Primal and weakly primal ideals in $C(X)$

Ahmad Yousefian Darani

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13A15, 13C05; Secondary 13F05.

Keywords and phrases: Prime ideal, Primal ideal, Weakly prime ideal, Weakly primal ideal, Prime $z$-filter, Primal $z$-filter, Weakly prime $z$-filter, Weakly primal $z$-filter.

Abstract Let $X$ be a completely regular Hausdorff space, and consider the ring of continuous functions $C(X)$. A $z$-filter $\mathcal{F}$ on $X$ is called primal if the set of all elements of $Z(X)$ that are not prime to $\mathcal{F}$ forms a $z$-filter on $X$: here an element $Z \in Z(X)$ is called prime to $\mathcal{F}$ if $Z \cup Z' \in \mathcal{F}$ implies that $Z' \in \mathcal{F}$. In this paper we consider the primal $z$-filters on $X$, and then we discuss on the relations between this class of $z$-filters on $X$ and the class of primal ideals of $C(X)$. We also define the concept weakly prime and weakly primal $z$-filters on $X$ and show that there is a one-to-one correspondence between the weakly prime (resp. weakly primal) $z$-filters on $X$ and the weakly prime (resp. weakly primal) ideals of $C(X)$.

1 introduction

Prime $z$-filters play a central role in the study of the rings of continuous functions. Of course, a prime $z$-filter $\mathcal{F}$ on a completely regular Hausdorff space $X$ is $z$-filter $\mathcal{F}$ on $X$ with the property that

$$Z, Z' \in Z(X), Z \cup Z' \in \mathcal{F} \Rightarrow Z \in \mathcal{F} \text{ or } Z' \in \mathcal{F}.$$ 

There are several ways to generalize the notion of a prime $z$-filters. We could either restrict or enlarge where $Z$ and/or $Z'$ lie or restrict or enlarge where $Z \cup Z'$ lies. In this paper we will be mostly interested in generalizations obtained by restricting where $Z \cup Z'$ lies.

Let $R$ be a commutative ring, $I$ an ideal of $R$ and $J$ a subset of $R$. We denote by $(I :_R J)$ the set of all elements $r \in R$ with $ra \in I$ for every $a \in J$. Then the annihilator of $J$, denoted by $Ann_R(J)$ is just $(0 :_R J)$. An element $a \in R$ is called a zero-divisor of $R$ provided that $Ann(a) \neq 0$. We denote by $Z(R)$, the set of all zero-divisors of $R$.

We recall from [4] that an element $a \in R$ is called prime to $I$ if $ra \in I$ (where $r \in R$) implies that $r \in I$, that is $(I :_R a) = I$. Denote by $S(I)$ the set of elements of $R$ that are not prime to $I$, that is

$$S(I) = \{ a \in R | ra \in I \text{ for some } r \in R \setminus I \}.$$ 

Then $I$ is said to be primal if $S(I)$ forms an ideal of $R$; this ideal is always a prime ideal, called the adjoint prime ideal $P$ of $I$. In this case we also say that $I$ is a $P$-primal ideal of $R$ ([4]). It is easy to see that $I$ is a $P$-primal ideal of $R$ if and only if $Z(R/I) = P/I$. Then ring $R$ is called coprimal if the zero ideal of $R$ is primal.

The concept of weakly primal ideals in a commutative ring $R$ studied in [2]. An element $a \in R$ is called weakly prime to the ideal $I$ if $0 \neq ra \in I$ ($r \in R$) implies $r \in I$. Clearly 0 is always weakly prime to $I$. Denote by $W(I)$ the set of all elements of $R$ which are not weakly prime to $I$, that is

$$W(I) = \{ a \in R | 0 \neq ra \in I \text{ for some } r \in R \setminus I \}.$$ 

