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Generalized weighted Ostrowski type inequalities for local fractional integrals

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Abstract. In this paper, we establish some generalized weighted Ostrowski inequalities for local fractional integrals on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers. The results presented here would provide extensions of those given in earlier works.

1 Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [14]:

Theorem 1.1 (Ostrowski inequality). *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to R$ is bounded on (a,b), i.e. $\|f'\|_{\infty}:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_{\infty}, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality.* For more information recent development on Ostrowski inequality, please refer to *[1]-[5],[7]-[11],[15]-[20] and so on.*

2 Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [26, 27] and so on.

Recently, the theory of Yang's fractional sets [26] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}:p,q\in Z,$ $q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}:p,q\in Z,$ $q\neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};

(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;

(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;

(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;

(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;

(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;

(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. [26] A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2.2. [26]The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{\left(x-x_0\right)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right).$

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 2.3. [26] Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 2.4 (Generalized convex function). [26] Let $f:I\subseteq R\rightarrow R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;

(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 2.5. *[12] Let $f\in D_{\alpha}(I)$, then the following conditions are equivalent*

a) f is a generalized convex function on I

b) $f^{(\alpha)}$ is an increasing function on I

c) for any $x_1,x_2\in I$,

$$f(x_2)-f(x_1)\geq\frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)}\left(x_2-x_1\right)^{\alpha}.$$

Corollary 2.6. *[12] Let $f\in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$f^{(2\alpha)}(x)\geq 0\left(or\ f^{(2\alpha)}(x)\leq 0\right)$$

for all $x\in(a,b)$.

Lemma 2.7. *[26]*

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, *then we have*

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$, *then we have*

$${}_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2.8. *[26] We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)}\int\limits_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right),\ k \in R.$

Lemma 2.9. *[26] Suppose that* $f(x) \in C_\alpha[a,b]$, *then*

$$\frac{d^\alpha\left({}_aI_x^\alpha f(t)\right)}{dx^\alpha} = f(x) \quad a < x < b.$$

Lemma 2.10 (Generalized Hölder's inequality). *[26] Let* $f, g \in C_\alpha[a,b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b |f(x)g(x)|\,(dx)^\alpha \le \left(\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b |f(x)|^p\,(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b |g(x)|^q\,(dx)^\alpha\right)^{\frac{1}{q}}.$$

In [21], Sarikaya and Budak proved the following generalized Ostrowski inequality:

Theorem 2.11 (Generalized Ostrowski inequality). *Let* $I \subseteq \mathbb{R}$ *be an interval,* $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *(*I^0 *is the interior of* I*) such that* $f \in D_\alpha(I^0)$ *and* $f^{(\alpha)} \in C_\alpha[a,b]$ *for* $a, b \in I^0$ *with* $a < b$ *Then. for all* $x \in [a,b]$, *we have the inequality*

$$\left|f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right| \le 2^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{1}{4^\alpha} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right](b-a)^\alpha\left\|f^{(\alpha)}\right\|_\infty. \tag{2.1}$$

For more information and recent developments on local fractional theory, please refer to [6],[12],[13],[21]-[30].

The aim of the this paper is to obtain some generalized weighted Ostrowski inequality for local fractional integrals.

3 Main Results

We will give a identity for local fractional integrals as follow:

Theorem 3.1. *Let* $I \subseteq \mathbb{R}$ *be an interval,* $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ *(*I^0 *is the interior of* I*) such that* $f \in D_\alpha(I^0)$ *and* $f^{(\alpha)} \in C_\alpha[a,b]$ *for* $a, b \in I^0$ *with* $a < b$ *and* $w : [a,b] \to \mathbb{R}^\alpha$, *non-negative and* $w(x) \in I_x^\alpha[a,b]$. *Then, for all* $x \in [a,b]$, *we have the identity*

$$\left[{}_aI_b^\alpha w(t)\right]f(x) - {}_aI_b^\alpha w(t)f(t) = \frac{1}{\Gamma(1+\alpha)}\int\limits_a^b p_w(x,t)f^{(\alpha)}(t)\,(dt)^\alpha \tag{3.1}$$

where

$$p(x,t) = \begin{cases} \frac{1}{\Gamma(1+\alpha)}\int\limits_a^t w(u)\,(du)^\alpha, & t \in [a,x] \\ \frac{1}{\Gamma(1+\alpha)}\int\limits_b^t w(u)\,(du)^\alpha, & t \in (x,b]. \end{cases}$$

