A REMARK ABOUT THE COMPOSITION OPERATORS IN THE SPACE OF BOUNDED $\Lambda$–VARIATION FUNCTIONS IN WATERMAN SENSE

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Abstract. In this paper, we demonstrate the generalization of uniform continuous of the composition operators in the space of the bounded $\Lambda$-variation functions [2, 3]. In this paper we extend the result obtained recently in [2, 3] and [11] the space of bounded $\Lambda$-variation in the sense Waterman [21]. Also we give some results about locally defined operators.

1 Introduction

Let $I$ be an interval of $\mathbb{R}$, $X$ a real normed space, $C$ a closed convex subset of $X$, $Y$ a real Banach space and $h : I \times C \to Y$. Denote by $X^I$ the algebra of all functions $f : I \to X$ and by $H : X^I \to Y^I$ the Nemytskii composition operator generated by the function $h$ defined by

$$(Hf)(t) = h(t, f(t)), \quad t \in I, \quad f \in X^I.$$ (1.1)

Let $(ABV(I, X), \| \cdot \|_{\Lambda})$ be the Banach space of functions $f : I \to X$ which are of bounded $\Lambda$-variation in the sense of Waterman, where the norm $\| \cdot \|_{\Lambda}$ is defined with the aid of Luxemburg-Nakano-Orlicz seminorm [16, 10, 18].

Assume that $H$ maps the set of functions $f \in ABV(I, X)$ such that $f(I) \subset C$ into $ABV(I, Y)$. In the present paper, we prove that, if $H$ is uniformly continuous, then the left and right regularization of its generator $h$ with respect for the first variable are affine functions in the second variable. This extends the main result of paper [2, 3].

2 Preliminaries

In this section we recall some facts which will be needed further on.

Denote by $\mathbb{R}$ the set of all real numbers and put $\mathbb{R}_+ = [0, \infty)$.

Next, let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive real numbers such that $\sum 1 / \lambda_n$ diverges.

If $\{I_n\}$ denote a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset I$ and we write $f(I_n) = f(b_n) - f(a_n)$. Throughout this paper, when we consider a collection of intervals, they will be assumed to be non-decreasing without further reference to that fact.

Let $I \subset \mathbb{R}$ be an interval. Then, for a set $X$ we denote by $X^I$ the set of all mappings $f : I \to X$ acting from $I$ into $X$. 
Definition 2.1 (21). A function \( f \in X^I \) is said to be of \( \Lambda \)-bounded variation (ABV), in the sense of Waterman in \( I \), if for every \( \{I_n\} \), we have
\[
v_\Lambda(f) = v_\Lambda(f, I) := \sup_n \frac{\|f(I_n)\|}{\lambda_n} < \infty,
\]
the supremum being taken over all \( \{I_n\}, \{I_n\} \subseteq I \).

Various spaces of the functions of generalized bounded variation which have been considered can be obtained by making special choices of the functions \( \lambda_n, n = 1, 2, \ldots \). If we take \( \Lambda = \{n\} \) rise to be class of functions of harmonic bounded variation \( HBV \). The definition (2.1) coincides with the classical concept of variation in the sense of Jordan. For \( \lambda_n = \varphi \), the condition (2.1) coincides with the classical concept of variation in the sense of Wiener [22], where \( \varphi : [0, +\infty) \to [0, +\infty) \) denote a continuous, convex and non-decreasing function, with \( \varphi(0) = 0 \), \( \varphi(x) > 0 \) for \( x > 0 \).

It is easily seen that \( ABV = BV \), the space of functions of ordinary Jordan bounded variation on \( I \), if and only if \( \Lambda \) is a bounded sequence. Consequently, if we suppose that \( \sup \lambda_i = \infty \), then BV is a proper subspace of ABV.

It is known that for all \( a, b, c \in I \), such that \( a \leq c \leq b \), we have \( v_\Lambda(f, [a, c]) < v_\Lambda(f, [a, b]) \) (that is, \( v_\Lambda \) is increasing with respect to the interval) and
\[
v_\Lambda(f, [a, c]) + v_\Lambda(f, [c, b]) < v_\Lambda(f, [a, b]).
\]
In what follows we denote by \( V_\Lambda(I, X) \) the set of all bounded \( \Lambda \)-variation functions \( f \in X^I \) in the Waterman sense. This is a symmetric and convex set; but it is not necessarily a linear space. In fact, Musielak-Orlicz proved the following statement: this class of functions \( (V_\Lambda(I, X) \supseteq V_\Lambda(I, X)) \) is a linear space if, and only if, \( \varphi \) satisfies the \( \delta_2 \) condition [15] (there exist \( a > 0 \) and \( k > 0 \) such that \( \varphi(2u) \leq k\varphi(u) \) for \( 0 < u \leq a \)). Denote by \( \Lambda BV(I, X) \) the linear space of all functions \( f \in X^I \) such that \( v_\Lambda(\lambda f) < \infty \) for some constant \( \lambda > 0 \).