$I$ is called weakly primal provided that the set $P := W(I) \cup \{0\}$ forms an ideal of $R$. Then the ideal is a weakly prime ideal of $R$, called the weakly prime adjoint ideal of $R$. We also say that $I$ is $P$-weakly primal ideal. Of course a proper ideal $P$ of $R$ is said to be weakly prime if $0 \neq ab \in R$ implies either $a \in P$ or $b \in P$ [1].
Throughout this paper $X$ is a completely regular Hausdorff space. Let $C(X)$ be the ring of all real-valued continuous functions on $X$. We list here some standard facts, terminology and notation for reference. The set of all positive integers is denoted by $N$. In any ring $C(X)$, the constant function whose value is $r$ is designated by $r$. For any $f \in C(X)$, we write $Z(f)$ for the set

$$\{ x \in X : f(x) = 0 \}$$

$Z(f)$ is called a zero-set in $X$. For $C' \subseteq C(X)$, we write $Z[C']$ to designate the family of zero-sets in $C'$, that is

$$Z[C'] = \{ Z(f) : f \in C' \}.$$ 

On the other hand, the family $Z[C(X)]$ of all zero-sets in $X$ will also be denoted, for simplicity, by $Z(X)$.

We shall occasionally refer to the ring $C(X)$ itself as an improper ideal. Thus, the word ideal, unmodified, will always mean proper ideal. For any ideal $I$ in $C(X)$ and $f \in C(X)$, the residue class of $f$ modulo $I$ is written $I[f]$. The ideal $I$ is called a $z$-ideal if $Z(f) \in Z[I]$ implies that $f \in I$. By a prime $z$-filter, we shall mean a $z$-filter $\mathcal{F}$ on $X$ with the property that whenever the union of two zero-sets belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$. For any undefined terms here the reader may consult [6].

The structure of the family of prime ideals in $C(X)$ has been extensively studied in [10, 11, 12]. In this paper we first study the basic properties of the family of primal ideals of the ring $C(X)$. We define primal, weakly prime and weakly primal $z$-filters on $X$. Then we show that there exists a one-to-one correspondence between the set of all primal $z$-ideals (resp. weakly prime, weakly primal) of $C(X)$ and the set of all primal (resp. weakly prime, weakly primal) $z$-filters on $X$.

2 Primal ideals in $C(X)$

In this section we discuss on primal ideals of $C(X)$ and consider the relations between primal ideals of $C(X)$ and primal $z$-filters on $X$.

**Definition 2.1.** Let $\mathcal{F}$ be a $z$-filter on $X$. An element $Z$ in $Z(X)$ is called $z$-prime to $\mathcal{F}$ provided that $Z \cup Z' \in \mathcal{F}$ implies $Z' \in \mathcal{F}$.

**Lemma 2.2.** Let $\mathcal{F}$ be a $z$-filter on $X$ and denote by $T(\mathcal{F})$ the set of all elements of $Z(X)$ that are not $z$-prime to $\mathcal{F}$, that is

$$T(\mathcal{F}) = \{ Z \in Z(X) | Z \cup Z' \in \mathcal{F} \text{ for some } Z' \in Z(X) \setminus \mathcal{F} \}.$$ 

Then:

1. $\mathcal{F} \subseteq T(\mathcal{F})$, and
2. If $T(\mathcal{F})$ forms a $z$-filter on $X$, then it is a prime $z$-filter.

**Proof.** (1) For every $Z(f) \in \mathcal{F}$, the relations

$$Z(f) \cup Z(1) \in \mathcal{F} \text{ with } Z(1) = \emptyset \notin \mathcal{F}$$

imply that $Z(f)$ is not $z$-prime to $\mathcal{F}$. Hence $Z(f) \notin T(\mathcal{F})$ and so $\mathcal{F} \subseteq T(\mathcal{F})$.

(2) Assume that $Z(f) \cup Z(g) \in T(\mathcal{F})$ but $Z(f) \notin T(\mathcal{F})$. There exists $Z(h) \in Z(X) \setminus \mathcal{F}$ such that $Z(f) \cup Z(g) \cup Z(h) \in \mathcal{F}$. As $Z(f)$ is $z$-prime to $\mathcal{F}$, we get $Z(g) \cup Z(h) \in \mathcal{F}$ with $Z(h) \in Z(X) \setminus \mathcal{F}$, that is $Z(g)$ is not $z$-prime to $\mathcal{F}$. So $Z(g) \in T(\mathcal{F})$. Therefore $T(\mathcal{F})$ is a prime $z$-filter.

