COEFFICIENT ESTIMATES FOR SOME SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

Pravati Sahoo and Saumya Singh

Communicated by Ayman Badawi

MSC 2010 Classifications: 30C45, 30C50.
Keywords and phrases: Bi-univalent, bi-starlike and bi-convex functions, subordination

Abstract. In the present paper, we introduce and investigate two new subclasses $\mathcal{B}_S(\alpha, \lambda, \mu)$ and $\mathcal{M}_S(\alpha, \lambda, \mu)$ of bi-valent functions in the unit disc $U$. For functions belonging to the classes $\mathcal{B}_S(\alpha, \lambda, \mu)$ and $\mathcal{M}_S(\alpha, \lambda, \mu)$, we obtain bounds of the first two Taylor-Maclaurin coefficients of $f(z)$.

1 Introduction and Preliminaries

Let $A$ be the class of analytic functions defined on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized conditions $f(0) = 0 = f'(0) - 1$. Let $S$ be the class of all functions $f \in A$ which are univalent in $U$. So $f(z) \in S$ has the form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, \quad z \in U. \quad (1.1)$$

Let $f^{-1}(z)$ be inverse of the function $f(z)$ and it is well known that every function $f \in S$ has an inverse $f^{-1}(z)$, defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f(f^{-1}(w)) = w, \quad \text{for} \quad |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_3^2 - a_3)w^3 - (5a_3^2 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (1.2)$$

A function $f \in A$ is said to be bi-univalent in $U$ if both $f(z)$ and $f^{-1}(w)$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions in $U$ given by (1.1).

Many interesting examples of the functions of the class $\Sigma$, together with various other properties and characteristics associated with bi-univalent functions (including also several open problems and conjectures involving bounds of the coefficients of the functions in $\Sigma$), can be found in the earlier work studied by Lewin[7], Brannan and Clunie [5], Netanyahu[8] and others. They introduced subclasses of $\Sigma$, like class of bi-starlike and bi-convex functions, bi-strongly starlike and bi-convex functions similar to the well-known subclasses $S^*(\alpha)$ and $K^*(\alpha)$ of starlike and convex functions of order $\alpha(0 < \alpha < 1)$, respectively (see [2]) and obtained non-sharp estimates on the initial coefficients in the Taylor-Maclaurin series expasion (1.1) see [4, 5, 13]. More recently, Srivastava et.al [12, 14, 15], Frasin and Aouf [6], R.M. Ali et.al [1] introduced some new subclasses of $\Sigma$ and obtained bounds for the initial coefficients of the function given by (1.1).

Motivated by the work of [12, 14, 15] and Sahoo et.al [11], we introduce and study some new subclasses $\mathcal{B}_S(\alpha, \lambda, \mu)$ and $\mathcal{M}_S(\alpha, \lambda, \mu)$.

Definition 1.1. A function $f$ given by (1.1) is said to be in the class $\mathcal{B}_S(\alpha, \lambda, \mu)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad 0 < \alpha \leq 1, \quad 0 < \mu < 1, \quad \lambda > \mu,$$

$$\arg \left( (1 - \lambda) \left( \frac{z}{f(z)} \right)^\mu + \lambda \left( \frac{z}{f(z)} \right)^{\mu+1} \right) < \frac{\alpha \pi}{2}, \quad z \in U, \quad (1.3)$$
and
\[
\left| \arg \left( (1 - \lambda) \left( \frac{w}{g(w)} \right)^\mu + \lambda \left( \frac{w}{g(w)} \right)^{\mu+1} \right) \right| < \frac{\alpha \pi}{2}, \quad w \in \mathbb{U},
\]  
(1.4)
where
\[
g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

\( B_\Sigma(\alpha, \lambda, -1) \) was introduced and studied in [6] and \( B_\Sigma(\alpha, 1, -1) \) was introduced and studied in [12]. In this paper, we found the estimates for the initial coefficients \( a_2 \) and \( a_3 \) of bi-univalent functions belonging to the class \( B_\Sigma(\alpha, \lambda, \mu) \). Our results generalizes several well-known results in [6, 12, 15].

In order to prove our main result we need the following lemma:

**Lemma 1.2.** [9] If \( p \in \mathcal{P} \), then \( |c_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p(z) \) analytic in \( \mathbb{U} \) for which \( \Re p(z) > 0 \), \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) for \( z \in \mathbb{U} \).