In the linear space \( \Lambda BV(I, X) \), the function \( \|f\|_\Lambda \) defined by
\[
\|f\|_\Lambda := |f(a)| + p_\Lambda(f), \quad f \in \Lambda BV(I, X),
\]
where
\[
p_\Lambda(f) := p_\Lambda(f, I) = \inf \left\{ \varepsilon > 0 : v_\Lambda(f/\varepsilon) \leq 1 \right\}, \quad f \in \Lambda BV(I, X),
\]
is a norm (see for instance [15, 6, 20]).

For \( X = \mathbb{R} \), the linear normed space \( (\Lambda BV(I, \mathbb{R}), \|\cdot\|_\Lambda) \) was studied by Daniel Waterman (21). Also he joint with Perlman shows that the space \( (\Lambda BV(I, \mathbb{R}), \|\cdot\|_\Lambda) \) is a Banach algebra (14, 19]). The functional \( p_\Lambda(\cdot) \) defined by (2.2) is called the Luxembourg-Nakano-Orlicz seminorm [16, 10, 18].

In the sequel, the symbol \( \Lambda BV(I, C) \) stands for the set of all functions \( f \in \Lambda BV(I, X) \) such that \( f : I \to C \) and \( C \) is a subset of \( X \).

Lemma 2.2. For \( f \in \Lambda BV(I, X) \), we have:

(a) if \( t, t' \in I \), then \( \|f(t) - f(t')\| \leq \lambda_1 p_\Lambda(f) \);
(b) if \( p_\Lambda(f) > 0 \) then \( v_\Lambda(f/p_\Lambda(f)) \leq 1 \);
(c) for \( \varepsilon > 0 \),

(c1) \( p_\Lambda(f) \leq \varepsilon \) if and only if \( v_\Lambda(f/\varepsilon) \leq 1 \);
(c2) if \( v_\Lambda(f/\varepsilon) = 1 \) then \( p_\Lambda(f) = \varepsilon \).

Proof. (a) Take \( \varepsilon > p_\Lambda(f) \); then for any \( t, s \in I \) and for any finite collection \( \{I_n\} \), by virtue (2.1) and (2.2), we have
\[
\frac{\|f(t) - f(s)\|}{\lambda_1 \varepsilon} \leq \sum_n \left( \frac{\|f(I_n)\|}{\lambda_n \varepsilon} \right) \leq v_\Lambda \left( \frac{f}{\varepsilon} \right) \leq 1.
\]
whence, taking the function $\lambda_1$ we obtain (a). Property (a) in Lemma 2.2 implies that any function $f \in ABV(I, X)$ is bounded. Indeed, we have $\|f\| \leq \|f(a)\| + \|f(t) - f(a)\|$, whence
\[
\|f\|_\infty \leq \|f(a)\| + \lambda_n^{-1}(1) p_n(f) < \infty.
\]

(b) Suppose that sequence of the numbers $\lambda_n > \lambda = p_n(f)$ converges a $\lambda$ as $n \to \infty$. If follows from the definition of the number $\lambda$ that $v_\lambda(f) \leq 1$ for all positive integers $n$. Since $f/\lambda_n$ pointwise converges to $f/\lambda$ on $I$ as $n \to \infty$, by the lower semicontinuity of the functional $v_\lambda(.)$, we obtain that $v_\lambda(f/\lambda) \leq \lim_{n \to \infty} v_\lambda(f/\lambda_n) \leq 1$.

(c) To prove (c.1), it suffices to show that if $0 < p_n(f) < \epsilon$, then $v_\lambda(f/\epsilon) < 1$, and this is directly implied by the convexity of $v_\lambda(.)$ and of the part (b), that is,
\[
v_\lambda(f/\epsilon) \leq \frac{p_n(f)}{\epsilon} v_\lambda\left(\frac{f}{p_n(f)}\right) \leq \frac{p_n(f)}{\epsilon} \leq 1.
\]

To prove the second assertion (c.2), it suffices to observe that the cases where $p_n(f) > \epsilon$ and $p_n(f) < \epsilon$ are impossible.