$\square$
**Definition 2.3.** A z-filter $\mathcal{F}$ on $X$ is called a primal z-filter if $T(\mathcal{F})$ forms a z-filter on $X$. In this case, by Lemma 2.2, the z-filter $\mathcal{G} := T(\mathcal{F})$ is a prime z-filter, called the adjoint prime z-filter of $\mathcal{F}$. We will also say that $\mathcal{F}$ is a $\mathcal{G}$-primal z-filter.

**Theorem 2.4.** (1) Let $I$ and $P$ be z-ideals of $C(X)$. If $I$ is a $P$-primal of $C(X)$, then $Z[I]$ is a primal z-filter on $X$.

(2) If $\mathcal{F}$ is a primal z-filter on $X$, then $Z^{-}[\mathcal{F}]$ is a primal z-ideal of $C(X)$.

**Proof.** (1) We know that $P$ is a prime ideal of $C(X)$. So, by [6, Theorem P.29], $Z[P]$ is a prime z-filter on $X$. Assume that $Z(f) \in Z[P]$ is not z-prime to $Z[I]$. There exists $Z(g) \in Z(X) \setminus Z[I]$ such that $Z(f) \cup Z(g) \in Z[I]$. So $Z(fg) \in Z[I]$ and $I$ z-ideal gives $fg \in I$. This implies that $g \in C(X) \setminus I$ with $fg \in I$, that is $f$ is not prime to $I$. So $f \in P$ and so $Z(f) \in Z[P]$. Now assume that $Z(h) \in Z[P]$. As $P$ is a z-ideal, $h \in P$. Therefore $hk \in I$ for some $k \in C(X) \setminus I$. It follows that $Z(h) \cup Z(k) \in Z[I]$ where $Z(k) \in Z(X) \setminus Z[I]$. Thus $Z(h)$ is not z-prime to $Z[I]$. We have already shown that $Z[P]$ consists exactly of elements of $Z(X)$ that are not z-prime to $Z[I]$. This shows that $Z[I]$ is a $Z[P]$-primal z-filter.

(2) Clearly $Z^{-}[\mathcal{F}]$ is a z-ideal of $C(X)$. Assume that $\mathcal{F}$ is $\mathcal{G}$-primal. By Lemma 2.2 and [6, Theorem p.29], $Z^{-}[\mathcal{G}]$ is a prime z-ideal of $C(X)$. It is enough to show that $Z^{-}[\mathcal{G}] = T(Z^{-}[\mathcal{F}])$. If $f \in C(X)$ is not prime to $Z^{-}[\mathcal{F}]$, then $fg \in Z^{-}[\mathcal{F}]$ for some $g \in C(X) \setminus Z^{-}[\mathcal{F}]$. This implies that $Z(f) \cup Z(g) \in \mathcal{F}$ with $Z(g) \notin \mathcal{F}$, that is $Z(f)$ is not z-prime to $\mathcal{F}$. Therefore $Z(f) \notin \mathcal{G}$ and so $f \in Z^{-}[\mathcal{G}]$. Conversely, assume that $h \in Z^{-}[\mathcal{G}]$. Then as $Z(h)$ is not z-prime to $\mathcal{F}$, there exists $Z(k) \in Z(X) \setminus \mathcal{F}$ with $Z(h) \cup Z(k) \in \mathcal{F}$. It follows that $hk \in Z^{-}[\mathcal{F}]$ with $k \in C(X) \setminus Z^{-}[\mathcal{F}]$, that is $h$ is not prime to $Z^{-}[\mathcal{F}]$.