2 Coefficient bounds for the function belonging to the class \( B_\Sigma(\alpha, \lambda, \mu) \)

**Theorem 2.1.** Let \( f(z) \) given by (1.1) be in the class \( B_\Sigma(\alpha, \lambda, \mu) \), \( 0 < \mu < \alpha \leq 1 \), \( \lambda > \mu \). Then
\[
|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda - \mu)^2 + \alpha(2\lambda - \lambda^2 - \mu)}}
\]  
(2.1)
and
\[
|a_3| \leq \frac{4\alpha^2}{(\lambda - \mu)^2} + \frac{2\alpha}{2\lambda - \mu}.
\]  
(2.2)

**Proof.** It follows from (1.3) and (1.4) that
\[
(1 - \lambda) \left( \frac{z}{f(z)} \right)^\mu + \lambda \left( \frac{z}{f(z)} \right)^{\mu+1} = (p(z))^\alpha
\]  
(2.3)
and
\[
(1 - \lambda) \left( \frac{w}{g(w)} \right)^\mu + \lambda \left( \frac{w}{g(w)} \right)^{\mu+1} = (q(w))^\alpha,
\]  
(2.4)
where \( p(z) = 1 + p_1z + p_2z^2 + \cdots \) and \( q(w) = 1 + q_1w + q_2w^2 + \cdots \) in \( \mathcal{P} \). Now on equating the coefficients in (2.3) and (2.4), we have
\[
(\lambda - \mu)a_2 = \alpha p_1
\]  
(2.5)
\[
(2\lambda - \mu)a_3 + \frac{(\mu - 2\lambda)(\mu + 1)}{2}a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2}p_1^2
\]  
(2.6)
\[-(\lambda - \mu)a_2 = \alpha q_1
\]  
(2.7)
and
\[-(2\lambda - \mu)a_3 + \frac{(3 - \mu)(2\lambda - \mu)}{2}a_2^2 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2}q_1^2.
\]  
(2.8)
From (2.5) and (2.7) we get
\[
p_1 = -q_1
\]  
(2.9)
and
\[2(\lambda - \mu)a_2^2 = \alpha^2(p_1^2 + q_1^2).
\]  
(2.10)
From (2.6), (2.8) and (2.10), we get
\[
[(\mu - 1)(\mu - 2\lambda)]a_2^2 = (p_2 + q_2)\alpha + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2)
\]  
\[
= (p_2 + q_2)\alpha + \frac{\alpha - 1}{\alpha}(\lambda - \mu)^2a_2^2.
\]
Therefore, we have

\[ a_2^2 = \frac{\alpha^2(p_2^2 + q_2^2)}{(\lambda - \mu)^2 + \alpha(2\lambda - \mu - \lambda^2)} \]  

(2.11)

Applying Lemma 1.2 for (2.11), we get

\[ |a_2| \leq \frac{2\alpha}{\sqrt{(\lambda - \mu)^2 + \alpha(2\lambda - \mu - \lambda^2)}} \]

which gives us desired estimate on \( |a_2| \) as asserted in (2.1).

Next in order to find the bound on \( |a_3| \), by subtracting (2.8) from (2.6), we get

\[ 2(2\lambda - \mu)a_3 - 2(2\lambda - \mu)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left( \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2 \right). \]

(2.12)

It follows from (2.9), (2.10) and (2.12)

\[ a_3 = \frac{\alpha^2(p_1^2 + q_1^2)}{2(\lambda - \mu)^2} + \frac{\alpha(\beta_2 - q_2)}{2(2\lambda - \mu)}. \]

(2.13)

Applying Lemma 1.2 for (2.13), we get

\[ |a_3| \leq \frac{4\alpha^2}{(\lambda - \mu)^2} + \frac{2\alpha}{2\lambda - \mu}. \]

This completes the proof of Theorem 2.1.

If we take \( \mu = 1 \) in Theorem 2.1, we have the following corollary.

**Corollary 2.2.** Let \( f(z) \) given by (1.1) be in the class \( B_2(\alpha, \lambda, 1) \), \( 0 < \alpha \leq 1, \lambda > 1 \). Then

\[ |a_2| \leq \frac{2\alpha}{\sqrt{(\lambda - 1)^2 + \alpha(2\lambda - \lambda^2 - 1)}} \]  

(2.14)

and

\[ |a_3| \leq \frac{4\alpha^2}{(\lambda - 1)^2} + \frac{2\alpha}{2\lambda - 1}. \]

(2.15)

### 3 Coefficient bounds for the function belonging to the class \( M_\Sigma(\beta, \lambda, \mu) \)

**Definition 3.1.** A function \( f \) given by (1.1) is said to be in the class \( M_\Sigma(\beta, \lambda, \mu) \) if the following conditions are satisfied:

\[ f \in \Sigma, \ 0 \leq \beta < 1, \ 0 < \mu < 1, \ \lambda > \mu \]

\[ \Re \left( (1 - \lambda) \left( \frac{z}{f(z)} \right)^\mu + \lambda \left( \frac{z}{f(z)} \right)^{\mu + 1} \right) > \beta \quad z \in \mathbb{U}, \]

(3.1)

and

\[ \Re \left( (1 - \lambda) \left( \frac{w}{g(w)} \right)^\mu + \lambda \left( \frac{w}{g(w)} \right)^{\mu + 1} \right) > \beta \quad w \in \mathbb{U}, \]

(3.2)

where

\[ g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \]