Proof. We have

$$\begin{aligned} K &= \frac{1}{\Gamma(1+\alpha)} \int_a^b p_w(x,t) f^{(\alpha)}(t)\,(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) f^{(\alpha)}(t)\,(dt)^\alpha \\ &\quad + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left(\frac{1}{\Gamma(1+\alpha)} \int_b^t w(u)\,(du)^\alpha \right) f^{(\alpha)}(t)\,(dt)^\alpha \\ &= K_1 + K_2. \end{aligned}$$

Using the local fractional integration by parts, we have

$$\begin{aligned} K_1 &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) f^{(\alpha)}(t)\,(dt)^\alpha \\ &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u)\,(du)^\alpha \right) f(t) \Big|_a^x - \frac{1}{\Gamma(1+\alpha)} \int_a^x w(t) f(t)\,(dt)^\alpha \\ &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^x w(u)\,(du)^\alpha \right) f(x) - \frac{1}{\Gamma(1+\alpha)} \int_a^x w(t) f(t)\,(dt)^\alpha \end{aligned} \tag{3.2}$$

and similarly,

$$K_2 = \left(\frac{1}{\Gamma(1+\alpha)} \int_x^b w(u)\,(du)^\alpha \right) f(x) - \frac{1}{\Gamma(1+\alpha)} \int_x^b w(t) f(t)\,(dt)^\alpha . \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\begin{aligned} K &= \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b w(u)\,(du)^\alpha \right) f(x) - \frac{1}{\Gamma(1+\alpha)} \int_a^b w(t) f(t)\,(dt)^\alpha \\ &= \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \end{aligned}$$

which completes the proof. □

Remark 3.2. If we take $w \equiv 1^\alpha$ in Theorem 3.1, then Theorem 3.1 reduces Theorem 3 in [21].

Theorem 3.3 (Generalized weighted Ostrowski inequality). *Suppose that the assumptions of Theorem 3.1 are satisfied,* $\left\| f^{(\alpha)} \right\|_\infty = \sup_{x \in [a,b]} \left| f^{(\alpha)}(x) \right|$, *then we have the following generalized weighted Ostrowski inequality*

$$\begin{aligned} &\left| \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \right| \\ &\leq \frac{2^\alpha (b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right] \|w\|_\infty \left\| f^{(\alpha)} \right\|_\infty . \end{aligned} \tag{3.4}$$

Proof. Taking modulus in Theorem 3.1, we have

$$\begin{aligned} & \left| \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \right| \\ \leq & \frac{1}{\Gamma(1+\alpha)} \int_a^b \left| p_w(x,t) \right| \left| f^{(\alpha)}(t) \right| (dt)^\alpha \\ = & \frac{1}{\Gamma(1+\alpha)} \int_a^x \left(\frac{1}{\Gamma(1+\alpha)} \int_a^t w(u) (du)^\alpha \right) \left| f^{(\alpha)}(t) \right| (dt)^\alpha \\ & + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left(\frac{1}{\Gamma(1+\alpha)} \int_t^b w(u) (du)^\alpha \right) \left| f^{(\alpha)}(t) \right| (dt)^\alpha . \end{aligned}$$

Then, it follows that

$$\begin{aligned} & \left| \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \right| \\ \leq & \frac{\left\| f^{(\alpha)} \right\|_\infty \|w\|_\infty}{\Gamma(1+\alpha)} \left[\frac{1}{\Gamma(1+\alpha)} \int_a^x (t-a)^\alpha (dt)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_a^x (b-t)^\alpha (dt)^\alpha \right] \\ = & \frac{\left\| f^{(\alpha)} \right\|_\infty \|w\|_\infty}{\Gamma(1+\alpha)} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[(x-a)^{2\alpha} + (b-x)^{2\alpha} \right] \\ = & \frac{2^\alpha (b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right] \left\| f^{(\alpha)} \right\|_\infty \|w\|_\infty . \end{aligned}$$

which completes the proof. □

Remark 3.4. If we take $w \equiv 1^\alpha$ in Theorem 3.3, then the inequality (3.4) reduces the inequality (2.1).