**Lemma 2.5.** Every prime z-filter is primal.

**Proof.** Let $\mathcal{F}$ be a prime $Z$-filter. Then $Z^{-}[\mathcal{F}]$ is a prime z-ideal of $C(X)$ by [6, Theorem p.29]. But in any commutative ring, every prime ideal is primal. Hence $Z^{-}[\mathcal{F}]$ is a primal z-ideal of $C(X)$. Now the result follows from Theorem 2.4. □

An annihilator condition on a commutative ring $R$ is property (A). $R$ is said to have property (A) if every finitely generated ideal $I$ contained in $Z(R)$ has a nonzero annihilator ([7]). Y. Quentzel introduced property (A) in [14], calling it condition (C). Faith in [4] studied rings with property (A) and called such rings McCoy. An example of a McCoy ring is a Noetherian ring. However, the property (A) fails for some non-Noetherian rings [9, p. 63]. To avoid the ambiguity we call such rings F-McCoy.

Recently the concept of rings with property (A) has been generalized to noncommutative rings [8]. Let $R$ be an associative ring with identity. We write $Z_l(R)$ and $Z_r(R)$ for the set of all left zero-divisors of $R$ and the set of all right zero-divisors of $R$, respectively. Then the ring $R$ has right (left) Property (A) if for every finitely generated two-sided ideal $I \subseteq Z_l(R)$ ($Z_r(R)$), there exists nonzero $a \in R (b \in R)$ such that $|a| = 0 (|b| = 0)$. A ring $R$ is said to have Property (A) if $R$ has right and left Property (A).

Nielsen in [13] defined another class of rings and called it McCoy. This paper is on the basis of some recent papers devoted to this new class of rings. Let $R$ be an associative ring with 1 (not necessarily commutative). $R$ is said to be right McCoy when the equation $f(x)g(x) = 0$ over $R[x]$, where $f(x), g(x) \neq 0$, implies there exists a nonzero $r \in R$ with $f(x)r = 0$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then $R$ is called a McCoy ring. This class of McCoy rings includes properly the class of Armendariz rings introduced in [15], which is extensively studied in the last years.

Let $R$ be a commutative ring with identity. Then concepts "F-McCoy ring" and "McCoy ring" are different concepts. In fact neither implies the other. For example, if $R$ is a reduced ring, then it is McCoy by [13, Theorem 2]. But we know that there are reduced rings which are not F-McCoy. Also if we let $Z_4$ to be the ring of integers modulo 4, then, by [8, Theorem 2.1], $M_2(Z_4)$, the set of all $2 \times 2$ matrices over $Z_4$, has Property (A) but it is not right McCoy by [16].

The commutative ring $R$ is called strongly co-primal (resp. Super co-primal) if for arbitrary $a, b \in Z(R)$ (resp. finite subset $E$ of $Z(R)$) the annihilator of $\{a, b\}$ (resp. annihilator of $E$) in

"Primal and weakly primal ideals in $C(X)$"
$R$ is non-zero. Clearly, $R$ is a strongly coprimal if and only if $R$ is both a coprimal and a $F$-McCoy ring. In the following Theorem, we give some conditions under which $C(X)$ is strongly coprimal (resp. super coprimal) [17].

**Theorem 2.6.** (1) The ring $C(X)$ is strongly primal if and only if $\text{int}Z(f_1) \cap \text{int}Z(f_2) \neq \emptyset$ for every $f_1$ and $f_2$ in $Z(C(X))$.

(2) The ring $C(X)$ is super primal if and only if $\text{int}Z(f_1) \cap \text{int}Z(f_2) \cap \ldots \cap \text{int}Z(f_n) \neq \emptyset$ for every $f_1, f_2, \ldots, f_n$ in $Z(C(X))$.

**Proof.** (1) Assume that $C(X)$ is strongly primal. Then, for every $f_1, f_2 \in Z(C(X))$, $\text{Ann}(f_1, f_2) \neq 0$. So there exists a nonzero element $g \in C(X)$ with $gf_1 = gf_2 = 0$. In this case, for every $x \in X$, if $g(x) \neq 0$ we have $f_1(x) = f_2(x) = 0$, that is $coZ(g) \subseteq Z(f_1) \cap Z(f_2)$. Therefore $\text{int}Z(f_1) \cap \text{int}Z(f_2) \neq \emptyset$. Now suppose that $\text{int}Z(f_1) \cap \text{int}Z(f_2) \neq \emptyset$ for every $f_1, f_2 \in Z(C(X))$. Set $Y = \text{int}Z(f_1) \cap \text{int}Z(f_2)$ and define the map $g : X \to R$ as follows:

$$g(x) = \begin{cases} 1, & x \in Y; \\ 0, & x \in X - Y. \end{cases}$$

Then $g$ is a continuous function. So $0 \neq C(X)$, and for every $x \in X$, $g(x)f_1(x) = 0$, $g(x)f_2(x) = 0$, that is $gf_1 = 0 = gf_2$. Consequently $\text{Ann} \{f_1, f_2\} \neq 0$. Thus $C(X)$ is strongly primal.