**Theorem 3.2.** Let \( f(z) \) given by (1.1) be in the class \( M_\Sigma(\beta, \lambda, \mu) \), \( 0 < \beta < 1, \ 0 < \mu < 1, \ \lambda > \mu \). Then

\[ |a_2| \leq \min \left\{ \frac{2(1 - \beta)}{\lambda - \mu}, \ 2\sqrt{\frac{1 - \beta}{(1 - \mu)(2\lambda - \mu)}} \right\} \]

(3.3)

and

\[ |a_3| \leq \min \left\{ \frac{4(1 - \beta)^2}{(\lambda - \mu)^2} + \frac{2(1 - \beta)}{2\lambda - \mu}, \ \frac{4(1 - \beta)}{(2\lambda - \mu)(1 - \mu)} \right\}. \]

(3.4)
Proof. It follows from (3.1) and (3.2) that
\[
(1 - \lambda) \left( \frac{z}{f(z)} \right)^\mu + \lambda \left( \frac{z}{f(z)} \right)^{\mu+1} = \beta + (1 - \beta) p(z) \tag{3.5}
\]
and
\[
(1 - \lambda) \left( \frac{w}{g(w)} \right)^\mu + \lambda \left( \frac{w}{g(w)} \right)^{\mu+1} = \beta + (1 - \beta) q(w), \tag{3.6}
\]
where \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) and \( q(w) = 1 + q_1 w + q_2 w^2 + \cdots \) in \( \mathcal{P} \). Now on equating the coefficients in (3.5) and (3.6), we have
\[
(\lambda - \mu) a_2 = (1 - \beta) p_1 \tag{3.7}
\]
and
\[
(2\lambda - \mu) a_3 - \frac{(2\lambda - \mu)(\mu + 1)}{2} a_2^2 = (1 - \beta) p_2 \tag{3.8}
\]
and
\[-(\lambda - \mu) a_2 = (1 - \beta) q_1 \tag{3.9}\]
and
\[-(2\lambda - \mu) a_3 + \frac{(3 - \mu)(2\lambda - \mu)}{2} a_2^2 = (1 - \beta) q_2. \tag{3.10}\]
From (3.7) and (3.9), we get
\[
p_1 = -q_1 \tag{3.11}\]
and
\[
2(\lambda - \mu) a_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2). \tag{3.12}\]
From (3.8) and (3.10), we get
\[
[(1 - \mu)(2\lambda - \mu)] a_2^2 = (p_2 + q_2)(1 - \beta). \tag{3.13}\]
From (3.12) and (3.13), we get
\[
|a_2|^2 \leq \frac{(1 - \beta)^2 (|p_2|^2 + |q_2|^2)}{2(\lambda - \mu)^2} \tag{3.14}\]
and
\[
|a_2|^2 \leq \frac{(1 - \beta)(|p_2| + |q_2|)}{(1 - \mu)(2\lambda - \mu)}. \tag{3.15}\]
Applying Lemma 1.2 for (3.14) and (3.15), we get
\[
|a_2| \leq \frac{2(1 - \beta)}{(1 - \mu)(2\lambda - \mu)} ,
\]
and
\[
|a_2| \leq 2 \sqrt{\frac{1 - \beta}{(1 - \mu)(2\lambda - \mu)}},
\]
which gives us desired estimate on \( |a_2| \) as asserted in (3.3).
Next in order to find the bound on \( |a_3| \), by subtracting (3.10) from (3.8), we get
\[
2(2\lambda - \mu) a_3 - 2(2\lambda - \mu) a_2^2 = (1 - \beta)(p_2 - q_2). \tag{3.16}\]
It follows from (3.12) and (3.16)
\[
a_3 = \frac{(1 - \beta)^2 (p_1^2 + q_1^2)}{2(\lambda - \mu)^2} + \frac{(1 - \beta)(p_2 - q_2)}{2(2\lambda - \mu)}. \tag{3.17}\]
Applying Lemma 1.2 for (3.17), we get
\[
|a_3| \leq \frac{4(1 - \beta)^2}{(\lambda - \mu)^2} + \frac{2(1 - \beta)}{2\lambda - \mu}.\]
On the other hand, by using (3.13) and (3.16), we obtain
\[ a_3 = \frac{1 - \beta}{2(2\lambda - \mu)} \left[ \frac{3 - \mu}{1 - \mu}p_2 + \frac{1 + \mu}{1 - \mu}q_2 \right], \tag{3.18} \]
which gives
\[ |a_3| = \frac{4(1 - \beta)}{(2\lambda - \mu)(1 - \mu)}. \tag{3.19} \]
This completes the proof of Theorem 3.2.

References


Author information

Pravati Sahoo and Saumya Singh, Department of Mathematics, Banaras Hindu University, Banaras, INDIA and Department of Mathematics, O. P. Jindal University, Raigarh, INDIA.
E-mail: pravatis@yahoo.co.in

Received: December 21, 2015.

Accepted: October 3, 2016.