We consider the following notation of interval $I^-$ by formula $I^- := I \setminus \{\inf I\}$. If $(X, |\cdot|)$ is a Banach space and $f \in ABV(I, X)$, then $f(t) := \lim_{s \uparrow t} f(s), \ t \in I^-$, exists and is called the left regularization of $f$ it was proved in ([6]).

Let $ABV^-(I, X)$ denote the subset in $ABV(I, X)$ that consists of those functions that are left continuous on $I^-.$

**Lemma 2.3.** If $X$ is a Banach space and $f \in ABV(I, X)$, then $f^- \in ABV^-(I, X).$ The prove is similar to the since by Chistyakov [4, Lemma 6].

Thus, if a function has a bounded $\Lambda$-variation, then its left regularization is a left continuous function.

**Lemma 2.4.** [6] If $f : I \to X$ is monotone, then $v_\lambda(f) = \frac{\|f(b) - f(a)\|}{\lambda}$.

### 3 The Composition Operator

Our main result reads as follows:

**Theorem 3.1.** Let $(X, |\cdot|)$ be a real norm space, $(Y, |\cdot|_Y)$ a real Banach space, $C \subset X$ a closed convex set. Suppose that $\Lambda_1 = \{\lambda_1\}, \Lambda_2 = \{\varphi_2\}$ two sequence in sense Waterman and $h : I \times C \to Y$. If a composition operator $H : C^1 \to Y^1$ is generated by $h$, maps $\Lambda_1 BV(I, C)$ into $\Lambda_2 BV(I, Y)$ and is uniformly continuous, then the left regularization of $h$, i.e. the function $h^- : I^- \times X \to Y$, defined by
\[
h^-(t, y) := \lim_{s \uparrow t} h(s, y), \quad t \in I^-; \quad y \in C,
\]
exists and
\[
h^-(t, y) = A(t)y + B(t), \quad t \in I^-, \quad y \in C,
\]
for some $A : I^- \to \mathcal{L}(X, Y)^1$ and $B \in \Lambda_2 BV(I^-, Y)$. Moreover the functions $A$ and $B$ are left-continuous in $I^-.$

**Proof.** For every $y \in C$, the constant function $f(t) = y$ ($t \in I$) belongs to $\Lambda_1 BV(I, C)$. Since $H$ maps $\Lambda_1 BV(I, C)$ into $\Lambda_2 BV(I, Y)$, it follows that the function $t \mapsto h(t, y)$ ($t \in I$) belongs to $\Lambda_2 BV(I, Y)$. Now, by Lemma 2.3, the completeness of $(Y, |\cdot|_Y)$ implies the existence of a left regularization $h^-$ of $h$.

By assumption, $H$ is uniformly continuous on $\Lambda_1 BV(I, C)$. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be the modulus continuity of $H$ that is
\[
\omega(\rho) := \sup \left\{ \frac{\|H(f_1) - H(f_2)\|}{\Lambda_2 BV(I, Y)} : \|f_1 - f_2\|_{\Lambda_1 BV(I, Y)} \leq \rho \right\},
\]

\[\mathcal{L}(X, Y)\text{ denote the space of all linear mappings } A : X \to Y.\]
for \( f_1, f_2 \in \Lambda_1 BV(I, C) \) and \( \rho > 0 \).

Hence we get
\[
\|H(f_1) - H(f_2)\|_{\Lambda_2 BV(I, Y)} \leq \omega(\|f_1 - f_2\|_{\Lambda_1 BV(I, C)}), \quad \text{for } f_1, f_2 \in \Lambda_1 BV(I, C). \tag{3.1}
\]

From the definition of the norm \( \| \cdot \|_\omega \), we obtain
\[
p_\omega(H(f_1) - H(f_2)) \leq \|H(f_1) - H(f_2)\|_{\Lambda_2 BV(I, Y)}, \quad \text{for } f_1, f_2 \in \Lambda_1 BV(I, C). \tag{3.2}
\]

From (3.1), (3.2) and Lemma 2.2 (c1), if \( \omega(\|f_1 - f_2\|_{\Lambda_1 BV(I, C)}) > 0 \), then
\[
v_\omega\left(\frac{H(f_1) - H(f_2)}{\omega(\|f_1 - f_2\|_{\Lambda_1 BV(I, C)})}\right) \leq 1. \tag{3.3}
\]

Therefore, for any \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m, \alpha_i, \beta_i \in I, \ i \in \{1, 2, \ldots, m\}, m \in \mathbb{N} \), the definitions of the operator \( H \) and the functional \( v_\omega(\cdot) \) imply
\[
\sum_{n=1}^{m} \left( \frac{|h(\beta_n, f_1(\beta_n)) - h(\beta_n, f_2(\beta_n)) - h(\alpha_n, f_1(\alpha_n)) + h(\alpha_n, f_2(\alpha_n))|}{\lambda_n \omega(\|f_1 - f_2\|_{\Lambda_1 BV(I, C)})} \right) \leq 1. \tag{3.4}
\]