Theorem 3.5. *Suppose that the assumptions of Theorem 3.1 are satisfied, then we have the inequality*

$$\begin{aligned} & \left| \left[{}_aI_b^\alpha w(t) \right] f(x) - {}_aI_b^\alpha w(t) f(t) \right| \\ \leq & \frac{\left\| f^{(\alpha)} \right\|_q \|w\|_p}{\Gamma(1+\alpha)} \left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)} \right)^{\frac{1}{p}} \left[(x-a)^{(p+1)\alpha} + (b-x)^{(p+1)\alpha} \right]^{\frac{1}{p}} \end{aligned}$$

where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\left\| f^{(\alpha)} \right\|_q$ *is defined by*

$$\left\| f^{(\alpha)} \right\|_q = \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left| f^{(\alpha)}(t) \right|^q (dt)^\alpha \right)^{\frac{1}{q}} .$$

Proof. Taking modulus in Theorem 3.1 and using the generalized Hölder's inequality (Lemma

2.10), we obtain

$$\begin{aligned}
&\left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right| \\
&\leq \frac{1}{\Gamma(1+\alpha)}\int_a^b |p_w(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha} \\
&\leq \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |p(x,t)|^p (dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f^{(\alpha)}(t)\right|^q (dt)^{\alpha}\right)^{\frac{1}{q}} \\
&= \left\|f^{(\alpha)}\right\|_q \left[\frac{1}{\Gamma(1+\alpha)}\int_a^x \left(\frac{1}{\Gamma(1+\alpha)}\int_a^t w(u)\,(du)^{\alpha}\right)^p (dt)^{\alpha}\right. \\
&\quad \left.+\frac{1}{\Gamma(1+\alpha)}\int_x^b \left(\frac{1}{\Gamma(1+\alpha)}\int_t^b w(u)\,(du)^{\alpha}\right)^p (dt)^{\alpha}\right]^{\frac{1}{p}} \\
&\leq \frac{\left\|f^{(\alpha)}\right\|_q \|w\|_p}{\Gamma(1+\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_a^x (t-a)^{p\alpha}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_x^b (b-t)^{p\alpha}(dt)^{\alpha}\right] \\
&= \frac{\left\|f^{(\alpha)}\right\|_q \|w\|_p}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]\right)^{\frac{1}{p}}
\end{aligned}$$

which completes the proof. □

Remark 3.6. If we take $w \equiv 1^{\alpha}$ in Theorem 3.5, then we have the inequality

$$\begin{aligned}
&\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right| \\
&\leq \frac{\left\|f^{(\alpha)}\right\|_q}{(b-a)^{\alpha}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[(x-a)^{(p+1)\alpha}+(b-x)^{(p+1)\alpha}\right]^{\frac{1}{p}}
\end{aligned}$$

which is proved by Sarikaya and Budak in [21].

Theorem 3.7. *The assumptions of Theorem 3.1 are satisfied. If* $\left|f^{(\alpha)}\right|^q$ *is a generalized convex, then we have the following inequality*

$$\begin{aligned}
&\left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right| \qquad (3.5)\\
&\leq \frac{\|w\|_{[a,b],p}}{(b-a)^{\frac{\alpha}{q}}\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}} \\
&\quad\times\left[(x-a)^{\left(\frac{p+1}{p}\right)\alpha}\left(\left[(b-a)^{2\alpha}-(b-x)^{2\alpha}\right]\left|f^{(\alpha)}(a)\right|^q+(x-a)^{2\alpha}\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+(b-x)^{\left(\frac{p+1}{p}\right)\alpha}\left((b-x)^{2\alpha}\left|f^{(\alpha)}(a)\right|^q+\left[(b-a)^{2\alpha}-(x-a)^{2\alpha}\right]\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\|w\|_{[a,b],p}$ *is defined by*

$$\|w\|_{[a,b],p}=\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |w(t)|^p (dt)^{\alpha}\right)^{\frac{1}{p}}.$$