(2) The proof of this part is completely to that of part (1). \qed

## 3 Weakly prime and Weakly primal ideals

The concept of weakly prime and weakly primal ideals in a commutative ring introduced in [1, 2]. In this section we define the weakly prime and weakly primal $z$-filters on $X$ and then we investigate the relations among these classes of $z$-filters, weakly prime and weakly primal ideals.

**Definition 3.1.** Assume that $F$ is a $z$-filter on $X$. $F$ is said to be a weakly prime $z$-filter whenever, for $Z, Z' \in Z(X)$, $X \neq Z \cup Z' \notin F$ implies that either $Z \in F$ or $Z' \in F$.

**Lemma 3.2.** Every prime $z$-filter is weakly prime.

**Theorem 3.3.** (1) If $P$ is a weakly prime $z$-ideal in $C(X)$, then $Z[P]$ is a weakly prime $z$-filter on $X$.

(2) If $F$ is a weakly prime $z$-filter on $X$, then $Z^+[F]$ is a weakly prime $z$-ideal of $C(X)$.

**Proof.** (1) Let $P$ be a weakly prime $z$-ideal in $C(X)$. Clearly $Z[P]$ is a $z$-filter on $X$. Assume that $X \neq Z(f) \cup Z(g) \in Z[P]$ for some $Z(f), Z(g) \in Z(X)$. Then $Z(0) \neq Z(fg) = Z(f) \cup Z(g) \in Z[P]$. Since $P$ is a $z$-ideal, we have $0 \neq fg \in P$. Therefore either $f \in P$ or $g \in P$ since $P$ is assumed to be weakly prime. It follows that either $Z(f) \in Z[P]$ or $Z(g) \in Z[P]$, that is $Z[P]$ is a weakly prime $z$-filter.

(2) Assume that $F$ is a weakly prime $z$-filter on $X$. In this case $P = Z^+[F]$ is a $z$-ideal of $C(X)$. Suppose that $f, g \in C(X)$ are such that $0 \neq fg \in P$. Then, $X \neq Z(f) \cup Z(g) = Z(fg) \in Z[Z^+[F]] = F$. Since $F$ is weakly prime, either $Z(f) \in F$ or $Z(g) \in F$. Thus either $f \in P$ or $g \in P$, and this implies that $P$ is a weakly prime $z$-ideal of $C(X)$.

**Definition 3.4.** Assume that $F$ is a $z$-filter on $X$. An element $Z$ in $Z(X)$ is called $z$-weakly prime to $F$ provided that $X \neq Z \cup Z' \in F (Z' \in Z(X))$ implies that $Z' \in F$.

**Remark 3.5.** Let $F$ be a $z$-filter on $X$. Then:

(1) $X$ (i.e. $Z(0)$) is always $z$-weakly prime to $F$

(2) If $Z \in Z(X)$ is $z$-prime to $F$, then it is $z$-weakly prime to $F$. 

Ahmad Yousefian Damani
Lemma 3.6. Let $\mathcal{F}$ be a $z$-filter on $X$ and denote by $W(\mathcal{F})$ the set of all elements of $Z(X)$ that are not $z$-weakly prime to $\mathcal{F}$. Then:

1. $\mathcal{F} \subseteq W(\mathcal{F}) \cup \{X\}$, and

2. If $W(\mathcal{F}) \cup \{X\}$ forms a $z$-filter on $X$, then it is a weakly prime $z$-filter.

Proof. (1) For every $Z(f) \in F - \{X\}$ we have:

$$X \neq Z(f) = Z(f) \cup Z(1) \in F$$

with $Z(1) = \emptyset \notin \mathcal{F}$. This implies that $Z(f)$ is not $z$-weakly prime to $\mathcal{F}$. Hence $Z(f) \in W(\mathcal{F})$. Therefore $\mathcal{F} \subseteq W(\mathcal{F}) \cup \{X\}$.