For \( \alpha, \beta \in \mathbb{R}, \alpha < \beta \), we define auxiliary Lipschitz functions \( \eta_{\alpha, \beta} : \mathbb{R} \rightarrow [0, 1] \) by
\[
\eta_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } t \leq \alpha \\ t - \alpha & \text{if } \alpha \leq t \leq \beta \\ 1 & \text{if } \beta \leq t. \end{cases} \tag{3.5}
\]

Let us fix \( t \in I^- \). For arbitrary finite sequence \( \inf I < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t \) and \( y_1, y_2 \in C \), \( y_1 \neq y_2 \), the functions \( f_1, f_2 : I \rightarrow X \) defined by
\[
f_\ell(t) := \frac{1}{2} \eta_{\alpha_\ell, \beta_\ell}(t)(y_1 - y_2) + y_\ell + y_2, \quad t \in I, \ \ell = 1, 2, \tag{3.6}
\]
belong to the space \( \Lambda_1 BV(I, C) \). From (3.6), we have
\[
f_1(t) - f_2(t) = \frac{y_1 - y_2}{2},
\]
therefore
\[
\|f_1 - f_2\|_{\Lambda_1 BV(I, C)} = \left| \frac{y_1 - y_2}{2} \right|;
\]

moreover
\[
f_1(\beta_\ell) = y_1; \quad f_2(\beta_\ell) = \frac{y_1 + y_2}{2}; \quad f_1(\alpha_\ell) = \frac{y_1 + y_2}{2}; \quad f_2(\alpha_\ell) = y_2.
\]

Using (3.4), we hence get
\[
\sum_{i=1}^{m} \left( \frac{|h(\beta_i, y_1) - h(\beta_i, \frac{y_1 + y_2}{2}) - h(\alpha_i, \frac{y_1 + y_2}{2}) + h(\alpha_i, y_2)|}{\lambda_i \omega(\left| \frac{y_1 - y_2}{2} \right|)} \right) \leq 1. \tag{3.7}
\]

It is of great importance remarks that the constants functions defined on the interval \( I \) belong to the space \( \Lambda_1 BV(I, C) \) since the composition operator \( H \) generate by \( h \) acts from \( \Lambda_1 BV(I, C) \) into \( \Lambda_2 BV(I, Y) \), it follows that the function \( t \mapsto h(t, y) \) \((t \in I)\) belong to \( \Lambda_2 BV(I, Y) \) for all \( y \in C \). From the continuity of \( \Lambda_2 \) and the definition of \( h^- \), passing to the limit in (3.7) when \( \alpha_1 \uparrow t \), we obtain that
\[
\sum_{i=1}^{m} \left( \frac{|h^-(t, y_1) - h^-(t, \frac{y_1 + y_2}{2}) - h^-(t, \frac{y_1 + y_2}{2}) + h^-(t, y_2)|}{\lambda_i \omega(\left| \frac{y_1 - y_2}{2} \right|)} \right) \leq 1,
\]
The sum of the left hand side suppose without lost generality fix \( i = n \) for \( n = 1, 2, \ldots, m \), such that

\[
m \cdot \left( \frac{|h^-(t, y_1) - 2h^-(t, \frac{y_1 + y_2}{2}) + h^-(t, y_2)|}{\lambda_n \omega\left(\left|\frac{y_1 - y_2}{2}\right|\right)} \right) \leq 1.
\]

we get

\[
\left( \frac{|h^-(t, y_1) - 2h^-(t, \frac{y_1 + y_2}{2}) + h^-(t, y_2)|}{\omega\left(\left|\frac{y_1 - y_2}{2}\right|\right)} \right) \leq \frac{1}{m}
\]

and since that \( m \in \mathbb{N} \) is arbitrary we derive

\[
\left( \frac{|h^-(t, y_1) - 2h^-(t, \frac{y_1 + y_2}{2}) + h^-(t, y_2)|}{\omega\left(\left|\frac{y_1 - y_2}{2}\right|\right)} \right) = 0,
\]

then

\[
|h^-(t, y_1) - 2h^-(t, \frac{y_1 + y_2}{2}) + h^-(t, y_2)| = 0.
\]

Or equivalently

\[
h^-\left(t, \frac{y_1 + y_2}{2}\right) = \frac{h^-(t, y_1) + h^-(t, y_2)}{2}
\]

for all \( t \in I^- \) and all \( y_1, y_2 \in C \).

