obtain

$$2a_2 = \alpha p_1, \tag{2.9}$$

$$3a_3 - B_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.10}$$

$$-2a_2 = \alpha q_1 \tag{2.11}$$

and

$$3\left(2a_2^2 - a_3\right) - C_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1 = -q_1 \tag{2.13}$$

and

$$8a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.14}$$

Also from (2.10), (2.12) and (2.14), we find that

$$\begin{aligned} 6a_2^2 &= B_3 + C_3 + \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 + q_1^2) \\ &= B_3 + C_3 + \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}\frac{8a_2^2}{\alpha^2}. \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha}{2\alpha+4}(B_3 + C_3) + \frac{\alpha^2}{2\alpha+4}(p_2 + q_2). \tag{2.15}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2 and considering the inequalities

$$|B_{2n-1}| \le 1 \text{ and } |C_{2n-1}| \le 1,$$

we obtain desired estimate for $|a_2|$ as asserted (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$6a_3 - 6a_2^2 + C_3 - B_3 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2)$$

or

$$a_3 = \frac{1}{6}(B_3 - C_3) + \frac{1}{4}\alpha^2 p_1^2 + \frac{1}{6}\alpha(p_2 - q_2). \tag{2.16}$$

Applying Lemma 1.1 one more time for the coefficients p_1, p_2, q_1 and q_2 and considering the inequalities

$$|B_{2n-1}| \le 1 \text{ and } |C_{2n-1}| \le 1,$$

we obtain

$$|a_3| \le \frac{\alpha(3\alpha+2)+1}{3}.$$

This completes the proof of the Theorem 2.1. □

3 Coefficient bounds for the function class $\mathcal{K}_\Sigma^s(\beta)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{K}_\Sigma^s(\beta)$ if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$, $h(w) = w + \sum_{n=2}^{\infty} c_n w^n \in \mathcal{S}^*(1/2)$ and the following conditions are satisfied:

$$f \in \Sigma \text{ and } Re\left\{\frac{-z^2 f'(z)}{g(z)g(-z)}\right\} > \beta \quad (0 \le \beta < 1; z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left\{\frac{-w^2 F'(w)}{h(w)h(-w)}\right\} > \beta \quad (0 \le \beta < 1; w \in \mathbb{U}) \tag{3.2}$$

where the function $F(w)$ is given by (1.5).

Theorem 3.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{K}_\Sigma^s(\beta)$ $(0 \leq \beta < 1\ ;\ z \in \mathbb{U})$, then*

$$|a_2| \leq \sqrt{\frac{3-2\beta}{3}} \qquad \textit{and} \qquad |a_3| \leq \frac{(1-\beta)(5-3\beta)+1}{3}. \tag{3.3}$$

Proof. It follows from (3.1) and (3.2) that there exists $p(z) \in \mathcal{P}$ and $q(z) \in \mathcal{P}$ such that

$$\frac{-z^2 f'(z)}{g(z)g(-z)} = \frac{zf'(z)}{\frac{-g(z)g(-z)}{z}} = \frac{zf'(z)}{G(z)} = \beta + (1-\beta)p(z) \tag{3.4}$$

and

$$\frac{-w^2 F'(w)}{G(w)G(-w)} = \frac{wF'(w)}{\frac{-h(w)h(-w)}{w}} = \frac{wF'(w)}{H(w)} = \beta + (1-\beta)q(w) \tag{3.5}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{3.6}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{3.7}$$

Equating coefficients in (3.4) and (3.5) yields

$$2a_2 = (1-\beta)p_1, \tag{3.8}$$

$$3a_3 - B_3 = (1-\beta)p_2, \tag{3.9}$$

$$-2a_2 = (1-\beta)q_1, \tag{3.10}$$

and

$$3(2a_2^2 - a_3) - C_3 = (1-\beta)q_2. \tag{3.11}$$

From (3.8) and (3.10), we get

$$p_1 = -q_1 \tag{3.12}$$

and

$$8a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.13}$$

Also from (3.9) and (3.11), we find that

$$6a_2^2 = B_3 + C_3 + (1-\beta)(p_2 + q_2).$$

Thus, we have

$$|a_2^2| \leq \frac{1}{6}(|B_3| + |C_3|) + \frac{(1-\beta)}{6}(|p_2| + |q_2|) \leq \frac{3-2\beta}{3}$$

which is the bound on $|a_2|$ as given in the (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.11) from (3.9), we get

$$6a_3 - 6a_2^2 + C_3 - B_3 = (1-\beta)\,(p_2 - q_2)$$

Upon substituting the value of a_2^2 from (3.13), we have

$$a_3 = \frac{1}{6}(B_3 - C_3) + \frac{1-\beta}{6}\,(p_2 - q_2) + \frac{1}{8}(1-\beta)^2(p_1^2 + q_1^2).$$

Applying Lemma 1.1 once again p_1, p_2, q_1 and q_2 and considering the inequalities

$$|B_{2n-1}| \leq 1 \text{ and } |C_{2n-1}| \leq 1,$$

we obtain

$$|a_3| \leq \frac{1+(1-\beta)(5-3\beta)}{3}.$$

which is the bound on $|a_3|$ as asserted in (3.3).

□

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Author information

Bilal ŞEKER, Department of Mathematics, Faculty of Science and Letters, Batman University TR-72060, Batman, Turkey.
E-mail: bilal.seker@batman.edu.tr

S. SÜMER EKER, Department of Mathematics, Faculty of Science, Dicle University TR-72060, Diyarbakir, Turkey.
E-mail: sevtaps35@gmail.com

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