(2) Let $Z(f), Z(g) \in Z(X)$ be such that $X \neq Z(f) \cup Z(g) \in W(\mathcal{F}) \cup \{X\}$. Suppose also that $Z(f) \notin W(\mathcal{F}) \cup \{X\}$, that is $Z(f)$ is $z$-weakly prime to $\mathcal{F}$. There exists $Z(h) \in Z(X) \setminus \mathcal{F}$ such that $X \neq Z(f) \cup Z(g) \cup Z(h) \notin \mathcal{F}$. Now $Z(f)$ is $z$-weakly prime to $\mathcal{F}$ implies that $X \neq Z(g) \cup Z(h) \in \mathcal{F}$ with $Z(h) \in Z(X) \setminus \mathcal{F}$, that is $Z(g)$ is not $z$-weakly prime to $\mathcal{F}$, hence $Z(g) \in W(\mathcal{F}) \cup \{X\}$, that is $W(\mathcal{F}) \cup \{X\}$ is a weakly prime $z$-filter on $X$.

Definition 3.7. Assume that $\mathcal{F}$ is a $z$-filter on $X$. $\mathcal{F}$ is called a weakly primal $z$-filter on $X$ if $W(\mathcal{F}) \cup \{X\}$ forms a $z$-filter on $X$. In this case, by Lemma 3.6, the $z$-filter $\mathcal{G} := W(\mathcal{F}) \cup \{X\}$ is a weakly prime $z$-filter, called the adjacent weakly prime $z$-filter of $\mathcal{F}$. In this case we sat that $\mathcal{F}$ is a $\mathcal{G}$-weakly primal $z$-filter.

Theorem 3.8. Every weakly prime $z$-filter on $X$ is weakly primal.

Proof. Assume that $\mathcal{F}$ is a weakly prime $z$-filter on $X$. Then $\mathcal{F} \subseteq W(\mathcal{F}) \cup \{X\}$ by Lemma 3.6. Now pick an element $Z(f) \in W(\mathcal{F}) \cup \{X\}$. If $Z(f) = X$, then $Z(f) \in \mathcal{F}$. So assume that $Z(f) \neq X$. Then $Z(f)$ is $z$-weakly prime to $\mathcal{F}$. So there exists $Z(g) \in Z(X) - \mathcal{F}$ with $X \neq Z(f) \cup Z(g) \in \mathcal{F}$. Since $\mathcal{F}$ is a weakly prime $z$-filter we get $Z(f) \in \mathcal{F}$, that is $W(\mathcal{F}) \cup \{X\} \subseteq \mathcal{F}$. Hence $\mathcal{F} = W(\mathcal{F}) \cup \{X\}$, and this implies that $\mathcal{F}$ is an $\mathcal{F}$-weakly primal $z$-filter.

Theorem 3.9. (1) Let $I$ be a $P$-weakly primal ideal of $\text{C}(X)$, where $I$ and $P$ are both $z$-ideals. Then $Z[I]$ is a primal $z$-filter on $X$ with the weakly prime adjoint $z$-filter $Z[P]$.

(2) If $\mathcal{F}$ is a $\mathcal{G}$-weakly primal $z$-filter on $X$, then $Z^-(\mathcal{F})$ is a weakly primal $z$-ideal of $\text{C}(X)$ with the weakly prime adjoint ideal $Z^-[\mathcal{F}]$.

Proof. The proof is completely similar to that of Theorem 2.4 and we omit it.

References


**Author information**

Ahmad Yousefian Danani, Department of Mathematics and Applications, University of Mohaghegh Ardabili, Arghbil 56199-11367, Iran.

E-mail: yousefian@uma.ac.ir

Received: July 12, 2016.

Accepted: December 11, 2016.