Thus, for each \( t \in I^- \), the function \( h^-(t, \cdot) \) satisfies the Jensen functional equation in \( C \). Modifying a little the standard argument (cf. Kuczma [9]), we conclude that, for each \( t \in I^- \), there exist \( A(t) : C \to L(X, Y) \) and \( B(t) \in Y \) such that \( h^-(t, y) = A(t)y + B(t) \).

The uniform continuity of the operator \( H : \Lambda_1 BV(I, C) \to \Lambda_2 BV(I, Y) \) implies the continuity of the additive function \( A(t) \). Consequently \( A(t) \in L(X, Y) \), for each \( t \in I^- \). \( \square \)

**Remark 3.2.** Obviously, the counterpart of Theorem 3.1 for the right regularization \( h^+ \) of \( h \) defined by

\[
h^+(t, y) := \lim_{s \uparrow t} h(s, y); \quad t \in I^+ := I \setminus \{ \sup I \},
\]

is also true.

**Remark 3.3.** Taking \( X = Z = \mathbb{R} \), \( A = \varphi := \begin{cases} 1 \end{cases} \) in Theorem 3.1 and \( C := J \) where \( J \subset \mathbb{R} \) is an interval we obtain the main result from [11].

**Remark 3.4.** Theorem 3.1 extends also the result of Guerrero ([2, 3]).

**Remark 3.5.** In the proof of Theorem 3.1 we apply the uniform continuity of the operator \( H \) only on the set of functions \( U \subset \Lambda_1 BV(I, C) \) such that \( f \in U \) if, and only if, there are \( \alpha, \beta \in I \), \( \alpha < \beta \), such that

\[
f(t) = \frac{1}{2} \left[ \eta_{\alpha, \beta}(t)(y_1 - y_2) + y + y_2 \right], \quad t \in I,
\]

where \( \eta_{\alpha, \beta} \) is defined by (3.5), \( y_1, y_2 \in C \) and \( y = y_1 \) or \( y = y_2 \).

Thus the assumption of the uniform continuity of \( H \) on \( \Lambda_1 BV(I, C) \) in Theorem 3.1 can be replaced by a weaker condition of the uniform continuity of \( H \) on \( U \).
4 Locally Defined Operators

It is well known that every Nemtyskii composition operator is locally defined (cf. [1], also [13, 23, 24]). To recall the definition of a local operator assume that $G = G(I, \mathbb{R})$ and $H = H(I, \mathbb{R})$ are two classes of functions $\varphi : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. A mapping $K : G \to H$ is said to be a locally defined operator or $(G, H)$-local operator if for any open interval $J \subset \mathbb{R}$ and for any functions $\varphi, \psi \in G$, 

$$\varphi\big|_{J\cap I} = \psi\big|_{J\cap I} \Rightarrow K(\varphi)\big|_{J\cap I} = K(\psi)\big|_{J\cap I},$$

where $\varphi\big|_{J\cap I}$ denotes the restriction of $\varphi$ to $J \cap I$.

The form of the locally defined operator strongly depends on the nature of the function spaces $G$ and $H$ which are its domains and ranges, respectively.

Let $C(I)$ be a family of real continuous functions defined on $I$ and $CM_+(I)$ and $CM_-(I)$ denote, respectively, a family of continuous nondecreasing and continuous nonincreasing functions $f : I \to \mathbb{R}$.

We write $CBV(I)$ for $C(I) \cap BV(I, \mathbb{R})$.

**Proposition 4.1.** If a locally defined operator $K$ maps $CBV(I)$ into $CM_+(I)$, then it is constant, that is, a function $b \in CM_+(I)$ such that

$$K(\varphi) = b, \quad \varphi \in CBV(I).$$

**Proof.** Let $K : CBV(I) \to CM_+(I)$ be a local operator. Since $CM_+(I) \subset CBV(I)$ and $CM_-(I) \subset CBV(I)$, an operator $K$ is $(CM_+, CM_+)$ and $(CM_-, CM_+)$-locally defined. Hence, $K$ is the Nemtyskii composition operator and by Theorem 1 and Theorem 4 from [24], we get our claim. \hfill \Box

Similarly, by [24, Remark 4], we can get the following

**Proposition 4.2.** If a locally defined operator $K$ maps $CBV(I)$ into $CM_-(I)$, then it is constant, that is there is a function $b \in CM_-(I)$ such that

$$K(\varphi) = b, \quad \varphi \in CBV(I).$$

References


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