Proof. Taking modulus in Theorem 3.1

$$\begin{aligned}
&\left|\left[{}_aI_b^{\alpha}w(t)\right]f(x)-{}_aI_b^{\alpha}w(t)f(t)\right| \qquad (3.6)\\
&\leq \frac{1}{\Gamma(1+\alpha)}\int_a^b |p_w(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\\
&= \frac{1}{\Gamma(1+\alpha)}\int_a^x\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t w(u)\,(du)^{\alpha}\right)\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_x^b\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b w(u)\,(du)^{\alpha}\right)\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}.\\
&= K_3+K_4.
\end{aligned}$$

Using the generalized Hölder's inequality, we obtain

$$\begin{aligned}
K_3 \leq& \left(\frac{1}{\Gamma(1+\alpha)}\int_a^x\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t w(u)\,(du)^{\alpha}\right)^p(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^x\left|f^{(\alpha)}(t)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(\alpha)}\right|^q$ is a generalized convex, we have

$$\begin{aligned}
\left|f^{(\alpha)}(t)\right|^q &= \left|f^{(\alpha)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\\
&\leq \left(\frac{b-t}{b-a}\right)^{\alpha}\left|f^{(\alpha)}(a)\right|^q+\left(\frac{t-a}{b-a}\right)^{\alpha}\left|f^{(\alpha)}(b)\right|^q.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
K_3 \leq& \|w\|_{[a,x],p}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^x\frac{(t-a)^{p\alpha}}{[\Gamma(1+\alpha)]^p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\times\left(\frac{\left|f^{(\alpha)}(a)\right|^q}{\Gamma(1+\alpha)}\int_a^x\left(\frac{b-t}{b-a}\right)^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(\alpha)}(b)\right|^q}{\Gamma(1+\alpha)}\int_a^x\left(\frac{t-a}{b-a}\right)^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
=& \frac{\|w\|_{[a,x],p}}{(b-a)^{\frac{\alpha}{q}}\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}(x-a)^{(p+1)\alpha}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\\
&\times\left(\left[(b-a)^{2\alpha}-(b-x)^{2\alpha}\right]\left|f^{(\alpha)}(a)\right|^q+(x-a)^{2\alpha}\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the similar way, we have

$$\begin{aligned}
K_4 \leq& \frac{\|w\|_{[x,b],p}}{(b-a)^{\frac{\alpha}{q}}\Gamma(1+\alpha)}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}(b-x)^{(p+1)\alpha}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\\
&\times\left((b-x)^{2\alpha}\left|f^{(\alpha)}(a)\right|^q+\left[(b-a)^{2\alpha}-(x-a)^{2\alpha}\right]\left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the fact that $\|w\|_{[a,x],p} \leq \|w\|_{[a,b],p}$ and $\|w\|_{[x,b],p} \leq \|w\|_{[a,b],p}$, then we obtain required result. □

Corollary 3.8. *Under assumptions of Theorem 3.7 with* $w \equiv 1$, *then we have the inequality*

$$\left|f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(t)\right| \tag{3.7}$$

$$\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\left(1+\frac{1}{q}\right)\alpha}} \left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}$$

$$\times \Bigg[(x-a)^{\left(\frac{p+1}{p}\right)\alpha} \left(\left[(b-a)^{2\alpha} - (b-x)^{2\alpha}\right] \left|f^{(\alpha)}(a)\right|^q + (x-a)^{2\alpha} \left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}$$

$$+ (b-x)^{\left(\frac{p+1}{p}\right)\alpha} \left((b-x)^{2\alpha} \left|f^{(\alpha)}(a)\right|^q + \left[(b-a)^{2\alpha} - (x-a)^{2\alpha}\right] \left|f^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Corollary 3.9. *If we choose* $x = \frac{a+b}{2}$ *in inequality (3.7), then we obtain the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(t)\right|$$

$$\leq \frac{\Gamma(1+\alpha)(b-a)^{\alpha}}{4^{\alpha}} \left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}$$

$$\times \left[\left(\frac{3^{\alpha}\left|f^{(\alpha)}(a)\right|^q + \left|f^{(\alpha)}(b)\right|^q}{4^{\alpha}}\right)^{\frac{1}{q}} + \left(\frac{\left|f^{(\alpha)}(a)\right|^q + 3^{\alpha}\left|f^{(\alpha)}(b)\right|^q}{4^{\alpha}}\right)^{\frac{1}{q}}\right].$$

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